

A Structural Characterization for Certifying Robinsonian Matrices

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Abstract

A symmetric matrix is Robinsonian if its rows and columns can be simultaneously reordered in such a way that entries are monotone nondecreasing in rows and columns when moving toward the diagonal. The adjacency matrix of a graph is Robinsonian precisely when the graph is a unit interval graph, so that Robinsonian matrices form a matrix analogue of the class of unit interval graphs. Here we provide a structural characterization for Robinsonian matrices in terms of forbidden substructures, extending the notion of asteroidal triples to weighted graphs. This implies the known characterization of unit interval graphs and leads to an efficient algorithm for certifying that a matrix is not Robinsonian.

Keywords: Robinsonian matrix; seriation; unit interval graph; asteroidal triple

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1 Introduction

1.1 Background

Robinsonian matrices are a special class of structured matrices introduced by Robinson [24] to model the seriation problem, a combinatorial problem arising in data analysis, which asks to sequence a set of objects according to their pairwise similarities in such a way that similar objects are ordered close to each other. Seriation has many applications in a wide range of different subjects, including archaeology [21, 13], data visualization and exploratory analysis [12, 1], bioinformatics (e.g., microarray gene expression [25]), machine learning (e.g., to pre-estimate the number and shape of clusters [6, 11]). We refer to the survey [18] for details and further references.

A symmetric $n \times n$ matrix A is called a *Robinson similarity* if its entries are monotone nondecreasing in the rows and columns when moving toward the main diagonal, i.e., if

$$A_{xz} \leq \min\{A_{xy}, A_{yz}\} \quad (1)$$

for all $1 \leq x < y < z \leq n$. Throughout we will call any ordered triple (x, y, z) satisfying the condition (1) a *Robinson triple*. If the rows and columns of A can be symmetrically reordered by a permutation π in such a way that the permuted matrix is a Robinson similarity, then A is said to be a *Robinsonian similarity* and π is called a *Robinson ordering* of A . By construction, π is a Robinson ordering of A if any triple (x, y, z) ordered in π as $x \prec_{\pi} y \prec_{\pi} z$ is Robinson. Hence Robinsonian matrices best achieve the goal of seriation, which is to order similar objects close to each other.

Several Robinsonian recognition algorithms are known, permitting to check whether a matrix is Robinsonian in polynomial time. Most of the known algorithms rely on characterizations of Robinsonian matrices in terms of interval (hyper)graphs and the consecutive-ones property (C1P). Specifically, call a *ball* of A any set of the form $B(x, \rho) := \{y \in V : A_{xy} \geq \rho\}$ for some $x \in V$ and scalar $\rho > 0$. Let the *ball hypergraph* of A be the hypergraph whose vertex set is V and with hyperedges the balls of A . Then A is Robinsonian if and only if its ball hypergraph is an interval hypergraph [20]. Equivalently, define the *ball intersection graph* of A as the graph whose vertex set is the set of balls of A , with an edge between two distinct balls if they intersect. Then, it is known that A is a Robinsonian matrix if and only if its ball intersection graph is an interval graph (see [22]). Using the above characterizations, distinct recognition algorithms were introduced by Mirkin and Rodin [20] with running time $O(n^4)$, by Chepoi and Fichet [2] with running time $O(n^3)$, and by Pr ea and Fortin [22] with running time $O(n^2)$, when applied to a $n \times n$ symmetric matrix.

Different algorithms were recently introduced in [14, 15], based on a link between Robinsonian matrices and unit interval graphs (pointed out in [23]) and exploiting the fact that unit interval graphs can be recognized efficiently using a simple graph search algorithm, namely Lexicographic Breadth-First Search (Lex-BFS) (see [4, 3]). The algorithm of [14] is based on expressing Robinsonian matrices as conic combinations of (adjacency matrices of) unit interval graphs and iteratively using Lex-BFS to check whether these

are unit interval graphs; its overall running time is $O(L(m+n))$, where L is the number of distinct values in the matrix and m is its number of nonzero entries. The algorithm of [15] relies on a new search algorithm, Similarity-First Search (SFS), which can be seen as a generalization of Lexicographic Breadth-First Search (Lex-BFS) to the setting of weighted graphs. The SFS algorithm runs in $O(n+m\log n)$ time and the recognition algorithm for Robinsonian matrices terminates after at most n iterations of SFS, thus with overall running time $O(n^2+nm\log n)$ [15].

All the current recognition algorithms return a certificate (i.e., a Robinson ordering) only if the matrix is Robinsonian, and otherwise they just return a negative answer. In this paper we give a new structural characterization of Robinsonian matrices in terms of forbidden substructures. We provide a simple, succinct certificate for *non*-Robinsonian matrices, which represents a natural extension of the known structural characterization for unit interval graphs. Specifically, our certificate involves the new notion of *weighted* asteroidal triple, which generalizes to the matrix setting the known obstructions for unit interval graphs (namely, chordless cycles, claws and asteroidal triples), see Section 1.3 for details. Moreover we also give a simple, efficient algorithm for finding such a certificate, running in time $O(n^3)$ for a matrix of size n .

Observe that other certificates could be obtained from the alternative characterizations of Robinsonian matrices in terms of interval (hyper)graphs. Indeed, as the minimal obstructions for interval graphs are known (namely, they are the chordless cycles and the asteroidal triples), we can derive from this an alternative structural characterization for Robinsonian matrices. However this characterization is expressed in terms of the ball intersection graph of A , whose vertex set is the set of balls rather than the index set of A , and thus it is not directly in terms of A as in our main result (Theorem 3 below).

In the rest of the introduction, we first recall some properties of unit interval graphs, which will also serve as motivation for the notions and results we will introduce for Robinsonian matrices, and then we state our main structural result for Robinsonian matrices.

1.2 Structural characterization of unit interval graphs

Recall that a graph $G = (V = [n], E)$ is a unit interval graph if one can label its vertices by unit intervals in such a way that adjacent vertices correspond to intersecting intervals. Roberts [23] observed that a graph G is a unit interval graph if and only if its adjacency matrix A_G is a Robinsonian similarity matrix.

In particular, G is a unit interval graph if and only if there exists an ordering π of its vertices such that $\{x, z\} \in E$ implies $\{x, y\} \in E$ and $\{y, z\} \in E$ whenever $x \prec_\pi y \prec_\pi z$. This condition, known as the *3-vertex condition*, is thus a specialization of the Robinson condition (1) (see, e.g., [19]).

We now mention an alternative characterization of unit interval graphs in terms of forbidden substructures. We recall some definitions. Let $G = (V, E)$ be a graph. Given

$x, y, z \in V$, a *path from x to y missing¹ z* is a path $P = (x = x_0, x_1, \dots, x_k, y = x_{k+1})$ in G which is disjoint from the neighborhood of z , i.e., all pairs $\{x_i, x_{i+1}\}$ ($0 \leq i \leq k$) are edges of G and z is not adjacent to any node of P . An *asteroidal triple* in G is a set of nodes $\{x, y, z\}$ which is independent in G (i.e., induces no edge) and such that between any two nodes in $\{x, y, z\}$ there exists a path in G between them which misses the third one. Asteroidal triples were introduced for the recognition of interval graphs in [16], where it is shown that a graph is an interval graph if and only if it is chordal and does not have an asteroidal triple. As unit interval graphs are precisely the claw-free interval graphs (see [23]) we get the following characterization.

Theorem 1. (see [23, 8]) *A graph G is a unit interval graph if and only if it satisfies the following three conditions:*

- (i) *G is chordal, i.e., G does not contain an induced cycle of length at least 4;*
- (ii) *G does not contain an induced claw $K_{1,3}$;*
- (iii) *G does not contain an asteroidal triple.*

1.3 Structural characterization of Robinsonian matrices

We now extend the above notion of asteroidal triple to the general setting of matrices and we then use it to state our main structural characterization for Robinsonian matrices.

Let V be a finite set and let A be a symmetric matrix indexed by V . Given $z \in V$, a *path avoiding z in A* is of the form $P = (v_0, v_1, \dots, v_k)$, where v_0, \dots, v_k are distinct elements of V and, for each $1 \leq i \leq k$, the triple (v_{i-1}, z, v_i) is not Robinson, i.e., $A_{v_{i-1}v_i} > \min\{A_{v_{i-1}z}, A_{v_i z}\}$. Throughout, for distinct elements $x, y, z \in V$, we will use the notation $x \overset{z}{\sim} y$ to denote that there exists a path from x to y avoiding z in A .

This concept was introduced in [15, Definition 2.3], where it is used as a key tool for analyzing the new recognition algorithm for Robinsonian matrices. Indeed, saying that the pair (x, y) avoids z means that the triple (x, z, y) is not Robinson (and the same for its reverse (y, z, x)), so that z cannot be placed between x and y in *any* Robinson ordering of A . An important consequence, as observed in [15, Lemma 2.4], is then that if there exists a path from x to y avoiding z , i.e., $x \overset{z}{\sim} y$, then z cannot be placed between x and y in any Robinson ordering of A .

Therefore, if there exist three distinct elements x, y, z with $x \overset{z}{\sim} y$, $y \overset{x}{\sim} z$ and $z \overset{y}{\sim} x$ in A , then we can conclude that no Robinson ordering of A exists and thus that A is not a Robinsonian similarity matrix. This motivates the following definition.

Definition 2. Let A be a symmetric matrix. A triple $\{x, y, z\}$ of distinct elements of V is called a *weighted asteroidal triple of A* if $x \overset{z}{\sim} y$, $y \overset{x}{\sim} z$ and $z \overset{y}{\sim} x$ hold, i.e., for any two elements of $\{x, y, z\}$ there exists a path between them avoiding the third one.

¹Sometimes one also says that the path *avoids* z (e.g. in [5]). We use here the word “miss” instead of “avoid”, in order to keep the word “avoid” for the context of matrices and to prevent possible confusion.

Our main result is that weighted asteroidal triples are the *only* obstructions to the Robinsonian property.

Theorem 3. *A symmetric matrix A is a Robinsonian similarity matrix if and only if there does not exist a weighted asteroidal triple in A .*

If we apply this result to the adjacency matrix A_G of a graph G we obtain that G is a unit interval graph if and only if there does not exist a weighted asteroidal triple in A_G . As we will show in Section 4.1, the structural characterization of unit interval graphs from Theorem 1 can in fact be derived from our main result in Theorem 3. Indeed, we will show that the notion of weighted asteroidal triple in A_G subsumes the notions of claw, chordless cycle and asteroidal triple in G .

1.4 Organization of the paper

In Section 2 we group preliminary notions and results that will be used in the rest of the paper, in particular, about homogeneous sets, critical elements, and similarity layer structures. Section 3 is devoted to the proof of our main result (Theorem 3). In Section 4, we first show how to recover the known characterization of unit interval graphs (Theorem 1) from our main result. Then we give a simple algorithm for finding all weighted asteroidal triples (or decide that none exists), that runs in $O(n^3)$. Finally we conclude with some remarks about the problem of finding the largest Robinsonian submatrix when the given matrix is not Robinsonian.

2 Preliminary results

In this section we introduce some notation and basic results that we will need throughout the paper.

2.1 Homogeneous sets and critical elements

We first introduce the notion of ‘homogeneous set’ for a given symmetric matrix A , which we then use to reduce the problem of checking whether A is Robinsonian to the same problem on two smaller submatrices of A .

Definition 4. Let A be a symmetric matrix indexed by V . A set $S \subseteq V$ is said to be:

- *homogeneous* for A if $A_{xy} = A_{xz}$ for all $x \in V \setminus X$ and $y, z \in X$;
- *strongly homogeneous* for A if $A_{xy} = A_{xz} \leq A_{yz}$ for all $x \in V \setminus X$ and $y, z \in X$;
- *proper* if $2 \leq |S| \leq |V| - 1$.

Assume that S is proper strongly homogeneous for A . We will consider the following two submatrices of A :

- the *restriction* $A[S]$ of A to S , which is the submatrix of A indexed by S ;

- the contraction A/S of A by S , which is defined as the submatrix $A[\overline{S} \cup \{s\}]$, where $\overline{S} = V \setminus S$ and s is an arbitrary element of S (thus contracting S to a single element).

These definitions are motivated by the following lemma which shows that, if S is a proper strongly homogeneous set for A , then the problem of recognizing whether A is Robinsonian and if not of finding a weighted asteroidal triple can be reduced to the same problem for the two matrices $A[S]$ and A/S .

Lemma 5. *Let A be a symmetric matrix and let S be a proper strongly homogeneous set for A . Then A is Robinsonian if and only if both $A[S]$ and A/S are Robinsonian. Moreover:*

- (i) *If $\sigma_1 = (x_1, \dots, x_p)$ is a Robinson ordering of S and $\sigma_2 = (y_1, \dots, y_{j-1}, s, y_j, \dots, y_q)$ is a Robinson ordering of A/S , then $\sigma = (y_1, \dots, y_{j-1}, x_1, \dots, x_p, y_j, \dots, y_q)$ is a Robinson ordering of A .*
- (ii) *Any weighted asteroidal triple of $A[S]$ or of A/S is a weighted asteroidal triple of A .*

Proof. Direct verification. □

In view of this lemma, the core difficulty in the proof of Theorem 3 is the case when A has no proper strongly homogeneous set. The following notion of *critical* element will play a key role for analyzing this case.

Definition 6. Let A be a symmetric matrix indexed by V . Then $a \in V$ is said to be *critical* for A if $x \overset{a}{\sim} y$ holds for all distinct elements $x, y \in V \setminus \{a\}$.

Note that if a is a critical element of a Robinsonian matrix A , then it must be an end point of *any* Robinson ordering of A . On the other hand, an end point of a Robinson ordering might not be critical. In this work, we will study critical elements for arbitrary (not necessarily Robinsonian) matrices. The following lemma shows that any symmetric matrix A has a critical element or a proper strongly homogeneous set.

Lemma 7. *Given a symmetric matrix A , one can find a critical element or a proper strongly homogeneous set for A .*

Proof. The proof relies on the following algorithm. Pick an arbitrary element $a \in V$ and construct a set Z as follows.

- Initially set $Z = V \setminus \{a\}$.
- Repeat the following until $|Z| = 1$.
 - (i) If there exists an element $v \in V \setminus Z$ such that $\operatorname{argmin}\{A_{vz} : z \in Z\} \neq Z$, then pick any such v and let $Z \leftarrow \operatorname{argmin}\{A_{vz} : z \in Z\}$.

- (ii) Otherwise Z is homogeneous for A . If there exist distinct elements x, y, z with $x \in V \setminus Z$ and $y, z \in Z$ such that $A_{xy} = A_{xz} > A_{yz}$, then let $Z \leftarrow Z \setminus \{z\}$. Otherwise Z is strongly homogeneous and output Z .

- If $|Z| = 1$ then output the element in Z .

The proof of the lemma will be complete if we can show that, if the final set Z is a singleton set with (say) $Z = \{b\}$, then b is critical for A . For this it suffices to show:

$$a \stackrel{b}{\sim} v \text{ for any } v \in V \setminus \{a, b\}, \quad (2)$$

since then, for any distinct $x, y \in V \setminus \{a, b\}$, we have $x \stackrel{b}{\sim} a \stackrel{b}{\sim} y$. We denote by Z_i the set Z obtained at the i -th step in the above algorithm. Then, $Z_0 = V \setminus \{a\}$ and $Z_{i+1} \subsetneq Z_i$ for all $i \geq 0$, with $Z_k = \{b\}$ at the last k -th iteration. Relation (2) follows if we can show that, for any $0 \leq i \leq k - 1$,

$$a \stackrel{b}{\sim} v \text{ for any } v \in Z_i \setminus Z_{i+1}. \quad (3)$$

We prove (3) using induction on i . Let $0 \leq i \leq k - 1$ and assume that (3) holds for all $j \leq i - 1$ (when $i \geq 1$); we show that (3) also holds for index i . For this let $v \in Z_i \setminus Z_{i+1}$.

Assume first that Z_{i+1} is constructed from Z_i as in (i). Then there exists $v_i \in V \setminus Z_i$ such that $Z_{i+1} = \operatorname{argmin}\{A_{v_i z} : z \in Z_i\} \subsetneq Z_i$. As $v \in Z_i \setminus Z_{i+1}$ and $b \in Z_k \subseteq Z_{i+1}$ we have $A_{v_i v} > A_{v_i b}$ and thus (v_i, v) avoids b . If $v_i = a$ we are done. Otherwise, v_i belongs to one of the sets $Z_{j-1} \setminus Z_j$ for some $0 \leq j \leq i - 1$ and thus, using the induction assumption, we can find a path from a to v_i avoiding b . Concatenating it with (v_i, v) we get a path from a to v avoiding b , i.e., $v \stackrel{b}{\sim} a$.

Assume now that Z_{i+1} is constructed from Z_i as in (ii). Then Z_i is homogeneous for A and there exist elements $x \notin Z_i$, $y \in Z_{i+1}$ and $v \in Z_i \setminus Z_{i+1}$ such that $Z_{i+1} = Z_i \setminus \{v\}$ and $A_{xy} = A_{xv} > A_{yv}$. As $b \in Z_{i+1}$ we have $v \neq b$. We first claim that $A_{yv} \geq A_{bv}$. For this assume for contradiction that $A_{yv} < A_{bv}$. As $b, y \in Z_{i+1}$ we have $i \leq k - 2$. Moreover, as $v \notin Z_{i+1}$ and $A_{yv} < A_{bv}$, the set $\operatorname{argmin}\{A_{vu} : u \in Z_{i+1}\}$ does not contain b and thus is a strict subset of Z_{i+1} . Note that v is the only element in $V \setminus Z_{i+1}$ with this property (i.e., $\operatorname{argmin}\{A_{wu} : u \in Z_{i+1}\} = Z_{i+1}$ for $w \in V \setminus (Z_{i+1} \cup \{v\})$) since Z_i is homogeneous. Hence, at the next step we would construct Z_{i+2} from Z_{i+1} as in (i) and thus we would have $Z_{i+2} \subsetneq Z_{i+1} \setminus \{b\}$, a contradiction. Therefore, $A_{xb} = A_{xy} = A_{xv} > A_{yv} \geq A_{vb}$ holds. Hence the path (x, v) avoids b . If $x = a$ we are done. Otherwise, as $x \notin Z_i$, x lies in $Z_j \setminus Z_{j-1}$ for some $0 \leq j \leq i - 1$ and thus, by the induction assumption, there is a path from a to x avoiding b . Concatenating it with (x, v) we get a path from a to v avoiding b , i.e., $v \stackrel{b}{\sim} a$. \square

2.2 Similarity layer partitions

We begin with the notion of ordered partition. An *ordered partition* $\psi = (X_1, \dots, X_k)$ of V is an ordered list of mutually disjoint subsets of V that cover V . Then ψ defines a

partial order \preceq_ψ on V such that $x \preceq_\psi y$ if and only if $x \in X_i$ and $y \in X_j$ with $i \leq j$. If $i = j$ then we denote $x =_\psi y$ while if $i < j$ we denote $x \prec_\psi y$. When all classes X_i are singletons then ψ is a linear order of V , usually denoted by σ .

Given a linear order σ and an ordered partition ψ of V , we say that σ is compatible with ψ if, for any $x, y \in V$, $x \prec_\psi y$ implies $x \prec_\sigma y$.

A key ingredient in the proof of Theorem 3 is the notion of *similarity layer structure*, which was introduced in [15, Section 4.2] and played a crucial role there in the study of the multisweep SFS algorithm. Fix an element $a \in V$. We define subsets X_i ($i \geq 0$) of V in the following iterative manner: set $X_0 = \{a\}$ and for $i \geq 1$

$$X_i = \{y \notin X_0 \cup \dots \cup X_{i-1} : A_{xy} \geq A_{xz} \ \forall x \in X_0 \cup \dots \cup X_{i-1}, \forall z \notin X_0 \cup \dots \cup X_{i-1}\}. \quad (4)$$

We let k denote the largest integer for which $X_k \neq \emptyset$. The sets X_0, \dots, X_k are called the *similarity layers of A rooted at a* and we denote by $\psi_a = (X_0, X_1, \dots, X_k)$ the ordered collection of the similarity layers and call it the *similarity layer structure* rooted at a . As we will see in Lemma 9 below, when A is Robinsonian and a is critical for A , ψ_a is an ordered partition of V . The following basic properties of the similarity layers follow easily from their definition.

Lemma 8. *The following properties hold for $\psi_a = (X_0, X_1, \dots, X_k)$, the similarity layer structure of A rooted at a :*

- (L1) *If $x \in X_i, y, z \in X_j$ with $0 \leq i < j \leq k$, then $A_{xy} = A_{xz}$.*
- (L2) *If $x \in X_i, y \in X_j, z \in X_h$ with $0 \leq i < j < h \leq k$, then $A_{xz} \leq A_{xy}$.*
- (L3) *If $x \in X_i$ with $0 \leq i \leq k$ and $y \in V \setminus (X_0 \cup \dots \cup X_i)$, then $a \overset{y}{\sim} x$.*
- (L4) *Assume $V = X_0 \cup \dots \cup X_k$. Then the set X_k is homogeneous for A . Moreover, if $|X_k| = 1$ with (say) $X_k = \{b\}$, then b is critical for A .*

In the above lemma no assumption is made on the root of the similarity layer structure. We now group further properties that hold when the root a is assumed to be critical.

Lemma 9. *Assume that a is critical for A and let $\psi_a = (X_0, X_1, \dots, X_k)$ be the ordered collection of similarity layers of A rooted at a . If $X_0 \cup X_1 \cup \dots \cup X_k \neq V$ then we can find a weighted asteroidal triple of A .*

Proof. Assume that $U := X_0 \cup X_1 \cup \dots \cup X_k \neq V$. By assumption, $X_{k+1} = \emptyset$. We use the following notation: for $x \in U$ set $M_x := \operatorname{argmax}\{A_{xv} : v \in V \setminus U\}$. We claim that there exist elements $x \neq x' \in U$ such that $M_x \setminus M_{x'} \neq \emptyset$ and $M_{x'} \setminus M_x \neq \emptyset$. For, if not, then $M_{x_1} \subseteq \dots \subseteq M_{x_p}$ for some ordering of the elements of $U = \{x_1, \dots, x_p\}$ and thus $X_{k+1} = \bigcap_{x \in U} M_x = M_{x_1}$, contradicting the assumption $X_{k+1} = \emptyset$. Let $u \in M_x \setminus M_{x'}$ and $v \in M_{x'} \setminus M_x$. Then $A_{xu} > A_{xv}$ implying $x \overset{v}{\sim} u$, and $A_{x'v} > A_{x'u}$, implying $x' \overset{u}{\sim} v$. Combining with $a \overset{v}{\sim} x$, and $a \overset{u}{\sim} x'$ obtained from (L3) since $x, x' \in U$ and $u, v \notin U$, we obtain $a \overset{v}{\sim} u$ and $a \overset{u}{\sim} v$. Finally, $u \overset{a}{\sim} v$ since a is critical, so that $\{a, u, v\}$ is a weighted asteroidal triple of A . \square

Lemma 10. *Assume that a is critical for A and let $\psi_a = (X_0, X_1, \dots, X_k)$ be the ordered collection of similarity layers of A rooted at a . Consider the properties:*

(L1*) *For all $x \in X_i$ and $y, z \in X_j$ with $0 \leq i < j \leq k$, we have $A_{xy} = A_{xz} \leq A_{yz}$.*

(L2*) *For all $x \in X_i, y \in X_j, z \in X_h$ with $0 \leq i < j < h \leq k$, $A_{xz} \leq \min\{A_{xy}, A_{yz}\}$.*

The following holds:

(i) *If (L1*) or (L2*) does not hold then we can find a weighted asteroidal triple in A .*

(ii) *If (L1*) holds then the last layer X_k is strongly homogeneous.*

Proof. (i) Assume (L1*) does not hold, i.e., in view of (L1), there exist $x \in X_i, y \neq z \in X_j$ with $0 \leq i < j \leq k$ and $A_{xy} = A_{xz} > A_{yz}$. Then $x \stackrel{z}{\sim} y$ and $x \stackrel{y}{\sim} z$. Combining with $a \stackrel{z}{\sim} x$ and $a \stackrel{y}{\sim} x$ from (L3) we deduce that $a \stackrel{z}{\sim} y$ and $a \stackrel{y}{\sim} z$. Finally, $y \stackrel{a}{\sim} z$ since a is critical and thus $\{a, y, z\}$ is a weighted asteroidal triple. Assume now that (L2*) does not hold, i.e., in view of (L2), there exist $x \in X_i, y \in X_j, z \in X_h$ with $0 \leq i < j < h \leq k$ and $A_{yz} < A_{xz} \leq A_{xy}$. Then, again we have $x \stackrel{z}{\sim} y$ and $x \stackrel{y}{\sim} z$ and thus $\{a, y, z\}$ is a weighted asteroidal triple for A . Finally, (ii) follows directly from (L4) and (L1*). \square

3 Structural characterization

3.1 Main result

We can now formulate the main technical result of this paper, from which our main Theorem 3 will easily follow. The proof of Theorem 11 will be given in the next section.

Theorem 11. *Let A be a symmetric matrix indexed by V and let a be a critical element for A . Then one can find one of the following three objects:*

(i) *a proper strongly homogeneous set;*

(ii) *a weighted asteroidal triple;*

(iii) *a Robinson ordering of A compatible with the similarity layer structure ψ_a of A rooted at a .*

As an application we obtain the following result, which directly implies Theorem 3.

Corollary 12. *Let A be a symmetric matrix indexed by V . Then one can find either a weighted asteroidal triple, or a Robinson ordering of A .*

Proof. The proof is by induction on the size $|V|$ of A . In view of Lemma 7, one can find either a critical element a , or a proper strongly homogeneous set for A . If we have a critical element a , then we can apply Theorem 11 to (A, a) and find either a proper strongly homogeneous set, or a weighted asteroidal triple, or a Robinson ordering compatible with ψ_a . In the latter two cases we obtain the desired conclusion. So we now only need to

consider the case when a proper strongly homogeneous set S has been obtained in one of the above steps. Then we consider the two matrices $A[S]$ and A/S which have smaller size than A since S is proper. By the induction assumption applied to both $A[S]$ and A/S , either we find a weighted asteroidal triple in $A[S]$ or in A/S , in which case we also have an asteroidal triple in A (by Lemma 5 (ii)), or we find Robinson orderings of $A[S]$ and A/S , which we can then combine to get a Robinson ordering of A (by Lemma 5 (i)). \square

3.2 Proof of Theorem 11

This section is devoted to the proof of Theorem 11. Let A be a symmetric matrix indexed by V and let $a \in V$ be a critical element for A . The proof is by induction on the size $|V|$ of A . Moreover, it is algorithmic. It will go through a number of steps where, either we stop and return a proper strongly homogeneous set or a weighted asteroidal triple, or we end up with constructing a Robinson ordering compatible with the similarity layer structure ψ_a rooted at a .

We start with computing the similarity layer structure $\psi_a = (X_0, \dots, X_k)$ rooted at a . If $X_0 \cup \dots \cup X_k \neq V$ or if property (L1*) or (L2*) does not hold, then we can find a weighted asteroidal triple (by Lemma 9 and Lemma 10 (i)) and we are done. Hence we now assume that $X_0 \cup \dots \cup X_k = V$ and that (L1*) and (L2*) hold for ψ_a . If $|X_k| \geq 2$ then X_k is proper strongly homogeneous (by Lemma 10 (ii)) and we are done. Hence we now assume that $|X_k| = 1$, say $X_k = \{b\}$, and, in view of property (L4), we know that b too is critical for A .

We can repeat the above reasoning to the similarity layer structure $\psi_b = (Y_0, \dots, Y_\ell)$ rooted at b . Hence we may now also assume that $Y_0 \cup \dots \cup Y_\ell = V$, (L1*) and (L2*) hold for ψ_b , and $|Y_\ell| = 1$.

Next we check if the similarity layer structure ψ_a is compatible with the reverse of the similarity layer structure ψ_b , which will imply in particular that the last layer of ψ_b is $Y_\ell = \{a\}$.

Claim 13. *If there exist two distinct elements $x, y \in V$ with $x \prec_{\psi_a} y$ and $x \prec_{\psi_b} y$, then one can find a weighted asteroidal triple of A .*

Proof. Assume $a \preceq_{\psi_a} x \prec_{\psi_a} y \preceq_{\psi_a} b$ and $b \preceq_{\psi_b} x \prec_{\psi_b} y$, so $y \neq b$. Assume first $x = a$. Then, using (L3) applied to ψ_a and ψ_b we get, respectively, $a \overset{b}{\sim} y$ and $b \overset{y}{\sim} a$. As a is critical we also have $b \overset{a}{\sim} y$ and thus $\{a, b, y\}$ is a weighted asteroidal triple.

Assume now $x \neq a$. Using again (L3) applied to ψ_a and ψ_b , we get $a \overset{y}{\sim} x$ and $b \overset{y}{\sim} x$, implying $a \overset{y}{\sim} b$. Moreover, $b \overset{a}{\sim} y$ since a is critical, and $a \overset{b}{\sim} y$ since b is critical. Hence $\{a, b, y\}$ is again a weighted asteroidal triple. \square

To recap, from now on we will assume that $\psi_a = (X_0, \dots, X_k)$ is compatible with the reverse of $\psi_b = (Y_0, \dots, Y_\ell)$, i.e., there do not exist $x, y \in V$ with $x \prec_{\psi_a} y$ and $x \prec_{\psi_b} y$, and therefore $X_0 = Y_\ell = \{a\}$ and $X_k = Y_0 = \{b\}$. We show the shape of the similarity layer partitions ψ_a and ψ_b in Figure 1, where the similarity layers X_i and Y_j are indicated by ellipses and rectangles, respectively. We also indicate the set $X_{k-1} \cap Y_{j^*}$, where j^* is

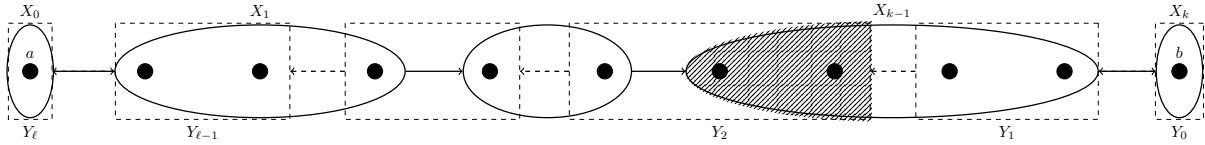


Figure 1: The similarity layer structures ψ_a, ψ_b and the set $X_{k-1} \cap Y_{j^*}$ for $j^* = 2$.

the largest integer $j \geq 1$ for which $X_{k-1} \cap Y_j \neq \emptyset$, which will play a crucial role in the rest of the proof.

Claim 14. *Let j^* be the largest integer $j \geq 1$ such that $X_{k-1} \cap Y_j \neq \emptyset$. If $|X_{k-1} \cap Y_{j^*}| \geq 2$ then $X_{k-1} \cap Y_{j^*}$ is proper strongly homogeneous for A .*

Proof. For this pick $x \notin X_{k-1} \cap Y_{j^*}$ and distinct elements $y, z \in X_{k-1} \cap Y_{j^*}$. If x lies in $X_{k-1} \cup X_k$ then $x \in Y_j$ for some $0 \leq j < j^*$ and thus $A_{xy} = A_{xz} \leq A_{yz}$ follows from property (L1*) applied to ψ_b . Otherwise x lies in some X_i with $i \leq k-2$ and $A_{xy} = A_{xz} \leq A_{yz}$ follows from property (L1*) applied to ψ_a . \square

From now on we assume that $|X_{k-1} \cap Y_{j^*}| = 1$ and we set $X_{k-1} \cap Y_{j^*} = \{c\}$. Thus we may partition the set V as

$$V = \underbrace{X_0 \cup \dots \cup X_{k-2}}_{=:X} \cup \{c\} \cup \underbrace{Y_{j^*-1} \cup \dots \cup Y_0}_{=:Y} = X \cup \{c\} \cup Y.$$

For further use we record the following consequence of (L3) applied to ψ_a and ψ_b :

$$\text{For any } u, v \in X \text{ (resp., } u, v \in Y), u \stackrel{c}{\sim} v \text{ in } A. \quad (5)$$

At this step we now need to work with two new matrices A^X and A^Y that are indexed, respectively, by $X \cup \{c\}$ and $Y \cup \{c\}$ and constructed by modifying the entries of A in the following way. Let M be a positive integer, chosen sufficiently large, so that

$$M > 2 \max\{|A_{uv}| : u, v \in V\}. \quad (6)$$

Let A^X be the symmetric matrix indexed by $X \cup \{c\}$, obtained from $A[X]$ by adjoining a new column/row indexed by c with entries:

$$A_{cv}^X = -M - j + \frac{A_{cv}}{M} \quad \text{for } v \in X \text{ with } v \in Y_j. \quad (7)$$

Similarly, let A^Y be the symmetric matrix indexed by $Y \cup \{c\}$, obtained from $A[Y]$ by adding a new column/row indexed by c with entries:

$$A_{cv}^Y = -M - i + \frac{A_{cv}}{M} \quad \text{for } v \in Y \text{ with } v \in X_i. \quad (8)$$

Note that $j^* \leq j \leq \ell$ in (7) and $k-1 \leq i \leq k$ in (8).

Claim 15. (i) *The element a is critical in A^X and the similarity layer structure of A^X rooted at a , denoted as ψ_a^X , is equal to $\psi_a^X = (\{a\}, X_1, \dots, X_{k-2}, \{c\})$.*

(ii) *The element b is critical in A^Y and the similarity layer structure of A^Y rooted at b , denoted as ψ_b^Y , is equal to $\psi_b^Y = (\{b\}, Y_1, \dots, Y_{j^*-1}, \{c\})$.*

Proof. (i) We show that a is critical for A^X . For any $v \in X \setminus \{a\}$, note first using definition (7) that $A_{vc}^X > A_{ac}^X$ holds, since

$$A_{vc}^X - A_{ac}^X = -j + \ell + \frac{A_{vc} - A_{ac}}{M} \geq 1 + \frac{A_{vc} - A_{ac}}{M} > 0,$$

where the first two relations follow from $v \in Y_j$ and $a \in Y_\ell$ with $j \leq \ell - 1$ and the third inequality follows from (6). Hence, the path (v, c) avoids a in A^X . Now, for $x \neq y \in X \setminus \{a\}$, the path (x, c, y) avoids a in A^X , which shows that a is critical for A^X . Moreover, as $A_{vc}^X < A_{xy}^X$ for all $v, x, y \in X$, it follows that the similarity layer structure of A^X rooted at a has indeed the desired form.

The proof of (ii) is analogous. □

Since the size of both matrices A^X and A^Y is smaller than that of A , we can apply the induction assumption to (A^X, a) and (A^Y, b) , which gives the following three cases:

Case 1: we find a proper strongly homogeneous set in A^X or in A^Y ;

Case 2: we find a weighted asteroidal triple in A^X or in A^Y ;

Case 3: or we find Robinson orderings of A^X and A^Y that are compatible with ψ_a^X and ψ_b^Y , respectively.

We now deal with each of these three cases separately.

Case 1: We assume that we have found a proper strongly homogeneous set S in A^X . (The case of A^Y is similar and thus omitted.) As we now show, either we can claim that S is strongly homogeneous in A , or we find a weighted asteroidal triple in A .

Claim 16. *Let S be a proper strongly homogeneous set in A^X . Then, either S is strongly homogeneous in A , or there exist $x \neq x' \in S$ such that $\{x, x', c\}$ is a weighted asteroidal triple for A .*

Proof. Let $S \subseteq X \cup \{c\}$ be a proper strongly homogeneous set in A^X . We first show $c \notin S$. For this, suppose for contradiction that $c \in S$. Since S is proper, we can take elements $x \in S \setminus \{c\}$ and $v \in X \setminus S$. Since S is homogeneous in A^X , we have $A_{vc}^X = A_{vx}^X$, which gives $-M - j + \frac{A_{vc}}{M} = A_{vx}$ if $v \in Y_j$. This however contradicts the choice of M in (6). Therefore, $c \notin S$.

Take any $x, x' \in S$. As $c \notin S$ and S is homogeneous in A^X , we have $A_{cx}^X = A_{cx'}^X$, which gives $-M - j + \frac{A_{cx}}{M} = -M - j' + \frac{A_{cx'}}{M}$, where $x \in Y_j$ and $x' \in Y_{j'}$. Using again (6) we derive that $j = j'$ and

$$A_{cx} = A_{cx'} \quad \text{for all } x, x' \in S. \tag{9}$$

Therefore, S is contained in some layer Y_j of ψ_b for some $j \geq j^*$. Moreover,

$$\text{if } j > j^* \quad \text{then } A_{cx} = A_{cx'} \leq A_{xx'} \quad \text{for all } x, x' \in S, \tag{10}$$

which follows from (L1*) applied to ψ_b (since $c \in Y_{j^*}$ and $x, x' \in Y_j$ with $j > j^*$). Next we claim that

$$A_{vx} = A_{vx'} \leq A_{xx'} \quad \text{for all } x, x' \in S, v \in V \setminus (S \cup \{c\}). \quad (11)$$

If $v \in X$, then $A_{vx} = A_{vx'} \leq A_{xx'}$ follows from $A_{vx}^X = A_{vx'}^X \leq A_{xx'}^X$ since A and A^X coincide on the triple $\{v, x, x'\} \subseteq X$. If $v \notin X$, then $v \in Y$ and thus $v \in Y_h$ for some $h < j$, in which case $A_{vx} = A_{vx'} \leq A_{xx'}$ follows from (L1*) applied to ψ_b since $x, x' \in Y_j$.

Hence, in view of relations (10) and (11), if the set S is not strongly homogeneous in A , then necessarily $j = j^*$ and there exist $x \neq x' \in S$ such that $A_{cx} = A_{cx'} > A_{xx'}$. Then $c \overset{x}{\sim} x'$ and $c \overset{x'}{\sim} x$. On the other hand, by (L3) applied to ψ_a , we have $x \overset{c}{\sim} a \overset{c}{\sim} x'$ (since $c \in X_{k-1}$, $x \in X_i$, $x' \in X_{i'}$ with $i, i' < k - 1$). Thus $\{x, x', c\}$ forms a weighted asteroidal triple in A . \square

Case 2: We now assume that we have found a weighted asteroidal triple in A^X . (The case of having a weighted asteroidal triple in A^Y is similar and thus omitted.) Our goal is to construct from it a weighted asteroidal triple in A . For this we use the following claim.

Claim 17. *Given $u, v \in X$ and a path from u to c avoiding v in A^X , the following holds:*

- (i) *If $v \neq_{\psi_b} c$, one can find a path from u to b avoiding v in A ;*
- (ii) *If $v =_{\psi_b} c$, one can find a path from u to c avoiding v in A ;*
- (iii) *If $v \prec_{\psi_b} u$, one can find a weighted asteroidal triple in A .*

Proof. Say, $P = (u, \dots, u', c)$ is a path from u to c avoiding v in A^X , with (u', c) as its last edge. Note that $P' = (u, \dots, u')$ is a path avoiding v in A and thus $u \overset{v}{\sim} u'$ not only in A^X but also in A . We claim:

$$\text{either } u' \prec_{\psi_b} v, \text{ or } u' =_{\psi_b} v \text{ and } A_{u'c} > A_{vc}. \quad (12)$$

Indeed, since (u', c) avoids v in A^X , we have $A_{u'c}^X > \min\{A_{u'v}^X, A_{vc}^X\} = A_{vc}^X$, where the last equality follows from the definition of A^X and the choice of M . Say $v \in Y_j$ and $u' \in Y_{j'}$. Then, $A_{u'c}^X > A_{vc}^X$ implies $j - j' > \frac{A_{vc} - A_{u'c}}{M}$. We cannot have $j' > j$ since then one would have

$$0 < j - j' - \frac{A_{vc} - A_{u'c}}{M} < -1 - \frac{A_{vc} - A_{u'c}}{M} < 0,$$

reaching a contradiction. Hence, either $j' < j$ (i.e., $u' \prec_{\psi_b} v$), or $j = j'$ (i.e., $v =_{\psi_b} u'$) and $A_{vc} < A_{u'c}$.

(i) Assume $v \neq_{\psi_b} c$. Then, $c \prec_{\psi_b} v$ and, by (L3) applied to ψ_b , $b \overset{v}{\sim} c$ in A . If $u' \prec_{\psi_b} v$, then $u' \overset{v}{\sim} b$ in A (again by (L3) applied to ψ_b) and thus $u \overset{v}{\sim} u' \overset{v}{\sim} b$ in A , giving the desired conclusion. Otherwise, in view of (12), $u' =_{\psi_b} v$ and $A_{u'c} > A_{vc}$, which implies $u' \overset{v}{\sim} c$ in A . We obtain $u \overset{v}{\sim} u' \overset{v}{\sim} c \overset{v}{\sim} b$ in A , giving again the desired conclusion.

(ii) Assume $v =_{\psi_b} c$. Then $u' \prec_{\psi_b} v$ cannot occur since $c \preceq_{\psi_b} u'$. In view of (12), we have $A_{u'c} > A_{vc}$, which implies $u' \overset{v}{\sim} c$ and thus $u \overset{v}{\sim} u' \overset{v}{\sim} c$ in A , giving the desired conclusion.

(iii) We assume $v \prec_{\psi_b} u$. We consider two cases depending whether $v =_{\psi_b} c$ or not. Assume first that $v \neq_{\psi_b} c$. By Claim 17(i), we have $u \overset{v}{\sim} b$ in A . Moreover, as $v \prec_{\psi_b} u$, we get $b \overset{u}{\sim} v$ in A (by (L3) applied to ψ_b). Finally, $u \overset{b}{\sim} v$ in A , since b is critical in A , and thus the triple $\{b, u, v\}$ is a weighted asteroidal triple in A . Finally assume $v =_{\psi_b} c$. Then, by Claim 17(ii), we have $u \overset{v}{\sim} c$ in A . Also, as $c =_{\psi_b} v \prec_{\psi_b} u$, we get $v \overset{u}{\sim} b \overset{u}{\sim} c$ in A (applying (L3) to ψ_b). Finally, using (5), we get $u \overset{c}{\sim} v$ in A . Hence, $\{c, u, v\}$ is a weighted asteroidal triple in A . \square

As a direct application of Claim 17, we get the following result, which we use to show Claim 19 below.

Corollary 18. *Consider distinct elements $u, v, w \in X$ and a path P from u to w avoiding v in A^X . If P does not contain c then P is also a path avoiding v in A . If P contains c then one can construct from it a path P' from u to w avoiding v in A .*

Claim 19. *Given a weighted asteroidal triple $\{x, y, z\}$ in A^X , one can construct from it a weighted asteroidal triple in A .*

Proof. We may assume without loss of generality that $x \preceq_{\psi_b} y \preceq_{\psi_b} z$. If $\{x, y, z\} \cap \{c\} = \emptyset$ then it follows directly from Corollary 18 that $\{x, y, z\}$ is also a weighted asteroidal triple in A . Without loss of generality, we now assume that $c = x$. By Claim 17(iii) one can find a weighted asteroidal triple in A if $y \prec_{\psi_b} z$. Hence we now assume that $c = x \preceq_{\psi_b} y =_{\psi_b} z$.

If $c = x =_{\psi_b} y =_{\psi_b} z$ then, by Claim 17(ii), we have $c \overset{y}{\sim} z$ and $c \overset{z}{\sim} y$ in A . Moreover, by (5), $y \overset{c}{\sim} z$ in A . Hence $\{c, y, z\}$ is a weighted asteroidal triple in A .

If $c = x \prec_{\psi_b} y =_{\psi_b} z$ then, by Claim 17(i), we have $b \overset{y}{\sim} z$ and $b \overset{z}{\sim} y$ in A . Moreover, as b is critical in A , $y \overset{b}{\sim} z$ in A . Hence $\{b, y, z\}$ is a weighted asteroidal triple in A . \square

Case 3: The remaining case is when we have found a Robinson order σ_X (resp., σ_Y) of A^X (resp., of A^Y), which is compatible with the similarity layer structure ψ_a^X of A^X rooted at a (resp., with the similarity layer structure ψ_b^Y of A^Y rooted at b). By Claim 15, we must have $\sigma_X = (a, \dots, c)$ and $\sigma_Y = (b, \dots, c)$. We define the linear order $\sigma = (\sigma_X, \sigma_Y^{-1})$ of V , obtained by concatenating σ_X and the reverse of σ_Y along the element c . In view of the form of ψ_a^X in Claim 15, it follows that σ is compatible with ψ_a . In order to complete the proof of Theorem 11 we need to show that, either σ is a Robinson ordering of A , or we can find a weighted asteroidal triple in A .

Recall that an ordered triple (x, y, z) is Robinson in A if $A_{xz} \leq \min\{A_{xy}, A_{yz}\}$ holds. We will show that for any triple (x, y, z) with $x \prec_{\sigma} y \prec_{\sigma} z$, either (x, y, z) is Robinson, or $\{x, y, z\}$ is a weighted asteroidal triple for A .

Assume $x \prec_{\sigma} y \prec_{\sigma} z$. Then, $x \preceq_{\psi_a} y \preceq_{\psi_a} z$, since σ is compatible with ψ_a . If $x \prec_{\psi_a} y \preceq_{\psi_a} z$, then we can conclude that (x, y, z) is Robinson (using (L1*)-(L2*) applied to ψ_a). Hence from now on we may assume that $x =_{\psi_a} y \preceq_{\psi_a} z$. In the next two claims we will consider separately the two cases: $x =_{\psi_a} y \prec_{\psi_a} z$ and $x =_{\psi_a} y =_{\psi_a} z$.

Note that we can analogously conclude that (x, y, z) is Robinson if $z \prec_{\psi_b} y \preceq_{\psi_b} x$ (using (L1*)-(L2*) applied to ψ_b).

Claim 20. Consider $x, y, z \in V$ such that $x \prec_\sigma y \prec_\sigma z$ and $x =_{\psi_a} y \prec_{\psi_a} z$. If the triple (x, y, z) is not Robinson in A then $\{x, y, z\}$ is a weighted asteroidal triple in A .

Proof. Assume $x \prec_\sigma y \prec_\sigma z$, $x =_{\psi_a} y \prec_{\psi_a} z$, and (x, y, z) is not a Robinson triple in A . Then $A_{xz} > \min\{A_{xy}, A_{yz}\}$ and thus $x \overset{y}{\sim} z$ in A . We first claim that $z \notin X$. Indeed, assume $z \in X$. Then, as $x =_{\psi_a} y \prec_{\psi_a} z$ we have $\{x, y, z\} \subseteq X$ and thus, as σ restricts to σ_X on X , $x \prec_{\sigma_X} y \prec_{\sigma_X} z$. As σ_X is a Robinson ordering of A^X , this implies that (x, y, z) is a Robinson triple in A^X and thus in A , a contradiction. Therefore, $z \in X_{k-1} \cup \{b\}$.

Next we claim that

$$z \prec_{\psi_b} x \prec_{\psi_b} y \text{ does not hold.} \quad (13)$$

For this assume $z \prec_{\psi_b} x \prec_{\psi_b} y$. As $x =_{\psi_a} y \prec_{\psi_a} z$, we have $x, y \in X_i$ for some $i \leq k-1$. We first claim that $i \leq k-2$. This is clear if $z \in X_{k-1}$. Assume now $z = b$ and $x, y \in X_{k-1}$. Then $\{x, y, z\} \subseteq Y \cup \{c\}$ and, as σ restricts to σ_Y^{-1} on $Y \cup \{c\}$ and σ_Y is compatible with ψ_b^Y , $x \prec_\sigma y \prec_\sigma z$ implies $z \preceq_{\psi_b} y \preceq_{\psi_b} x$, a contradiction. So we have shown that $x, y \in X_i$ for some $i \leq k-2$. This implies $x \prec_\sigma y \prec_\sigma c$ and thus $x \prec_{\sigma_X} y \prec_{\sigma_X} c$. Say $x \in Y_j$, $y \in Y_h$, with $h > j$ as $x \prec_{\psi_b} y$. As σ_X is a Robinson ordering of A^X we can conclude $A_{xc}^X \leq \min\{A_{xy}^X, A_{yc}^X\} = A_{yc}^X$, which implies $1 \leq h - j \leq \frac{A_{cy} - A_{cx}}{M} < 1$ (by the choice of M in (6)), a contradiction. So we have shown (13).

Recall that the reverse of ψ_b is compatible with ψ_a . Hence it follows from $x =_{\psi_a} y \prec_{\psi_a} z$ that $z \preceq_{\psi_b} x$ and $z \preceq_{\psi_b} y$. Moreover, we claim that

$$z =_{\psi_b} x \text{ or } z =_{\psi_b} y. \quad (14)$$

Indeed, if (14) does not hold, then $z \prec_{\psi_b} x \prec_{\psi_b} y$ or $z \prec_{\psi_b} y \preceq_{\psi_b} x$. The former does not hold by (13), while the latter does not hold since (x, y, z) is not Robinson (as observed just before Claim 20). Thus (14) holds. Then, by $x =_{\psi_a} y \prec_{\psi_a} z$, (14) implies $z = c$.

We next claim that $y \preceq_{\psi_b} x$. For this assume for contradiction that $x \prec_{\psi_b} y$. As above, let $x \in Y_j$, $y \in Y_h$ with $h > j$. From this (and the definition of M in (6)), it follows that $A_{cx}^X > \min\{A_{cy}^X, A_{xy}^X\} = A_{cy}^X$. As σ_X is a Robinson ordering of A^X we must have $y \prec_{\sigma_X} x \prec_{\sigma_X} c$, which implies $y \prec_\sigma x \prec_\sigma c$, a contradiction.

In total we have shown that the following relation holds:

$$c = z =_{\psi_b} y \preceq_{\psi_b} x.$$

We will now show that $\{x, y, z\}$ is a weighted asteroidal triple in A . We already have $x \overset{y}{\sim} z$, since by assumption the triple (x, y, z) is not Robinson in A . Moreover, as $x =_{\psi_a} y \prec_{\psi_a} z$ we have $x \overset{z}{\sim} y$. Indeed, this follows from (L3) applied to ψ_a : if $a \notin \{x, y\}$ then $x \overset{z}{\sim} a \overset{z}{\sim} y$ and, if (say) $a = x$, then $x = a \overset{z}{\sim} y$. Remains to show $y \overset{x}{\sim} z$. We distinguish two cases. If $c = z =_{\psi_b} y \prec_{\psi_b} x$ then $y \overset{x}{\sim} b \overset{x}{\sim} z$ follows (using (L3) applied to ψ_b). Assume now $c = z =_{\psi_b} y =_{\psi_b} x$. Then, we have $\{x, y, z\} \subseteq X \cup \{c\}$, and $x \prec_\sigma y \prec_\sigma z$ implies $x \prec_{\sigma_X} y \prec_{\sigma_X} z$. This in turn implies $A_{xz}^X \leq \min\{A_{xy}^X, A_{yz}^X\} = A_{yz}^X$ and thus $A_{xz} \leq A_{yz}$. Combining with $A_{xz} > \min\{A_{xy}, A_{yz}\}$ we get $A_{xz} > A_{xy}$ and thus $A_{yz} \geq A_{xz} > A_{xy}$ which gives $y \overset{x}{\sim} z$. So we have shown that $\{x, y, z\}$ is a weighted asteroidal triple in A and this concludes the proof. \square

Claim 21. Consider $x, y, z \in V$ such that $x \prec_\sigma y \prec_\sigma z$ and $x =_{\psi_a} y =_{\psi_a} z$. If (x, y, z) is not a Robinson triple in A then $\{x, y, z\}$ is a weighted asteroidal triple in A .

Proof. By assumption, $x, y, z \in X_i$ for some $i \leq k - 1$ and the triple (x, y, z) is not Robinson in A . We first claim that $i = k - 1$. For this assume $i \leq k - 2$. Then $\{x, y, z\} \subseteq X$ and thus $x \prec_\sigma y \prec_\sigma z$ implies $x \prec_{\sigma_X} y \prec_{\sigma_X} z$. As σ_X is a Robinson ordering of A^X and thus of $A[X]$ it follows that (x, y, z) is Robinson in A , contradicting our assumption. Hence we know that $x, y, z \in X_{k-1}$. Then $c \preceq_\sigma x \prec_\sigma y \prec_\sigma z$, which implies $z \prec_{\sigma_Y} y \prec_{\sigma_Y} x \preceq_{\sigma_Y} c$ and thus the triple (z, y, x) is Robinson in A^Y (since σ_Y is a Robinson ordering of A^Y). If $x \neq c$ then (z, y, x) is also a Robinson triple in A , contradicting our assumption. Therefore we must have $x = c$. In turn, this gives $A_{xz}^Y \leq \min\{A_{xy}^Y, A_{yz}^Y\} = A_{xy}^Y$, which implies $A_{xz} \leq A_{xy}$. On the other hand, $A_{xz} > \min\{A_{xy}, A_{yz}\}$, since (x, y, z) is not Robinson in A . This implies $A_{xz} > A_{yz}$ and thus $A_{xy} \geq A_{xz} > A_{yz}$, so that $x \overset{z}{\sim} y$ and $x \overset{y}{\sim} z$. Lastly, $y \overset{c=x}{\sim} z$ follows from (5) and thus we have shown that $\{x, y, z\}$ is a weighted asteroidal triple in A . \square

This concludes the proof of Theorem 11.

4 Applications

In this section we group some applications of our characterization of Robinsonian matrices in terms of weighted asteroidal triples. First we indicate how we can derive from it the known structural characterization of unit interval graphs from Theorem 1. As we will see, weighted asteroidal triples offer a common framework to formulate the three types of obstructions for the graph case: chordless cycles, claws and asteroidal triples.

As another application, in order to decide whether a matrix A is Robinsonian it suffices to check whether it has a weighted asteroidal triple. The proof of Theorem 11 is algorithmic and yields a polynomial time algorithm for finding a weighted asteroidal triple (if some exists), however we can give a much simpler, direct algorithm permitting to find all weighted asteroidal triples in time $O(n^3)$ for a $n \times n$ symmetric matrix A .

Finally we mention a possible application of our characterization for identifying large Robinsonian submatrices. In particular we obtain an explicit characterization of the maximal subsets I for which the principal submatrix $A[I]$ is Robinsonian in terms of forbidden ‘weighted asteroidal cycles’.

4.1 Recovering the structural characterization of unit interval graphs

In this section we indicate how to recover from our main result (Theorem 3) the known structural characterization of unit interval graphs in Theorem 1.

Let $G = (V, E)$ be a graph. Given $x, y, z \in V$ and a path P from x to y in G , recall that P misses z if P is disjoint from the neighborhood of z . In other words, if A_G denotes the adjacency matrix of G , then $P = (x = x_0, x_1, \dots, x_k, y = x_{k+1})$ misses z if $(A_G)_{x_i, x_{i+1}} > \max\{(A_G)_{x_i z}, (A_G)_{x_{i+1} z}\}$ for all $0 \leq i \leq k$. Hence if P misses z , then it also avoids z in A_G , but the converse is not true in general.

An asteroidal triple in G is a set of nodes $\{x, y, z\}$ containing no edge and such that there exists a path in G between any two nodes in $\{x, y, z\}$ that misses the third one. Hence, if $\{x, y, z\}$ is an asteroidal triple in G , then it is also a weighted asteroidal triple in the adjacency matrix A_G of G , but the converse is not true in general.

As was recalled earlier, G is a unit interval graph if and only if its adjacency matrix A_G is Robinsonian. In view of Theorem 3, A_G is Robinsonian if and only if it does not contain a weighted asteroidal triple. Combining those two facts with Theorem 1, we have the following.

Theorem 22. *Let G be a graph and A_G its adjacency matrix. The conditions (i), (ii), and (iii) in Theorem 1 hold if and only if A_G has no weighted asteroidal triple.*

We now give a direct proof of Theorem 22, which in turn implies an alternative proof of Theorem 1.

Lemma 23. *Let G be a graph and A_G its adjacency matrix. Assume that one of the conditions (i), (ii), or (iii) in Theorem 1 is violated. Then one can find a weighted asteroidal triple in A_G .*

Proof. Assume first that we have an induced cycle $C = (x_1, \dots, x_k)$ of length $k \geq 4$ in G . Then, (x_1, x_2) avoids x_k in A_G and thus $x_1 \overset{x_k}{\sim} x_2$, and (x_1, x_k) avoids x_2 in A_G and thus $x_1 \overset{x_2}{\sim} x_k$. In addition, the path $(x_2, x_3, \dots, x_{k-1}, x_k)$ avoids x_1 in A_G , which gives $x_2 \overset{x_1}{\sim} x_k$. Hence $\{x_1, x_2, x_k\}$ is a weighted asteroidal triple in A_G .

Assume now that we have a claw $K_{1,3}$ in G , say u is adjacent to x, y, z and $\{x, y, z\}$ is independent in G . Then, $x \overset{z}{\sim} u \overset{z}{\sim} y$, $x \overset{y}{\sim} u \overset{y}{\sim} z$, $y \overset{x}{\sim} u \overset{x}{\sim} z$ in A_G , and thus $\{x, y, z\}$ is a weighted asteroidal triple in A_G .

Finally, if $\{x, y, z\}$ is an asteroidal triple in G then clearly it is also a weighted asteroidal triple in A_G . □

To prove the converse we will use the following result.

Lemma 24. *Let G be a graph with adjacency matrix A_G and consider distinct elements $x, y, z \in V$. Assume P is a path from x to y avoiding z in A_G which has the smallest possible number of nodes. Then one of the following holds:*

- (i) P is a path in G that misses z (i.e., z is not adjacent to any node of P);
- (ii) we find a claw or an induced cycle of length at least 4 in G ;
- (iii) P is an induced path in G , z is adjacent to exactly one node u of P and $u \in \{x, y\}$.

Proof. Let $P = (x = x_0, x_1, \dots, x_k, x_{k+1} = y)$ be a path satisfying the assumptions of the lemma. As P avoids z in A_G , it follows that z cannot be adjacent to two consecutive nodes in P . If z is not adjacent to any node of P then we are in case (i). Assume first that z is adjacent to at least two nodes of P . Say, z is adjacent to x_i and x_j with $0 \leq i \leq j - 2 \leq k - 1$, and z is not adjacent to any x_h with $i < h < j$. Consider the subpath (x_i, \dots, x_j) of P . If this subpath is not induced in G then we could find a

shorter path than P going from x to y and avoiding z in A_G , contradicting our minimality assumption on P . Hence this subpath is induced, so we find an induced cycle of length at least 4 and we are in case (ii).

We can now assume that z is adjacent to exactly one node x_i of P . Then the path P is induced in G (for if not one would contradict the minimality of P). If x_i is not the first or last node of P then we find a claw and thus we are again in case (ii). Hence we can conclude that z is adjacent to exactly one of x and y . Hence we are in case (iii). \square

Lemma 25. *Let G be a graph and A_G its adjacency matrix. If there exists a weighted asteroidal triple in A_G then one of the conditions (i), (ii), (iii) in Theorem 1 is violated.*

Proof. Assume that $\{x, y, z\}$ is a weighted asteroidal triple in A_G . Select paths P_{xy} , P_{xz} and P_{yz} that avoid, respectively, z , y , x in A_G and have the smallest possible lengths. We apply Lemma 24 to each of the three paths P_{xy} , P_{xz} and P_{yz} . If for some of these three paths we are in case (ii) of Lemma 24, then we find a claw or an induced cycle and we are done. Hence, for each of the three paths we are in case (i) or (iii) of Lemma 24. If for all the three paths we are in case (i), then we can conclude that $\{x, y, z\}$ is an asteroidal triple of G and we are done.

Therefore, we may now assume that (say) for the path P_{xy} , we are in case (iii). Then P_{xy} is an induced path in G and (say) z is adjacent to x . In turn, this implies that we are in case (iii) also for the path P_{yz} and thus P_{yz} is induced in G . Together with the edge $\{x, z\}$ the two paths P_{xy} and P_{yz} form a cycle C with at least 4 nodes. If C is induced in G then we are done. So let us now assume that C has a chord. As both paths P_{xy} and P_{yz} are induced there must exist an edge of the form $\{u, v\}$ where u belongs to P_{xy} and v belongs to P_{yz} . First we choose u to be the ‘first’ node on P_{xy} which is adjacent to some node v of P_{yz} , ‘first’ when traveling from x to y on P_{xy} . After that, we choose for v the ‘last’ node of P_{yz} which is adjacent to u , ‘last’ when traveling from y to z on P_{yz} . Note that $u \neq x$, $u \neq y$, $v \neq z$, and $v \neq y$ since we are in case (iii). Thus, from the choice of u and v , it follows that the cycle obtained by traveling along P_{xy} from x to u , then traversing edge $\{u, v\}$, then traveling along P_{yz} from v to z , and finally traversing edge $\{z, x\}$, is an induced cycle in G of length at least 4. This concludes the proof. \square

Combining Lemmas 23 and 25 completes the proof of Theorem 22.

4.2 An algorithm for enumerating the weighted asteroidal triples

As an application of our main theorem we obtain an alternative algorithm to decide whether a given matrix A is Robinsonian, namely by checking the existence of a weighted asteroidal triple for A . We indicate a simple algorithm for doing this.

A first observation is that, given distinct elements $x, y, v \in V$, one can check whether $x \overset{v}{\sim} y$, i.e., whether there exists a path from x to y avoiding v in A , in time $O(n^2)$. For this consider the graph H_v with vertex set $V \setminus \{v\}$, where two nodes $u, w \in V \setminus \{v\}$ are adjacent if $A_{vw} > \min\{A_{uv}, A_{vw}\}$. Then $x \overset{v}{\sim} y$ precisely when x and y lie in the same connected component of H_v . Building H_v and checking the existence of a path from x to z in H_v can be done in time $O(n^2)$.

A first elementary algorithm to decide existence of a weighted asteroidal triple in A would be to test all possible triples, which can be done in time $O(n^5)$. The following simple algorithm permits to check existence of a weighted asteroidal triple more efficiently, in time $O(n^3)$. It computes a function f defined on the set $\binom{V}{3}$ of all triples of elements of V , whose value records whether a triple is a weighted asteroidal triple.

Algorithm 1:

input: a symmetric matrix A (indexed by V)

output: A weighted asteroidal triple $\{x, y, z\}$ or “ A has no weighted asteroidal triple”.

```

1 Initialize  $f : \binom{V}{3} \rightarrow \mathbb{Z}$  by  $f(\{x, y, z\}) = 0$  for every  $\{x, y, z\} \in \binom{V}{3}$ .
2 for each  $v \in V$  do
3   | Compute the graph  $H_v$  with vertex set  $V \setminus \{v\}$  and with edges the pairs  $\{u, w\}$ 
4   | such that  $A_{uw} > \min\{A_{uv}, A_{vw}\}$ .
5   |  $f(\{x, y, v\}) \leftarrow f(\{x, y, v\}) + 1$  for every pair  $\{x, y\}$  of elements lying in the
6   | same connected component of  $H_v$ .
7 end
8 if there exists  $\{x, y, z\} \in \binom{V}{3}$  with  $f(\{x, y, z\}) = 3$  then
9   | return  $\{x, y, z\}$ 
10 else
11   | return “ $A$  has no weighted asteroidal triple”
12 end

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In fact the final function f returned by the above algorithm permits to return all the weighted asteroidal triples, which are precisely the triples $\{x, y, z\}$ with $f(\{x, y, z\}) = 3$.

4.3 Maximal Robinsonian submatrices

When a given symmetric matrix A indexed by V is not Robinsonian, one might be interested in the maximal subsets indexing a Robinsonian submatrix or, equivalently, in the minimal subsets whose deletion leaves a Robinsonian submatrix. Note that finding a Robinsonian submatrix of largest possible size is in fact a hard problem, already for binary matrices. Indeed it is known that finding in a given graph a smallest cardinality set of nodes whose deletion leaves a unit interval graph is an NP-complete problem (see [17, 10]).

Let \mathcal{I}_A denote the collection of all maximal subsets $I \subseteq V$ for which $A[I]$ is a Robinsonian matrix and let \mathcal{F}_A consist of the minimal subsets $F \subseteq V$ for which $A[V \setminus F]$ is Robinsonian, i.e., $\mathcal{F}_A = \{V \setminus I : I \in \mathcal{I}_A\}$. Let also \mathcal{C}_A denote the collection of minimal transversals of \mathcal{F}_A (i.e., the minimal sets intersecting all sets in \mathcal{F}_A). In other words, \mathcal{I}_A coincides with the collection of maximal independent sets of the hypergraph $\mathcal{H}_A = (V, \mathcal{C}_A)$, whose dual (or transversal) hypergraph is $\mathcal{H}_A^d = (V, \mathcal{F}_A)$ (see, e.g., [7]).

In order to describe the minimal transversals of \mathcal{F}_A we introduce the following definition. A set $C \subseteq V$ is called a *weighted asteroidal cycle* of A if there exists a weighted aster-

oidal triple $\{x, y, z\}$ of A and paths P_{xy}, P_{xz}, P_{yz} such that $C = V(P_{xy}) \cup V(P_{xz}) \cup V(P_{yz})$, where P_{xy} (resp., P_{xz}, P_{yz}) is a path from x to y avoiding z (resp., from x to z avoiding y , from y to z avoiding x). Then, as a direct application of Theorem 3, for sets $I, C \subseteq V$, we have:

$A[I]$ is Robinsonian $\iff I$ does not contain a weighted asteroidal cycle,

$C \in \mathcal{C}_A \iff C$ is a minimal weighted asteroidal cycle of A .

Furthermore, a set C is a minimal weighted asteroidal cycle of A if and only if $A[C]$ is not Robinsonian, but $A[C \setminus \{x\}]$ is Robinsonian for any $x \in C$, and thus one can check in polynomial time membership in the collection \mathcal{C}_A . Analogously, one can check membership in the collection \mathcal{I}_A in polynomial time, since $I \in \mathcal{I}_A$ if and only if $A[I]$ is Robinsonian, but $A[I \cup \{x\}]$ is not Robinsonian for any $x \in V \setminus I$.

As mentioned above, it is of interest to generate the elements of \mathcal{I}_A (which correspond to the maximal Robinsonian submatrices of A) as well as the sets in \mathcal{C}_A (which correspond to the minimal obstructions to the Robinsonian property). For this one can apply the algorithmic approach developed in [9], which gives a quasi-polynomial time incremental algorithm for the joint generation of the collections $(\mathcal{F}_A, \mathcal{I}_A)$. Namely, given $\mathcal{X} \subseteq \mathcal{F}_A$ and $\mathcal{Y} \subseteq \mathcal{I}_A$, the algorithm of [9] permits to decide whether $(\mathcal{X}, \mathcal{Y}) = (\mathcal{F}_A, \mathcal{I}_A)$ and if not to find a new set in $(\mathcal{F}_A \setminus \mathcal{X}) \cup (\mathcal{I}_A \setminus \mathcal{Y})$, in time $O(n^4 + m^{O(\log m)})$, where $m = |\mathcal{F}_A| + |\mathcal{I}_A|$. Then, starting $(\mathcal{X}, \mathcal{Y}) = (\emptyset, \emptyset)$, this incremental algorithm can find $(\mathcal{F}_A, \mathcal{I}_A)$ in $|\mathcal{F}_A| + |\mathcal{I}_A|$ iterations.

On a more practical point of view, when A is not Robinsonian, one may try to remove some of its rows/columns and/or modify some of its entries in order to eliminate its weighted asteroidal cycles. Investigating whether this may lead to useful heuristics to get good Robinsonian approximations will be the subject of future research.

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