Quantum State Transfer in Coronas

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Abstract

We study state transfer in quantum walks on graphs relative to the adjacency matrix. Our motivation is to understand how the addition of pendant subgraphs affect state transfer. For two graphs G and H, the Frucht-Harary corona product $G \circ H$ is obtained by taking $|G|$ copies of the cone $K_1 + H$ and by identifying the conical vertices according to G. Our work explores conditions under which the corona $G \circ H$ exhibits state transfer. We also describe new families of graphs with state transfer based on the corona product. Some of these constructions provide a generalization of related known results.

Keywords: Quantum walk; state transfer; corona.

1 Introduction

Quantum walk is a natural generalization of classical random walk on graphs. It has received strong interest due to its important applications in quantum information and computation. Farhi and Gutmann [15] introduced quantum walk algorithms for solving search problems on graphs. In their framework, given a graph G with adjacency matrix A, the time-varying unitary matrix $U(t) := e^{-\mathbf{i}tA}$ defines a continuous-time quantum walk on G. Subsequently Childs *et al.* [8] showed that these algorithms may provide exponential speedup over classical probabilistic algorithms. The work by Farhi *et al.* [14] described an intriguing continuous-time quantum walk algorithm with nontrivial speedup for a concrete problem called Boolean formula evaluation.

In quantum information, Bose [5] studied the problem of information transmission in quantum spin chains. Christandl *et al.* [10, 9] showed that this problem may be reduced to the following phenomenon in quantum walk. Given two vertices u and v in G, we say perfect state transfer occurs between u and v in G at time τ if the (u, v)-entry of $U(\tau)$ has unit magnitude. It was quickly apparent that perfect state transfer is an exotic phenomenon. Godsil [20] proved that for every positive integer k there are only finitely many graphs of maximum degree k which have perfect state transfer. Therefore, the following relaxation of this notion is often more useful to consider. The state transfer between u and v is called "pretty good" (see Godsil [19]) or "almost perfect" (see Vinet and Zhedanov [30]) if the (u, v)-entry of the unitary matrix $U(t)$ can be made arbitrarily close to one by varying t.

Two vertices in a graph are antipodal if their distance is the diameter of the graph. Christandl *et al.* [10, 9] observed that the path P_n on n vertices has antipodal perfect state transfer if and only if $n = 2, 3$. In a striking result, Godsil *et al.* [22] proved that P_n has antipodal pretty good state transfer if and only if $n + 1$ is a prime, twice a prime, or a power of two. This provides the first family of graphs with pretty good state transfer which correspond to the quantum spin chains originally studied by Bose.

Shortly after, Fan and Godsil [13] studied a family of graphs obtained by taking two cones $K_1 + \overline{K}_m$ and then connecting the two conical vertices. They showed that these graphs, which are called double stars, have no perfect state transfer, but have pretty good state transfer between the two conical vertices if and only if $4m + 1$ is a perfect square. These graphs provide the second family of graphs known to have pretty good state transfer.

In this work, we provide new families of graphs with pretty good state transfer. Our constructions are based on a natural generalization of Fan and Godsil's double stars. The corona product of an n-vertex graph G with another graph H, typically denoted $G \circ H$, is obtained by taking n copies of the cone $K_1 + H$ and by identifying the conical vertices according to G. In a corona product $G \circ H$, we sometimes call G the *base* graph and H the *pendant* graph. This graph product was introduced by Frucht and Harary [16] in their study of automorphism groups of graphs which are obtained by wreath products.

We first observe that perfect state transfer on corona products is extremely rare. This is mainly due to the specific forms of the corona eigenvalues (which unsurprisingly resemble the eigenvalues of cones) coupled with the fact that periodicity is a necessary condition for perfect state transfer. Our negative results apply to corona families $G \circ H$ when H is either the empty or the complete graph, under suitable conditions on G. In a companion work [1], we observed an optimal negative result which holds for all H but in a Laplacian setting.

Given that perfect state transfer is rare, our subsequent results mainly focus on pretty good state transfer. We prove that the family of graphs $K_2 \circ K_m$, which are called barbell graphs (see Ghosh *et al.* [17]), admit pretty good state transfer for all m. Here, state transfer occurs between the two vertices of K_2. This is in contrast to the double stars $K_2 \circ \overline{K}_m$ where pretty good state transfer requires number-theoretic conditions on m.

We observe something curious for corona products when the base graph is complete. It is known that the complete graphs K_n, $n \geqslant 3$, are periodic but have no perfect state transfer, and hence have no pretty good state transfer. We observe that although $K_n \circ \overline{K}_2$ has no pretty good state transfer, $(K_n + I) \circ \overline{K}_2$ has pretty good state transfer, where $K_n + I$ denotes the graph obtained by adding loops to each vertex of K_n. This provides another example where loops are useful for state transfer (see Casaccino *et al.* [7] and Kempton *et al.* [26]).

Our other results involve graphs of the form $G \circ K_1$ which are called thorny graphs (see Gutman [24]). We show that if G is a graph with perfect state transfer at time π/g, for some positive integer g, then $G \circ K_1$ has pretty good state transfer, provided that the adjacency matrix of G is nonsingular. On the other hand, if the adjacency matrix of G is singular, we derive the same result if G has perfect state transfer at time $\pi/2$. Taken together, this proves that the thorny cube $Q_d \circ K_1$ has pretty good state transfer for all d. This confirms some of the numerical observations of Makmal *et al.* [28] on embedded hypercubes albeit for the continuous-time setting (since their results were stated for discrete-time quantum walks).

It turns out that perfect state transfer is not necessary for a thorny graph to have state transfer. Coutinho *et al.* [12] proved that the cocktail party graph $\overline{nK_2}$ has perfect state transfer if and only if n is even. We show that the thorny graph $\overline{nK_2} \circ K_1$ has pretty good state transfer for all n. This observation holds in the Laplacian setting as well (see [1]).

Returning to the double stars, we also observe that $C_2 \circ \overline{K}_m$ has *perfect* state transfer whenever $m + 1$ is an even square. Here, C_2 is the digon (which is a multigraph on two vertices connected by two parallel edges). This shows that certain double stars are one additional edge away from having perfect state transfer. This is akin to using weighted edges on paths for perfect state transfer (see Christandl *et al.* [10]). Here, we merely use an integer weight in a non-unimodal weighting scheme on a path.

2 Preliminaries

Given an undirected graph $G = (V, E)$ on n vertices, the adjacency matrix of G is an $n \times n$ symmetric matrix $A(G)$ where, for all vertices u and v, the (u, v) entry of $A(G)$ is 1, if $(u, v) \in E$, and 0, otherwise. The spectrum of G, which we denote as $\mathrm{Sp}(G)$, is the set of distinct eigenvalues of $A(G)$. We use $\rho(G)$ to denote the spectral radius of G. By the spectral theorem, we may write

$$A(G) = \sum_{\lambda \in \mathrm{Sp}(G)} \lambda E_\lambda(G) \tag{1}$$

where $E_\lambda(G)$ is the eigenprojector (orthogonal projector onto an eigenspace) corresponding to eigenvalue λ.

The *eigenvalue support* of a vertex u in G, denoted $\mathtt{supp}_G(u)$, is the set of eigenvalues λ of G for which $E_\lambda(G)\mathbf{e}_u \neq 0$, where $\mathbf{e}_u$ is the characteristic vector corresponding to vertex u. Two vertices u and v of G are called *strongly cospectral* if $E_\lambda(G)\mathbf{e}_u = \pm E_\lambda(G)\mathbf{e}_v$ for each eigenvalue λ of G.

A continuous-time quantum walk on G is defined by the time-varying unitary matrix $U(t) = e^{-\mathtt{i}tA(G)}$. We say such a quantum walk has *perfect state transfer* between vertices u and v at time τ if

$$\mathbf{e}_v^T U(\tau)\mathbf{e}_u = \gamma, \tag{2}$$

for some complex unimodular γ. We call γ the *phase* of the perfect state transfer. Note that $U(t)$ is symmetric for all t. In the special case where perfect state transfer occurs with $u = v$, we say the quantum walk is *periodic at vertex* u. Moreover, if $U(\tau)$ is a scalar multiple of the identity matrix, then the quantum walk is *periodic*.

Given that perfect state transfer is rare, we consider the following relaxation of this phenomenon proposed by Godsil. The quantum walk on G has *pretty good state transfer* between u and v if for all $\epsilon > 0$ there is a time τ so that

$$\left\| e^{-\mathtt{i}\tau A(G)}\mathbf{e}_u - \gamma \mathbf{e}_v \right\| < \epsilon, \tag{3}$$

for some complex unimodular γ.

In what follows, we state some useful facts about state transfer on graphs. The following formulation by Coutinho (which summarized the most relevant facts) will be useful for our purposes.

Theorem 2.1 (Coutinho [11], Theorem 2.4.4)**.** *Let G be a graph and let u, v be two vertices of G. Then there is perfect state transfer between u and v at time t with phase γ if and only if all of the following conditions hold.*

i) Vertices u and v are strongly cospectral.

ii) There are integers a, Δ where Δ is square-free so that for each eigenvalue λ in $\mathtt{supp}_G(u)$:

(a) $\lambda = \frac{1}{2}(a + b_\lambda\sqrt{\Delta})$, for some integer b_λ.

(b) $\mathbf{e}_u^T E_\lambda(G)\mathbf{e}_v$ is positive if and only if $(\rho(G) - \lambda)/g\sqrt{\Delta}$ is even, where

$$g := \gcd\left(\left\{\frac{\rho(G) - \lambda}{\sqrt{\Delta}} : \lambda \in \mathtt{supp}_G(u)\right\}\right). \tag{4}$$

Moreover, if the above conditions hold, then the following also hold.

i) There is a minimum time of perfect state transfer between u and v given by

$$t_0 := \frac{\pi}{g\sqrt{\Delta}}. \tag{5}$$

ii) The time of perfect state tranfer t is an odd multiple of t_0.

iii) The phase of perfect state transfer is given by $\gamma = e^{-\mathtt{i}t\rho(G)}$.

We state some strong properties proved by Godsil which relate perfect state transfer with periodicity in a fundamental way.

Lemma 2.2 (Godsil [18]). *If G has perfect state transfer between vertices u and v at time t, then G is periodic at u at time $2t$.*

Theorem 2.3 (Godsil [20]). *A graph G is periodic at vertex u if and only if either:*

i) all eigenvalues in $\mathtt{supp}_G(u)$ are integers; or

ii) there is a square-free integer Δ and an integer a so that each eigenvalue λ in $\mathtt{supp}_G(u)$ is of the form $\lambda = \frac{1}{2}(a + b_\lambda\sqrt{\Delta})$, for some integer b_λ.

We also state some necessary conditions for perfect state transfer and pretty good state transfer in terms of the automorphisms of G.

Theorem 2.4 (Godsil [20, 21]). *Let G be a graph with state transfer (perfect or pretty good) between vertices u and v. Then u and v are strongly cospectral, and each automorphism of G fixing u must fix v.*

In our analysis, we will need some tools from number theory. For example, we will need the following form of Kronecker's approximation theorem.

Theorem 2.5. *(Kronecker's Theorem: Hardy and Wright [25], Theorem 442)*
Let $1, \lambda_1, \ldots, \lambda_m$ be linearly independent over $\mathbb{Q}$. Let $\alpha_1, \ldots, \alpha_m$ be arbitrary real numbers, and let N, ϵ be positive real numbers. Then there are integers $\ell > N$ and $q_1, \ldots, q_m$ so that

$$|\ell\lambda_k - q_k - \alpha_k| < \epsilon, \tag{6}$$

for each $k = 1, \ldots, m$.

For brevity, whenever we have an inequality of the form $|\alpha - \beta| < \epsilon$, for arbitrarily small ϵ, we will write instead $\alpha \approx \beta$ and omit the explicit dependence on ϵ. For example, (6) will be represented as $\ell\lambda_k - q_k \approx \alpha_k$.

In our applications of Kronecker's Theorem, we will use the following lemma to identify sets of numbers which are linearly independent over the rationals.

Lemma 2.6 (Newman and Flanders [29]). *Let $a_1, \ldots, a_n$ be positive integers, coprime in pairs, no one of which is a perfect square. Then the 2^n algebraic integers*

$$\sqrt{a_1^{e_1} \ldots a_n^{e_n}}, \qquad e_j = 0, 1, \tag{7}$$

are linearly independent over the field $\mathbb{Q}$ of rationals.

Notation: We describe some notation used throughout the rest of the paper.

The all-one and all-zero vectors of dimension n are denoted $\mathbf{j}_n$, $\mathbf{0}_n$, respectively. The $m \times n$ all-one matrix is denoted $J_{m,n}$ or simply J_n if $m = n$. The identity matrix of size n is denoted I_n.

For standard families of graphs, we use K_n for the complete graph on n vertices, P_n for the path on n vertices, and Q_n for the n-dimensional cube. In what follows, let G and H be graphs. The complement of G is denoted $\overline{G}$. The disjoint union of G and H is written as $G \cup H$, while the disjoint union of n copies G is denoted nG. The Cartesian product of G and H is denoted $G \square H$. If G, H have n, m vertices, respectively, the adjacency matrix of $G \square H$ is given by $A(G) \otimes I_m + I_n \otimes A(H)$. The join $G + H$ of G and H is the graph $\overline{\overline{G} \cup \overline{H}}$. Further background on algebraic graph theory may be found in Godsil and Royle [23].

3 Corona of Graphs

We define the Frucht-Harary corona product of two graphs (see [16]). Let G be a graph on the vertex set $\{v_1, \ldots, v_n\}$ and H be a graph on the vertex set $\{1, \ldots, m\}$. The latter is chosen for notational convenience. The *corona* $G \circ H$ is formed by taking the disjoint union of G and n copies of H and then adding an edge from each vertex of the jth copy of H to the vertex v_j in G. Formally, the corona $G \circ H$ has the vertex set

$$V(G \circ H) = V(G) \times (\{0\} \cup V(H)), \tag{8}$$

and the adjacency relation

$$((v,w),(v',w')) \in E(G \circ H) \iff \begin{cases} w = w' = 0 \text{ and } (v,v') \in E(G) & \text{or} \\ v = v' \text{ and } (w,w') \in E(H) & \text{or} \\ v = v' \text{ and exactly one of } w \text{ and } w' \text{ is } 0. \end{cases} \tag{9}$$

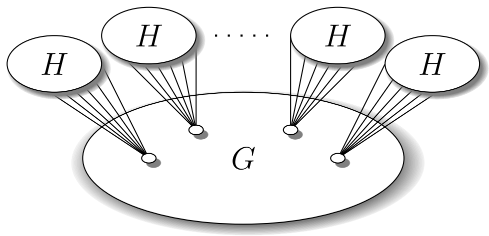

Figure 1: A corona $G \circ H$.

The adjacency matrix of $G \circ H$ where H has m vertices is given by

$$A(G \circ H) = A(G) \otimes \mathbf{e}_0 \mathbf{e}_0^T + I_n \otimes \begin{bmatrix} 0 & \mathbf{j}_m^T \\ \mathbf{j}_m & A(H) \end{bmatrix}. \tag{10}$$

The spectrum of $G \circ H$ when H is a k-regular graph is known and we state this in the following.

Theorem 3.1 (Barik *et al.* [4])**.** *Suppose that G is a graph and H is a k-regular graph on m vertices. Suppose G has eigenvalues $\lambda_0 > \ldots > \lambda_p$ with multiplicities $r_0, \ldots, r_p$, and H has eigenvalues $k = \mu_0 > \mu_1 > \ldots > \mu_q$ with multiplicities $s_0, \ldots, s_q$. Then $G \circ H$ has the following spectrum.*

i) k is an eigenvalue with multiplicity $n(s_0 - 1)$.

ii) μ_j is an eigenvalue with multiplicity ns_j for $j = 1, \ldots, q$.

iii) $\frac{1}{2}(\lambda_j + k \pm \sqrt{(\lambda_j - k)^2 + 4m})$ are eigenvalues each with multiplicity r_j for $j = 0, \ldots, p$.

Theorem 3.1 gives the spectrum of $G \circ H$ when H is k-regular. We now provide the corresponding eigenprojectors which will be crucial in our subsequent analyses.

Proposition 3.2. *Let G be a graph on n vertices and let H be a k-regular graph on m vertices. Then the eigenprojectors of $G \circ H$ are given by the following.*

i) For each eigenvalue μ of H, let

$$E_\mu := I_n \otimes \begin{bmatrix} 0 & \mathbf{0}_m^T \\ \mathbf{0}_m & E_\mu(H) - \delta_{\mu,k} \frac{1}{m} J_m \end{bmatrix}. \tag{11}$$

ii) For each eigenvalue λ of G, define a pair of eigenvalues

$$\lambda_\pm := \frac{\lambda + k \pm \sqrt{(\lambda - k)^2 + 4m}}{2}, \tag{12}$$

and let

$$E_{\lambda_\pm} := E_\lambda(G) \otimes \frac{1}{(\lambda_\pm - k)^2 + m} \begin{bmatrix} (\lambda_\pm - k)^2 & (\lambda_\pm - k)\mathbf{j}_m^T \\ (\lambda_\pm - k)\mathbf{j}_m & J_m \end{bmatrix}. \tag{13}$$

Then the spectral decomposition of the corona $G \circ H$ is given by

$$A(G \circ H) = \sum_{\lambda \in \mathrm{Sp}(G)} \sum_{\pm} \lambda_\pm E_{\lambda_\pm} + \sum_{\mu \in \mathrm{Sp}(H)} \mu E_\mu. \tag{14}$$

Proof. Recall that the adjacency matrix of $G \circ H$ is given in (10).

For an eigenvalue μ of H, let B_μ be an orthonormal basis of the μ-eigenspace that is orthogonal to $\mathbf{j}_m$. If H is a connected k-regular graph, then B_k is empty. Suppose μ has an eigenvalue multiplicity of r. Then the cardinality of B_μ is r if $\mu < k$, and is $r - 1$ if $\mu = k$. Moreover, μ is also an eigenvalue of $G \circ H$ since for each $\mathbf{x} \in B_\mu$ and for every $\mathbf{y}$ in $\mathbb{C}^n$, we have

$$A(G \circ H)\mathbf{y} \otimes \begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix} = \mu \mathbf{y} \otimes \begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix}. \tag{15}$$

An orthonormal basis for the μ-eigenspace of $G \circ H$ is given by

$$\left\{ \mathbf{e}_v \otimes \begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix} : v \in V(G), \mathbf{x} \in B_\mu \right\}. \tag{16}$$

The cardinality of this set is nr if $\mu < k$, and is $n(r-1)$ if $\mu = k$. These match the multiplicities given in Theorem 3.1. Note that the eigenprojector of μ in H is given by

$$E_\mu(H) = \begin{cases} \sum_{\mathbf{x}\in B_\mu} \mathbf{x}\mathbf{x}^\dagger & \text{if } \mu < k \\ \frac{1}{m}J_m + \sum_{\mathbf{x}\in B_\mu} \mathbf{x}\mathbf{x}^\dagger & \text{if } \mu = k \end{cases} \tag{17}$$

Therefore, the eigenprojector of μ in $G \circ H$ is given by

$$E_\mu(G\circ H) = I_n \otimes \begin{bmatrix} 0 & \mathbf{0}_m^T \\ \mathbf{0}_m & E_\mu(H) - \delta_{\mu,k}\frac{1}{m}J_m \end{bmatrix}. \tag{18}$$

Next, for each eigenvalue λ of G with a corresponding (normalized) eigenvector $\mathbf{x}$, we have

$$A(G\circ H)\left(\mathbf{x}\otimes \begin{bmatrix} \lambda_\pm - k \\ \mathbf{j}_m \end{bmatrix}\right) = \lambda_\pm \mathbf{x}\otimes \begin{bmatrix} \lambda_\pm - k \\ \mathbf{j}_m \end{bmatrix} \tag{19}$$

provided that

$$\lambda_\pm := \frac{\lambda + k \pm \sqrt{(\lambda+k)^2 + 4m - 4\lambda k}}{2} = \frac{\lambda + k \pm \sqrt{(\lambda-k)^2+4m}}{2}. \tag{20}$$

After normalizing the above eigenvectors, we see that the eigenprojectors of $\lambda_\pm$ are given by

$$E_{\lambda_\pm}(G\circ H) = E_\lambda(G) \otimes \frac{1}{(\lambda_\pm - k)^2 + m}\begin{bmatrix} (\lambda_\pm - k)^2 & (\lambda_\pm - k)\mathbf{j}_m^T \\ (\lambda_\pm - k)\mathbf{j}_m & J_m \end{bmatrix} \tag{21}$$

These provide the remaining $2n$ eigenvectors of $G\circ H$. $\square$

For state transfer in $G\circ H$, we need to analyze the elements of the transition matrix $e^{-\mathrm{i}tA(G\circ H)}$. We use Proposition 3.2 to derive a convenient form for our analysis.

Proposition 3.3. *Let G be a graph and let H be a k-regular graph on m vertices. For vertices v and v' of G, we have*

$$\mathbf{e}_{(v,0)}^T e^{-\mathrm{i}tA(G\circ H)}\mathbf{e}_{(v',0)} = \sum_{\lambda\in\mathrm{Sp}(G)} e^{-\mathrm{i}t(\lambda+k)/2}\mathbf{e}_v^T E_\lambda(G)\mathbf{e}_{v'}\left(\cos(t\Lambda_\lambda/2) - \frac{(\lambda-k)}{\Lambda_\lambda} i\sin(t\Lambda_\lambda/2)\right) \tag{22}$$

where $\Lambda_\lambda = \sqrt{(\lambda-k)^2+4m}$.

Proof. Let $\lambda_\pm = \frac{1}{2}(\lambda + k \pm \Lambda_\lambda)$. From Proposition 3.2, we get that

$$\begin{aligned} \mathbf{e}_{(v,w)}^T e^{-\mathrm{i}tA(G\circ H)}\mathbf{e}_{(v',w')} = &\sum_{\lambda\in\mathrm{Sp}(G)} e^{-\mathrm{i}t(\lambda+k)/2}\mathbf{e}_v^T E_\lambda(G)\mathbf{e}_{v'}\left(\sum_{\pm} e^{\mp\mathrm{i}t\Lambda_\lambda/2}\mathbf{e}_w^T M_{\lambda_\pm}\mathbf{e}_{w'}\right) \\ &+ \delta_{v,v'}(1-\delta_{w,0})(1-\delta_{w',0})\sum_{\mu\in\mathrm{Sp}(H)} e^{-\mathrm{i}t\mu}\left(\mathbf{e}_w^T E_\mu(H)\mathbf{e}_{w'} - \frac{1}{m}\delta_{\mu,k}\right), \end{aligned} \tag{23}$$

where

$$M_{\lambda_\pm} = \frac{1}{(\lambda_\pm - k)^2 + m} \begin{bmatrix} (\lambda_\pm - k)^2 & (\lambda_\pm - k)\mathbf{j}_m^T \\ (\lambda_\pm - k)\mathbf{j}_m & J_m \end{bmatrix}. \tag{24}$$

Therefore,

$$\mathbf{e}_{(v,0)}^T e^{-\mathrm{i}tA(G\circ H)} \mathbf{e}_{(v',0)} = \sum_{\lambda \in \mathrm{Sp}(G)} e^{-\mathrm{i}t(\lambda+k)/2} \mathbf{e}_v^T E_\lambda(G) \mathbf{e}_{v'} \sum_{\pm} e^{\mp \mathrm{i}t\Lambda_\lambda/2} \frac{(\lambda_\pm - k)^2}{(\lambda_\pm - k)^2 + m}. \tag{25}$$

But, the inner summation in (25) simplifies to

$$\sum_{\pm} e^{\mp \mathrm{i}t\Lambda_\lambda/2} \frac{(\lambda_\pm - k)^2}{(\lambda_\pm - k)^2 + m} \;=\; \cos(t\Lambda_\lambda/2) - \frac{(\lambda - k)}{\Lambda_\lambda} i \sin(t\Lambda_\lambda/2), \tag{26}$$

since $\prod_\pm((\lambda_\pm - k)^2 + m) = m\Lambda_\lambda^2$ and $\prod_\pm(\lambda_\pm - k) = -m$. This proves the claim. □

4 Perfect State Transfer

Lemma 2.2 shows that periodicity is a necessary condition for perfect state transfer. We show that there is no perfect state transfer in most coronas since their vertices are not periodic.

4.1 Conditions on Periodicity

To investigate periodicity in a corona $G \circ H$, the following lemma shows that it is sufficient to consider base vertices of the form $(v, 0)$, where v is a vertex in G.

Lemma 4.1. *Let G be a graph and let H be a regular graph. If (v, w) is periodic in $G \circ H$, then $(v, 0)$ is periodic in $G \circ H$.*

Proof. The eigenvalue support of $(v, 0)$ is contained within the eigenvalue support of (v, w), which is evident from the eigenprojectors given in Proposition 3.2. □

Theorem 2.3 provides necessary and sufficient conditions for periodicity. In the following, we show that periodicity in a corona places a strong condition on the eigenvalues of the base graph.

Lemma 4.2. *Suppose that G is a connected graph on at least two vertices and H is a k-regular graph on m vertices. Then a vertex $(v, 0)$ is periodic in $G \circ H$ if and only if there exists a positive square-free integer Δ such that for all eigenvalues $\lambda \in \mathtt{supp}_G(v)$, both $\lambda - k$ and $\sqrt{(\lambda - k)^2 + 4m}$ are integer multiples of $\sqrt{\Delta}$. Moreover, if this holds, we have that Δ divides $2m$.*

Proof. For each eigenvalue λ of G, let

$$\lambda_\pm := \frac{1}{2}\left(\lambda + k \pm \sqrt{(\lambda - k)^2 + 4m}\right). \tag{27}$$

By Proposition 3.2, the eigenvalue support of $(v,0)$ in $G \circ H$ is given by $\{\lambda_\pm : \lambda \in \mathtt{supp}_G(v)\}$. If there is a positive square-free integer Δ so that for each eigenvalue λ in the support of v in G, both $\lambda - k$ and $\sqrt{(\lambda-k)^2+4m}$ are integer multiples of $\sqrt{\Delta}$, then $(v,0)$ is periodic in $G \circ H$, by Theorem 2.3. This shows sufficiency.

Now for necessity. Suppose that $(v,0)$ is periodic in $G \circ H$. By Theorem 2.3, either all eigenvalues in the support of $(v,0)$ in $G \circ H$ are integers or there is an integer a and a square-free integer Δ so that all eigenvalues in the support of $(v,0)$ in $G \circ H$ are of the form $\lambda = \frac{1}{2}(a + b_\lambda\sqrt{\Delta})$, for some integers b_λ.

If the eigenvalue support of $(v,0)$ is entirely integers, then for each eigenvalue λ in the support of v, we conclude that $\lambda + k = \lambda_+ + \lambda_-$ and $\sqrt{(\lambda-k)^2+4m} = \lambda_+ - \lambda_-$ are integers.

Now suppose there is an integer a and a square-free integer $\Delta > 1$ so that all eigenvalues $\lambda_\pm$ in the support of $(v,0)$ are of the form

$$\lambda_\pm = \frac{1}{2}(a + b_\pm\sqrt{\Delta}), \tag{28}$$

where $b_\pm$ are integers (which depend on $\lambda_\pm$). Recall from the proof of Proposition 3.3 that $(\lambda_+ - k)(\lambda_- - k) = -m$. Thus,

$$-m \;=\; \frac{1}{4}((a-2k)^2 + (b_+b_-)\Delta) + \frac{1}{4}(a-2k)(b_+ + b_-)\sqrt{\Delta}. \tag{29}$$

Since $\sqrt{\Delta}$ is not rational, either $a - 2k$ or $b_+ + b_-$ is zero. If $b_+ = -b_-$ then we conclude that $a = \lambda_+ + \lambda_- = \lambda + k$. This implies that $|\mathtt{supp}_G(v)| = 1$ which contradicts that G is connected on at least two vertices. Otherwise, we have $a = 2k$, from which we conclude that $\lambda_\pm = k + \frac{1}{2}b_\pm\sqrt{\Delta}$. From the definition of $\lambda_\pm$ in (27), we have

$$(\lambda_+ - k) + (\lambda_- - k) = \lambda - k, \tag{30a}$$

$$(\lambda_+ - k) - (\lambda_- - k) = \sqrt{(\lambda-k)^2+4m}. \tag{30b}$$

Given the alternate form in (28), this also implies that $\lambda - k$ and $\sqrt{(\lambda-k)^2+4m}$ are half-integer multiples of $\sqrt{\Delta}$. Since their squares are rational algebraic integers, their squares must be integers. Thus, $\lambda - k$ and $\sqrt{(\lambda-k)^2+4m}$ are integer multiples of $\sqrt{\Delta}$.

It remains to show that Δ divides $2m$. The condition that $\sqrt{(\lambda-k)^2+4m}$ is an integer multiple of $\sqrt{\Delta}$ implies that

$$\sqrt{\frac{(\lambda-k)^2+4m}{\Delta}} \tag{31}$$

is an integer. Furthermore, since Δ divides $(\lambda-k)^2$, it must be that Δ divides $4m$. Since Δ is square-free, Δ divides $2m$. □

Remark 1. The conditions for periodicity at (v,w) in the corona $G \circ H$ imply that $\lambda \in k + \mathbb{Z}\sqrt{\Delta}$ for all λ in the eigenvalue support of v in G. In particular, v must be periodic in G.

Remark 2. The eigenvalues of G need not be integers for $G \circ H$ to have a periodic vertex. Consider P_3 whose spectrum is $\{0, \pm\sqrt{2}\}$. The eigenvalue support of the middle vertex is $\{\pm\sqrt{2}\}$. Thus we may apply Lemma 4.2 to see that the middle vertex of P_3 is periodic in $P_3 \circ \overline{K}_m$ if and only if $\sqrt{2+4m}$ is an integral multiple of $\sqrt{2}$, i.e. if and only if $\sqrt{1+2m}$ is an integer. Let us take the specific example of $m = 4$. If we let v denote the center vertex of P_3, then the eigenvalue support of $(v, 0)$ in $P_3 \circ \overline{K}_4$ is $\{\pm\sqrt{2}, \pm 2\sqrt{2}\}$, so $(v, 0)$ is periodic.

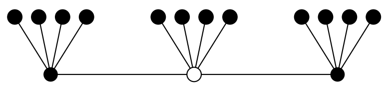

Figure 2: The white vertex in $P_3 \circ \overline{K}_4$ is periodic.

The heart of our applications of Lemma 4.2 is the following. For the conditions of Lemma 4.2 to be satisfied, it must be that $4m/\Delta$ is a difference of squares. Since there are only finitely many pairs of squares whose difference is $4m/\Delta$, this allows us to restrict possible values of $\lambda - k$ in various ways. Our first application is to show that the eigenvalues in the support of a periodic vertex can not be too close together.

Theorem 4.3. *Let G be a graph and H be a k-regular graph on $m \geqslant 1$ vertices. Suppose v is a vertex of G for which there are two distinct eigenvalues $\lambda, \mu \in \mathtt{supp}_G(v)$ such that*

$$|\lambda - k| - |\mu - k| \ \in \ \{\sqrt{\Delta}, 2\sqrt{\Delta}\} \tag{32}$$

for some square-free integer Δ. Then (v, w) is not periodic vertex in $G \circ H$, for all $w \in V(H) \cup \{0\}$.

Proof. By Lemma 4.1, it suffices to show $(v, 0)$ is not periodic. Suppose towards contradiction that $(v, 0)$ is periodic. By Lemma 4.2, there exists a square-free integer Δ such that for each eigenvalue λ in the support of v, both $\lambda - k$ and $\sqrt{(\lambda - k)^2 + 4m}$ are integer multiples of $\sqrt{\Delta}$. We define

$$\sigma := \frac{1}{\sqrt{\Delta}} \min \left\{ \Big| |\lambda_1 - k| - |\lambda_2 - k| \Big| \ : \ \lambda_1, \lambda_2 \in \mathtt{supp}_G(v) \right\}. \tag{33}$$

Let λ and μ be the eigenvalues in the support of v which achieve the above minimum. Now, define

$$n_\lambda := \frac{|\lambda - k|}{\sqrt{\Delta}}, \qquad n_\mu := \frac{|\mu - k|}{\sqrt{\Delta}}, \tag{34}$$

and suppose that $\sigma := n_\lambda - n_\mu$.

By the assumption in (32), we have two cases to consider: $\sigma = 1$ and $\sigma = 2$.

By Lemma 4.2, both $n_\lambda^2 + 4m/\Delta$ and $n_\mu^2 + 4m/\Delta$ are squares. If we let

$$p := \sqrt{n_\mu^2 + \frac{4m}{\Delta}} \quad \text{and} \quad q := \sqrt{n_\lambda^2 + \frac{4m}{\Delta}}, \tag{35}$$

then

$$q + p > n_\lambda + n_\mu = 2n_\mu + \sigma \quad \text{and} \quad q^2 - p^2 = (2n_\mu + \sigma)\sigma. \tag{36}$$

Hence $q - p < \sigma$ which is impossible when $\sigma = 1$.

When $\sigma = 2$, p and q have the same parity which contradicts $0 < q - p < 2$. □

Corollary 4.4. *Suppose G is a graph and H is a k-regular graph on $m \geqslant 1$ vertices. Let $v \in V(G)$ and suppose there are eigenvalues $\lambda, \mu \in \mathtt{supp}_G(v)$ such that $0 < |\lambda - k| - |\mu - k| < 3$. Then (v, w) is not periodic in $G \circ H$ for every $w \in V(H) \cup \{0\}$.*

Proof. Suppose towards contradiction that $(v, 0)$ is periodic. Then by Lemma 4.2, we have that $\lambda - k$ and $\mu - k$ are integer multiples of $\sqrt{\Delta}$, for some square-free integer Δ. Since $0 < |\lambda - k| - |\mu - k| < 3$, we have

$$|\lambda - k| - |\mu - k| \in \{\sqrt{1}, \sqrt{2}, \sqrt{3}, 2\sqrt{1}, \sqrt{5}, \sqrt{6}, \sqrt{7}, 2\sqrt{2}\}. \tag{37}$$

Note that Theorem 4.3 applies to all of these cases. □

We apply our machinery above to show that the corona products of certain distance-regular graphs with an arbitrary regular graph have no perfect state transfer. In particular, we show this for some families of distance-regular graphs which *have* perfect state transfer (see Coutinho *et al.* [12]). These examples confirm that perfect state transfer is highly sensitive to perturbations.

Corollary 4.5. *Let G be a graph from one of the following families:*

- *d-cubes Q_d, for $d \geqslant 2$.*
- *Cocktail party graphs $\overline{nK_2}$, for $n \geqslant 2$.*
- *Halved $2d$-cubes $\frac{1}{2}Q_{2d}$, for $d \geqslant 1$, where the vertices are elements of $\mathbb{Z}_2^{2d}$ of even Hamming weight and two vertices are adjacent if their Hamming distance is exactly two.*

Then the corona product $G \circ H$, where H is a regular graph, has no perfect state transfer.

Proof. The spectra of these graphs are well known (see Brouwer *et al.* [6]) and are given by:

- *Spec*$(Q_d) = \{d - 2\ell : 0 \leqslant \ell \leqslant d\}$.
- *Spec*$(\overline{nK_2}) = \{2n - 2, 0, -2\}$.
- *Spec*$(\frac{1}{2}Q_{2d}) = \left\{\binom{2d}{2} - 2\ell(2d - \ell) : 0 \leqslant \ell \leqslant d\right\}$.

Since these graphs are distance-regular, every eigenvalue is in the support of every vertex. Moreover, the eigenvalues satisfy the conditions of Corollary 4.4. In particular, $2 - d$ and $-d$ are always eigenvalues of the d-cube and the halved $2d$-cube. Therefore, the corona of these graphs with an arbitrary regular graph do not have periodic vertices. Hence, by Lemma 2.2, they do not have perfect state transfer. □

4.2 Corona with the Complete Graph

The main result of this section is that there is no perfect state transfer on $G \circ K_m$ when G is connected on at least two vertices. This is achieved by bounding the number of vertices in H for the corona product $G \circ H$ to have a periodic vertex.

Lemma 4.6. *Suppose that G is a graph and H is a k-regular graph on $m \geqslant 1$ vertices. If $(v, 0)$ is periodic in $G \circ H$, then for all $\lambda \in \mathtt{supp}_G(v)$ we have*

$$m \;\geqslant\; |\lambda - k| + 1. \tag{38}$$

Proof. Suppose that $(v, 0)$ is periodic in $G \circ H$. By Lemma 4.2, there exists a positive square-free integer Δ such that for all $\lambda \in \mathtt{supp}_G(v)$, both $(\lambda - k)^2/\Delta$ and $((\lambda - k)^2 + 4m)/\Delta$ are squares. Moreover, Δ divides $2m$, and therefore, these squares have the same parity. This yields the bound

$$\frac{4m}{\Delta} \geqslant \left(\frac{|\lambda - k|}{\sqrt{\Delta}} + 2\right)^2 - \left(\frac{|\lambda - k|}{\sqrt{\Delta}}\right)^2 = 4\left(\frac{|\lambda - k|}{\sqrt{\Delta}} + 1\right). \tag{39}$$

After rearranging, we see that

$$m \;\geqslant\; |\lambda - k|\sqrt{\Delta} + \Delta \;\geqslant\; |\lambda - k| + 1, \tag{40}$$

since $\Delta \geqslant 1$. $\square$

Theorem 4.7. *Let G be a connected graph on at least 2 vertices. Then for all $m \geqslant 1$, the corona product $G \circ K_m$ has no periodic vertices, and, therefore, has no perfect state transfer.*

Proof. Suppose towards contradiction that the vertex (v, w) of $G \circ K_m$ is periodic. If the eigenvalue support of v in G contains a negative eigenvalue λ, then $\lambda - (m-1) < 0$. Thus, Lemma 4.6 yields

$$m \;\geqslant\; |\lambda - (m-1)| + 1 \;=\; -\lambda + (m-1) + 1 \;>\; m. \tag{41}$$

So, it suffices to show that the support of v contains a negative eigenvalue.

Since G has no loops, we have

$$\mathbf{e}_v^T A \mathbf{e}_v = \sum_{\lambda \in \operatorname{Sp} G} \lambda \mathbf{e}_v^T E_\lambda \mathbf{e}_v = 0. \tag{42}$$

If v has no negative eigenvalue in its support, then $E_\lambda \mathbf{e}_v = 0$ for all $\lambda \neq 0$. In this case, $A\mathbf{e}_v$ is the zero vector and G is not connected. $\square$

4.3 Corona with the Empty Graph

In this section, we show that the corona of any graph with $\overline{K}_m$, where m is one or a prime number, has no perfect state transfer. We will need the following spectral characterization of when a vertex is conical in a star graph.

Lemma 4.8. *Let v be a vertex in a connected graph G. If $\mathtt{supp}_G(v) = \{\pm\lambda\}$, for some $\lambda > 0$, then $\lambda^2 \in \mathbb{Z}$ and v is the conical vertex of the star graph K_{1,λ^2}.*

Proof. Suppose that $\mathtt{supp}_G(v) = \{\pm\lambda\}$. If $A = \sum_\theta \theta E_\theta$ is the spectral decomposition of the adjacency matrix of G, then $(E_\lambda + E_{-\lambda})\mathbf{e}_v = \mathbf{e}_v$. Moreover, $A^2\mathbf{e}_v = \lambda^2\mathbf{e}_v$. Since A^2 has integer entries, we observe that $\lambda^2 \in \mathbb{Z}$. This means that every walk of length 2 starting from v must return to v, i.e. that every neighbor of v has degree 1. The number of closed walks of length two from v is then exactly the degree of v, and thus we see that v is the center vertex of K_{1,λ^2}. □

Lemma 4.9. *$K_{1,n} \circ \overline{K}_m$ has no perfect state transfer, for every $n, m \geqslant 1$.*

Proof. The case for $n = 1$ is a result of Fan and Godsil [13]. For $n = 2$, that is, $P_3 \circ \overline{K}_m$, we may apply Theorems 2.4 and 2.1 and Lemma 4.2 to rule out perfect state transfer. For $n > 2$ and $m \neq 2$, we may apply Theorems 2.4 and 2.1 to show no perfect state transfer exists.

So, we consider $K_{1,n} \circ \overline{K}_2$ where $n > 2$, and assume that it has perfect state transfer between vertices $(v, 1)$ and $(v, 2)$ where v is a vertex of $K_{1,n}$. Note that perfect state transfer between other pairs of vertices are ruled out by Theorems 2.4 and 2.1. Thus, $(v, 0)$ is a periodic vertex. By Lemma 4.2, there is a square-free integer Δ so that for each eigenvalue $\lambda \in \mathtt{supp}(v)$, both λ and $\sqrt{\lambda^2 + 8}$ are integer multiples of $\sqrt{\Delta}$. Moreover, Δ divides 4 and thus $\Delta \in \{1, 2\}$.

If $\Delta = 1$, both λ and $\sqrt{\lambda^2 + 8}$ are integers for each eigenvalue $\lambda \in \mathtt{supp}(v)$. Thus, $\mathtt{supp}(v) = \{\pm 1\}$. By Lemma 4.8, v is a conical vertex of $K_{1,1}$. But, $K_{1,1} \circ \overline{K}_2$ has no perfect state transfer (see [13]).

If $\Delta = 2$, both λ and $\sqrt{\lambda^2 + 8}$ are integer multiples of $\sqrt{2}$ for each eigenvalue $\lambda \in \mathtt{supp}(v)$. Suppose $\lambda = \ell\sqrt{2}$ for some integer ℓ. Then, $\ell^2 + 4$ is a square which implies $\ell = 0$. Thus, $\mathtt{supp}(v) = \{0\}$. But, this implies v is an isolated vertex, which is a contradiction. □

Theorem 4.10. *If G is a connected graph on at least two vertices and m is either 1 or a prime number, then $G \circ \overline{K}_m$ has no perfect state transfer.*

Proof. By Lemma 4.9, we may assume G is connected and is not a star. By Lemma 2.2, it suffices to show $G \circ \overline{K}_m$ has no periodic vertices for $m \geqslant 1$. Suppose for contradiction that vertex (v, w) is periodic in $G \circ \overline{K}_m$. Then by Lemma 4.1, $(v, 0)$ is periodic. Since $G \neq K_{1,n}$ is connected, by Lemma 4.8, the eigenvalue support of v in G contains eigenvalues λ and μ such that $|\lambda| \neq |\mu|$. Assume $|\mu| < |\lambda|$. By Lemma 4.2, there must exist a square-free integer Δ dividing $2m$ such that λ, μ, $\sqrt{\lambda^2 + 4m}$, and $\sqrt{\mu^2 + 4m}$ are all integer multiples of $\sqrt{\Delta}$.

We define the integers

$$n_\lambda := \frac{|\lambda|}{\sqrt{\Delta}}, \qquad n_\mu := \frac{|\mu|}{\sqrt{\Delta}}, \qquad \ell := \frac{2m}{\Delta}. \tag{43}$$

Since $\sqrt{\mu^2 + 4m}$ and $\sqrt{\lambda^2 + 4m}$ are integer multiples of $\sqrt{\Delta}$, both $n_\mu^2 + 2\ell$ and $n_\lambda^2 + 2\ell$ are perfect squares. So, let $N_\mu^2 := n_\mu^2 + 2\ell$ and $N_\lambda^2 := n_\lambda^2 + 2\ell$. Therefore, we have four perfect squares: n_μ^2, N_μ^2, n_λ^2, and N_λ^2. Since $|\mu| < |\lambda|$, we know $n_\mu < n_\lambda$.

If $m = 1$ or $m = 2$, then ℓ equals one, two or four and 2ℓ equals two, four or eight. First note that a difference of two squares is never two or four. For the last case, the only way for $N_\mu^2 - n_\mu^2 = N_\lambda^2 - n_\lambda^2 = 8$ is $N_\lambda = N_\mu = 3$ and $n_\mu = n_\lambda = 1$, contradicting $n_\lambda > n_\mu$.

Now, consider when m is an odd prime. Since a squarefree integer Δ divides $2m$, we have ℓ equals one, two, m or $2m$. Or, equivalently, 2ℓ equals two, four, $2m$ or $4m$. We may rule out the first two cases as before. Since $N_\mu^2 - n_\mu^2$ is even, N_μ and n_μ share the same parity. If $2\ell = 2m$, then our odd prime m can be written as $m = (N_\mu - n_\mu)(N_\mu + n_\mu)/2$. This shows $N_\mu - n_\mu = 1$ or that N_μ and n_μ differ in parity, a contradiction. If $2\ell = 4m$, then $m = \frac{1}{2}(N_\mu - n_\mu)\frac{1}{2}(N_\mu + n_\mu)$. This shows $N_\mu - n_\mu = 2$, and furthermore, $m = n_\mu + 1$. But using the other difference of squares, we also have $m = n_\lambda + 1$. This contradicts $n_\lambda > n_\mu$. $\square$

4.4 Multigraphs

Fan and Godsil [13] proved that the double star $K_2 \circ \overline{K}_m$ has no perfect state transfer for all m. Here, we show that $C_2 \circ \overline{K}_m$ *has* perfect state transfer for some m where C_2 is the *digon* (a multigraph on two vertices which are connected by two parallel edges).

This result follows from the next proposition. For a real number α, let $K_2(\alpha)$ denote the clique on two vertices whose edge has weight α. In what follows, we consider the weighted path $K_2(\alpha) \circ K_1$.

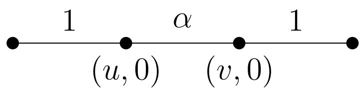

Figure 3: The weighted path $K_2(\alpha) \circ K_1$.

Proposition 4.11. *In $K_2(\alpha) \circ K_1$, perfect state transfer occurs between the vertices of K_2 if and only if*

$$\alpha = \frac{2(2s+1)}{\sqrt{(2\ell)^2 - (2s+1)^2}} \qquad \textit{for some integers } \ell > s \geqslant 0. \tag{44}$$

In this case, perfect state transfer occurs at time $t = \frac{\pi}{2}\sqrt{(2\ell)^2 - (2s+1)^2}$.

Proof. Let u, v denote the vertices of K_2. The eigenvalues of $K_2(\alpha)$ are $\pm\alpha$ with corresponding eigenprojectors $E_{\pm\alpha}$ which satisfy $\mathbf{e}_u^T E_{\pm\alpha} \mathbf{e}_v = \pm\frac{1}{2}$. By Proposition 3.3, the transition element between $(u, 0)$ and $(v, 0)$ is given by

$$\mathbf{e}_{(u,0)}^T e^{-\mathrm{i}tA(K_2(\alpha)\circ K_1)}\mathbf{e}_{(v,0)} = -i\sin\left(\frac{t}{2}\alpha\right)\cos\left(\frac{t}{2}\Lambda_\alpha\right) - i\frac{\alpha}{\Lambda_\alpha}\cos\left(\frac{t}{2}\alpha\right)\sin\left(\frac{t}{2}\Lambda_\alpha\right), \quad (45)$$

where $\Lambda_\alpha := \sqrt{\alpha^2+4}$. Since $|\alpha/\Lambda_\alpha| < 1$, there is perfect state transfer at time t if and only if

$$\left|\sin\left(\frac{t}{2}\alpha\right)\cos\left(\frac{t}{2}\Lambda_\alpha\right)\right| = 1. \quad (46)$$

Equivalently, this gives the conditions

$$\frac{t}{2}\sqrt{\alpha^2+4} = \ell\pi, \qquad \text{and} \qquad \frac{t}{2}\alpha = \left(s+\frac{1}{2}\right)\pi, \quad (47)$$

for some integers $\ell > s \geqslant 0$. Hence perfect state transfer occurs at time t if and only if $\alpha = 2(2s+1)/\sqrt{(2\ell)^2 - (2s+1)^2}$. $\square$

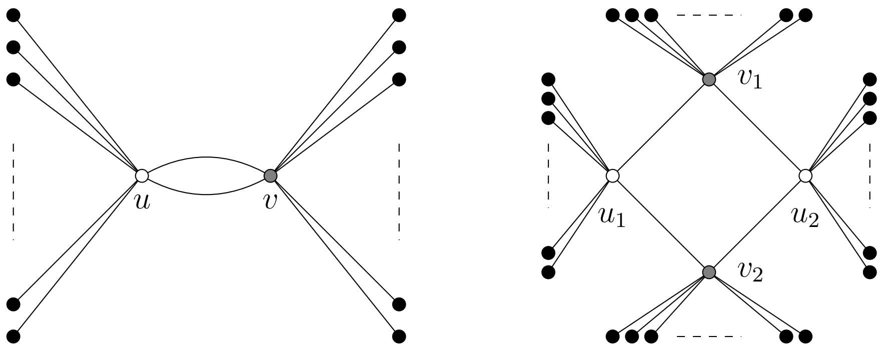

Figure 4: For a positive integer r, $C_2 \circ \overline{K}_{4r^2-1}$ has perfect state transfer between u and v, and $C_4 \circ \overline{K}_{4r^2-1}$ has *real* perfect state transfer between $\{u_1, u_2\}$ and $\{v_1, v_2\}$.

In a graph $G = (V, E)$, for a subset $U \subseteq V$ of vertices, we denote the real uniform superposition of vertices in U as $\mathbf{e}_U = |U|^{-1/2}\sum_{v\in U}\mathbf{e}_v$. We say G has *real* perfect state transfer between two subsets $U_1, U_2 \subseteq V$ if $|\mathbf{e}_{U_1}^T e^{-\mathrm{i}tA(G)}\mathbf{e}_{U_2}| = 1$ (see [21]).

Theorem 4.12. *For each positive integer r, the following hold.*

i) $C_2 \circ \overline{K}_{4r^2-1}$ has perfect state transfer between the vertices of C_2 at time $\pi/2$.

ii) $C_4 \circ \overline{K}_{4r^2-1}$ has real perfect state transfer between its two antipodal pairs of vertices at time $\pi/2$.

Proof. We apply Proposition 4.11 with $s = 0$ and $\ell = r$ for a positive integer r. This shows that $P_4(r) := K_2(2/\sqrt{4r^2-1}) \circ K_1$ has perfect state transfer between the inner two vertices at time $\tau = \frac{\pi}{2}\sqrt{4r^2-1}$. Hence there exists a unimodular complex number γ such that $e^{-i\tau A(P_4(r))}e_{(u,0)} = \gamma e_{(v,0)}$. If follows from

$$A(P_4(r))\left(e^{-i\tau A(P_4(r))}e_{(u,0)}\right) = e^{-i\tau A(P_4(r))}A(P_4(r))e_{(u,0)} = \gamma A(P_4(r))e_{(v,0)} \quad (48)$$

that perfect state transfer also occurs between the antipodal vertices.

By multiplying all weights of this graph by $\sqrt{4r^2-1}$, we obtain a weighted graph $\mathcal{G}_r$ whose adjacency matrix is $A(\mathcal{G}_r) = \sqrt{4r^2-1}A(P_4(r))$; see Figure 5. Note $\mathcal{G}_r$ has perfect state transfer at time $\pi/2$ between the two inner vertices.

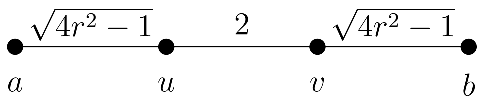

Figure 5: A family of weighted paths $\mathcal{G}_r$, for positive integer r, with perfect state transfer (between u and v, and between a and b).

Moreover, $\mathcal{G}_r$ is the quotient of both $C_2 \circ \overline{K}_{4r^2-1}$ and $C_4 \circ \overline{K}_{4r^2-1}$ under natural equitable partitions. Since perfect state transfer is closed under taking quotient and lifting (see Bachman *et al.* [2], Theorem 1), this completes both claims. □

Consider the family of weighted paths $\mathcal{G}_r$ given in Figure 5. The weighted path $\mathcal{G}_1$ corresponds to the distance quotient of the cube Q_3 (see also Figure 6). It is known that the distance quotients of the cube Q_n provide a family of weighted paths with perfect state transfer. The weighting schemes of these weighted paths are unimodal (single peaked), where the edge weight between the kth and $(k+1)$th vertices is $\sqrt{(k+1)(n-k)}$ for $k = 0, \ldots, n-1$. In contrast, the weighted paths $\mathcal{G}_r$, for $r > 1$, exhibit weighting schemes that are not unimodal.

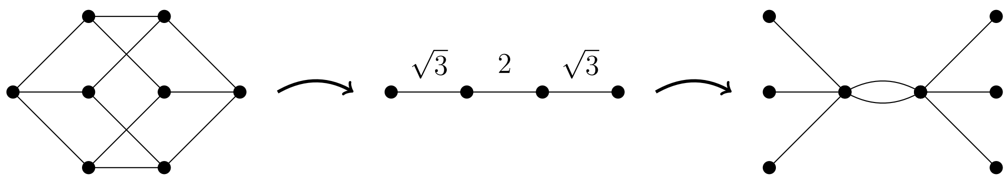

Figure 6: Example: a quotient and a lift of the hypercube Q_3.

5 Pretty Good State Transfer

We consider several families of corona graphs and explore their pretty good state transfer properties. In our analysis, we use the following characterization of pretty good state transfer observed by Banchi *et al.* [3]. We state a version that appeared in Kempton *et al.* [27].

Lemma 5.1. *([27], see Lemma 2.2)*
Let G be a graph with adjacency matrix A and let u, v be two vertices of G. Then, there is pretty good state transfer from u to v in G if and only if the following conditions hold:

1. *For each eigenvalue θ in the support of u and v, we have $E_\theta \mathbf{e}_u = \pm E_\theta \mathbf{e}_v$.*

2. *Let $\{\lambda_i\}$ be the eigenvalues of A for which $E_{\lambda_i}\mathbf{e}_u = E_{\lambda_i}\mathbf{e}_v$. Let $\{\mu_j\}$ be the eigenvalues of A for which $E_{\mu_j}\mathbf{e}_u = -E_{\mu_j}\mathbf{e}_v$. Then, for all integers ℓ_i and m_j, if*

$$\sum_i \ell_i\lambda_i + \sum_j m_j\mu_j = 0, \qquad \sum_i \ell_i + \sum_j m_j = 0 \tag{49}$$

then

$$\sum_j m_j \equiv 0 \pmod 2. \tag{50}$$

5.1 Barbell Graphs

We consider the family of barbell graphs (see Ghosh *et al.* [17]) and show that they exhibit pretty good state transfer.

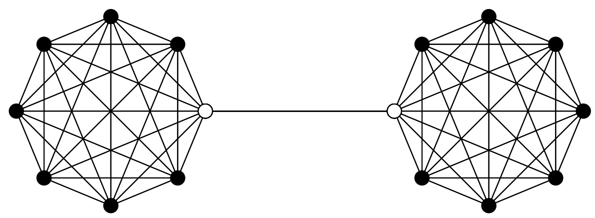

Figure 7: $K_2 \circ K_7$ has pretty good state transfer between vertices marked white.

Lemma 5.2. *For real numbers a, b, let G be a weighted path with adjacency matrix A where*

$$A = \begin{bmatrix} b & 1 & 0 & 0 \\ 1 & 0 & a & 0 \\ 0 & a & 0 & 1 \\ 0 & 0 & 1 & b \end{bmatrix}. \tag{51}$$

If $\{a, \sqrt{(b \pm a)^2 + 4}\}$ is linearly independent over $\mathbb{Q}$, then G has pretty good state transfer between the inner vertices.

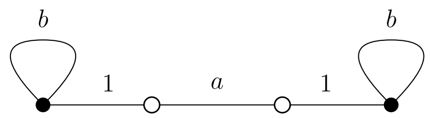

Figure 8: The quotient of $K_2 \circ K_m$ where $a = 1/\sqrt{m}$ and $b = (m-1)/\sqrt{m}$.

Proof. The eigenvalues of A are given by

$$\lambda_\pm = \frac{1}{2}(b + a \pm \sqrt{(b-a)^2 + 4}), \qquad \mu_\pm = \frac{1}{2}(b - a \pm \sqrt{(b+a)^2 + 4}). \tag{52}$$

It can be verified that $E_{\lambda_\pm}\mathbf{e}_u = E_{\lambda_\pm}\mathbf{e}_v$ and $E_{\mu_\pm}\mathbf{e}_u = -E_{\mu_\pm}\mathbf{e}_v$.

Suppose there are integers $\ell_\pm$ and $m_\pm$ so that $\sum_\pm \ell_\pm\lambda_\pm + \sum_\pm m_\pm\mu_\pm = 0$ and $\sum_\pm \ell_\pm + \sum_\pm m_\pm = 0$. So, we have

$$-2a\sum_\pm m_\pm + (\ell_+ - \ell_-)\sqrt{(b-a)^2+4} + (m_+ - m_-)\sqrt{(b+a)^2+4} = 0. \tag{53}$$

Since $\{a, \sqrt{(b\pm a)^2+4}\}$ is linearly independent over $\mathbb{Q}$, we have $\sum_\pm m_\pm = 0$. By Lemma 5.1, G has pretty good state transfer between the inner vertices. □

Theorem 5.3. *$K_2 \circ K_m$ has pretty good state transfer between the vertices of K_2 for all $m \geqslant 1$.*

Proof. We take the quotient of $K_2 \circ K_m$ and apply Lemma 5.2. Here, we have $a = 1/\sqrt{m}$ and $b = (m-1)/\sqrt{m}$. Thus, $\{a, \sqrt{(b\pm a)^2+4}\}$ is given by

$$\{\tfrac{1}{\sqrt{m}}, \tfrac{1}{\sqrt{m}}\sqrt{m^2+4}, \tfrac{1}{\sqrt{m}}\sqrt{m^2+4m}\},$$

which is linearly independent over $\mathbb{Q}$ for $m \geqslant 1$. □

5.2 Hairy Cliques

By Theorem 4.10, we know that there is no perfect state transfer between $(v,1)$ and $(v,2)$ in $G \circ \overline{K}_2$ unless v is isolated in G. We show a stronger result that there is no pretty good state transfer when G is the complete graph.

Let $K_n + I$ denote the graph whose adjacency matrix is the all-one matrix J_n. That is, $K_n + I$ is obtained from the complete graph by adding loops to each vertex. Although the quantum walks on K_n and on $K_n + I$ are equivalent up to phase factors, we show that in contrast to Theorem 4.10 with $K_n \circ \overline{K}_2$, the corona $(K_n + I) \circ \overline{K}_2$ has pretty good state transfer between the vertices of $\overline{K}_2$,

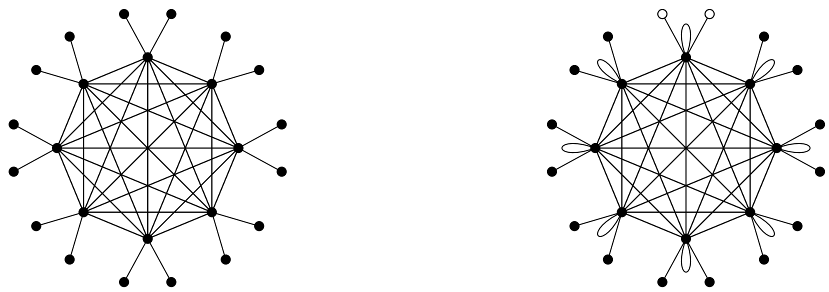

Figure 9: $K_8 \circ \overline{K}_2$ has no pretty good state transfer, while $(K_8 + I) \circ \overline{K}_2$ has pretty good state transfer between every two vertices of degree 1 at distance 2.

We will derive the above results from the following observation.

Fact 1. For an integer $n \geqslant 1$, let G be a weighted graph with adjacency matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & \sqrt{n} & 0 \\ 0 & 0 & \sqrt{n} & k & \sqrt{2} \\ 0 & 0 & 0 & \sqrt{2} & 0 \end{bmatrix}. \tag{54}$$

The eigenvalues of A are given by $\mu_1 = 0$ and by

$$\lambda_{a,b} = \frac{1}{4}\left(k + (-1)^a\sqrt{k^2+4n} + (-1)^b\sqrt{2k^2+4n+32+(-1)^a 2k\sqrt{k^2+4n}}\right), \tag{55}$$

for $a, b = 0, 1$. Moreover, $E_{\mu_1}\mathbf{e}_u = -E_{\mu_1}\mathbf{e}_v$ and that $E_{\lambda_{a,b}}\mathbf{e}_u = E_{\lambda_{a,b}}\mathbf{e}_v$, for all $a, b = 0, 1$.

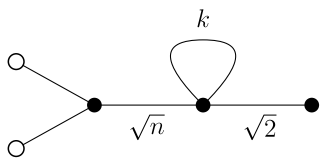

Figure 10: The quotient of $K_n \circ \overline{K}_2$ where $k = n - 1$.

Theorem 5.4. *For $n \geqslant 3$, there is no pretty good state transfer in $K_n \circ \overline{K}_2$.*

Proof. By Theorem 2.4, we may rule out pretty good state transfer from $(v, 0)$ to (v, a) for $a = 1, 2$, and from (v, a) to (w, b) for $a, b = 0, 1, 2$ and $v \neq w$.

Next, we rule out pretty good state transfer between $(v, 1)$ and $(v, 2)$ for each vertex v of K_n. We take the quotient of $K_n \circ \overline{K}_2$ and apply Fact 1 with $k = n - 1$. The eigenvalues are $\mu_1 = 0$ and $\lambda_{0,b} = \frac{1}{2}(n + (-1)^b\sqrt{n^2+8})$, for $b = 0, 1$, and $\lambda_{1,0} = 1$, $\lambda_{1,1} = -2$. Take $m_1 = -3$, $\ell_{0,b} = 0$, for $b = 0, 1$, and $\ell_{1,0} = 2$, $\ell_{1,1} = 1$. Note that

$$\sum_{a,b=0,1} \ell_{a,b}\lambda_{a,b} + m_1\mu_1 = 0, \qquad \sum_{a,b=0,1} \ell_{a,b} + m_1 = 0 \tag{56}$$

but $m_1 = -3$ is odd. By Lemma 5.1, there is no pretty good state transfer. $\square$

Theorem 5.5. *Let $K_n + I$ denote the graph obtained by adding loop to each vertex of the complete graph K_n, where $n \geqslant 1$. Then for $n \geqslant 2$, there is pretty good state transfer in $(K_n + I) \circ \overline{K}_2$ between vertices $(v, 1)$ and $(v, 2)$, for each vertex v of K_n, where the vertices of $\overline{K}_2$ are denoted by 1 and 2.*

Proof. Take the quotient of $(K_n + I) \circ \overline{K}_2$ and apply Fact 1 with $k = n$. The eigenvalues are $\mu_1 = 0$ and

$$\lambda_{a,b} = \frac{1}{4}\left(n + (-1)^a\sqrt{n^2+4n} + (-1)^b\sqrt{2n^2+4n+32+(-1)^a 2n\sqrt{n^2+4n}}\right), \tag{57}$$

for $a, b = 0, 1$. Suppose m_1 and $\ell_{a,b}$ are integers for which $\sum_{a,b=0,1} \ell_{a,b}\lambda_{a,b} = 0$ and $m_1 = -\sum_{a,b=0,1} \ell_{a,b}$. Let $\alpha = \sqrt{n^2+4n}$ and $\beta_\pm = \sqrt{2n^2+4n+32 \pm 2n\sqrt{n^2+4n}}$. We restate the first condition as

$$n \sum_{a,b} \ell_{a,b} + \alpha \sum_{a,b} (-1)^a \ell_{a,b} + \beta_+(\ell_{0,0} - \ell_{0,1}) + \beta_-(\ell_{1,0} - \ell_{1,1}) = 0. \tag{58}$$

Since α and $\beta_\pm$ are irrational, we have $\sum_{a,b} \ell_{a,b} = -m_1 = 0$. By Lemma 5.1, there is pretty good state transfer between u and v. □

We observe that $(K_1 + I) \circ \overline{K}_2$ is periodic but has no pretty good state transfer. The eigenvalues of the corona $(K_1 + I) \circ \overline{K}_2$ are $\lambda_1 = 2$, $\lambda_2 = -1$, and $\mu = 0$, which shows periodicity. However, there is no pretty good state transfer, since we may apply Lemma 5.1. with $\ell_1 = 1$, $\ell_2 = 2$ and $m = -3$.

5.3 Thorny Graphs

The corona product of a graph with K_1 is called a *thorny* graph (see Gutman [24]).

Theorem 5.6. *Let G be a graph and let u, v be two of its vertices. Suppose there is perfect state transfer between u and v at time $t = \pi/g$, for some positive integer g, and that 0 is not in the eigenvalue support of u. Then there is pretty good state transfer between $(u, 0)$ and $(v, 0)$ in $G \circ K_1$.*

Proof. Let S be the eigenvalue support of u in G. By Theorem 2.1, if G has perfect state transfer at time π/g between the vertices u and v, for some integer g, all eigenvalues in S must be integers. For each eigenvalue λ in S, let c_λ be the square-free part of $\lambda^2 + 4$, so that $\Lambda_\lambda = \sqrt{\lambda^2+4} = s_\lambda\sqrt{c_\lambda}$ for some integers s_λ. Since 0 is not in the eigenvalue support of u, then Λ_λ is irrational and $c_\lambda > 1$ for each λ in S. By Lemma 2.6,

$$\{\sqrt{c_\lambda} : \lambda \in \mathtt{supp}_G(u)\} \cup \{1\} \tag{59}$$

is linearly independent over $\mathbb{Q}$. Kronecker's Theorem implies that there exist integers ℓ, q_λ such that

$$\ell\sqrt{c_\lambda} - q_\lambda \;\approx\; -\frac{\sqrt{c_\lambda}}{2g}. \tag{60}$$

Multiplying by $4s_\lambda$ yields that

$$\left(4\ell + \frac{2}{g}\right)\Lambda_\lambda \;\approx\; 4q_\lambda s_\lambda. \tag{61}$$

Therefore, at $t = (4\ell + 2/g)\pi$, we have $\cos(\Lambda_\lambda t/2) \approx 1$ for each λ in S. By Proposition 3.3,

$$\begin{aligned}
\mathbf{e}_{(u,0)}^T e^{-\mathtt{i}tA(G\circ K_1)}\mathbf{e}_{(v,0)} &= \sum_{\lambda\in \mathrm{Sp}(G)} e^{-\mathtt{i}t\lambda/2}\left(\cos(\Lambda_\lambda t/2) - i\frac{\lambda}{\Lambda_\lambda}\sin(\Lambda_\lambda t/2)\right)\mathbf{e}_u^T E_\lambda(G)\mathbf{e}_v \\
&\approx \sum_{\lambda\in \mathrm{Sp}(G)} e^{-\mathtt{i}(2\pi)\ell\lambda} e^{-\mathtt{i}\frac{\pi}{g}\lambda}\mathbf{e}_u^T E_\lambda(G)\mathbf{e}_v \\
&= \mathbf{e}_u^T e^{-\mathtt{i}(\pi/g)A(G)}\mathbf{e}_v,
\end{aligned} \tag{62}$$

because all the eigenvalues λ in the support of u are integers. Since G has perfect state transfer between u and v at time π/g, this completes the proof. $\square$

When zero is in the eigenvalue support of u, we need a slightly stronger condition to get pretty good state transfer in $G \circ K_1$.

Theorem 5.7. *Let G be a graph having zero as an eigenvalue. Suppose that G has perfect state transfer at time $\pi/2$ between vertices u and v. Then there is pretty good state transfer between $(u,0)$ and $(v,0)$ in the corona $G \circ K_1$.*

Proof. Similar to the proof of Theorem 5.6, for each $\lambda \in \mathtt{supp}_G(u)$, we can write $\Lambda_\lambda = s_\lambda\sqrt{c_\lambda}$ where c_λ is the square-free part of λ^2+4 and s_λ is an integer. Note that $c_\lambda = 1$ if and only if $\lambda = 0$. The set

$$\{\sqrt{c_\lambda} : \lambda \in \mathtt{supp}_G(u), \lambda \neq 0\} \cup \{1\} \tag{63}$$

is linearly independent over $\mathbb{Q}$.

For $\lambda \neq 0$, Kronecker's Theorem implies that there exist integers l and q_λ such that

$$\ell\sqrt{c_\lambda} - q_\lambda \;\approx\; -\frac{\sqrt{c_\lambda}}{4} + \frac{1}{2s_\lambda}. \tag{64}$$

At time $t = (4\ell+1)\pi$, we have $\cos(\Lambda_0 t/2) = -1$, and $\cos(\Lambda_\lambda t/2) \approx -1$ for $\lambda \neq 0$. By Proposition 3.3,

$$\begin{aligned}
\mathbf{e}_{(u,0)}^T e^{-\mathtt{i}tA(G\circ K_1)}\mathbf{e}_{(v,0)} &= \sum_{\lambda\in \mathrm{Sp}(G)} e^{-\mathtt{i}t\lambda/2}\left(\cos(\Lambda_\lambda t/2) - i\frac{\lambda}{\Lambda_\lambda}\sin(\Lambda_\lambda t/2)\right)\mathbf{e}_u^T E_\lambda(G)\mathbf{e}_v \\
&\approx -\sum_{\lambda\in \mathrm{Sp}(G)} e^{-\mathtt{i}(2\pi)\ell\lambda} e^{-\mathtt{i}\frac{\pi}{2}\lambda}\mathbf{e}_u^T E_\lambda(G)\mathbf{e}_v \\
&= -\mathbf{e}_u^T e^{-\mathtt{i}(\pi/2)A(G)}\mathbf{e}_v.
\end{aligned} \tag{65}$$

Since G has perfect state transfer between u and v at time $\pi/2$, this completes the proof. $\square$

In contrast to Corollary 4.5, the following shows that certain thorny distance-regular graphs have pretty good state transfer, but not perfect state transfer.

Corollary 5.8. *Let G be a graph from one of the following families:*

- *d-cubes Q_d, for $d \geqslant 2$.*
- *Cocktail party graphs $\overline{nK_2}$ when n is even.*
- *Halved $2d$-cubes $\frac{1}{2}Q_{2d}$, for $d \geqslant 1$.*

Then $G \circ K_1$ has pretty good state transfer between antipodal vertices in G.

For a distance-regular graph G with diameter d, let G_ℓ be a graph obtained from G by connecting two vertices u and v if they are at distance ℓ from each other, where ℓ ranges from 0 to d. It is customary to denote $A_\ell(G)$ as the adjacency matrix of graph G_ℓ. We say G is *antipodal* if G_d is a disjoint union of cliques of the same size; here, these cliques are called the antipodal *classes* or *fibres* of G.

Lemma 5.9 (Coutinho *et al.* [12], Lemma 4.4)**.** *Let G be a distance-regular graph of diameter d with eigenvalues $\lambda_0 > \lambda_1 > \ldots > \lambda_d$. Let the spectral decomposition of the adjacency matrix of G be given by $A(G) = \sum_{j=0}^{d} \lambda_j E_j(G)$. Suppose that G is antipodal with classes of size two. Then*

$$A_d(G)E_j(G) = (-1)^j E_j(G). \tag{66}$$

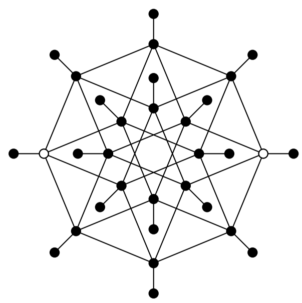

Figure 11: $Q_4 \circ K_1$ has PGST between the white vertices.

The following observation shows that it is not necessary for G to have perfect state transfer in order for $G \circ H$ to have pretty good state transfer.

Theorem 5.10. *Let $n \geqslant 3$ be an odd integer, and let u and v be antipodal vertices of the cocktail party graph $\overline{nK_2}$. Then there is pretty good state transfer between $(u, 0)$ and $(v, 0)$ in $\overline{nK_2} \circ K_1$.*

Proof. The eigenvalues of $\overline{nK_2}$ are $\lambda_0 = 2n-2$, $\lambda_1 = 0$, and $\lambda_2 = -2$. Since the cocktail-party graph is an antipodal distance-regular graph with fibers of size two, from Lemma 5.9, the eigenprojectors satisfy

$$\mathbf{e}_u^T E_j(\overline{nK_2})\mathbf{e}_v = (-1)^j \mathbf{e}_u^T E_j(\overline{nK_2})\mathbf{e}_u \tag{67}$$

for $j = 0, 1, 2$. From Proposition 3.3, letting $\Lambda_j = \sqrt{\lambda_j^2 + 4}$, it suffices to approximate $e^{-\mathtt{i}t\lambda_j/2} \approx 1$ and

$$\cos(\Lambda_j t/2) \approx (-1)^{j+1}. \tag{68}$$

For all integers ℓ, at time $t = 4\pi\ell$, we have $e^{-\mathtt{i}t\lambda_j/2} = 1$ and $\cos(\Lambda_1 t/2) = 1$. We will apply Kronecker's Theorem to Λ_0 and Λ_2. Note that $\Lambda_2 = 2\sqrt{2}$ while

$$\Lambda_0 = \sqrt{4 + 4(n-1)^2} = 2\sqrt{1 + (n-1)^2}, \tag{69}$$

and $1 + (n-1)^2$ is always odd when n is odd. Let c_0 denote the square-free part of Λ_0^2; then, $\Lambda_0 = 2s_0\sqrt{c_0}$ for some odd integer s_0. Choose integers ℓ, q_0, and q_2 such that

$$\ell\sqrt{c_0} - q_0 \approx \frac{1}{4}, \tag{70}$$

$$\ell\sqrt{2} - q_2 \approx \frac{1}{4}. \tag{71}$$

At time $t = 4\pi\ell$, we note $t\Lambda_0/2 \approx 4\pi q_0 s_0 + \pi s_0$ and $t\Lambda_2/2 \approx 4\pi q_2 + \pi$, implying that (68) is satisfied. □

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Corrigendum – Added October 7, 2020

The statement and proof of Theorem 5.4 (along with the caption in Figure 10) should refer to $K_{n+1} \circ \overline{K}_2$ with $n \geqslant 2$ (instead of $K_n \circ \overline{K}_2$). Theorem 5.5 is incorrect as stated which is revised below.

Theorem 5.5. Let $K_n + I$ denote the graph obtained by adding loop to each vertex of the complete graph K_n, where $n \geqslant 1$. Then for even $n \geqslant 2$, there is pretty good state transfer in $(K_{n+1} + I) \circ \overline{K}_2$ between vertices $(v, 1)$ and $(v, 2)$, for each vertex v of K_{n+1}, where the vertices of $\overline{K}_2$ are denoted by 1 and 2.

Proof. Take the quotient of $(K_{n+1} + I) \circ \overline{K}_2$ and denote its weighted adjacency matrix by A (see Figure 12).

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & \sqrt{n} & 0 \\ 0 & 0 & \sqrt{n} & n & \sqrt{2} \\ 0 & 0 & 0 & \sqrt{2} & 0 \end{bmatrix}$$

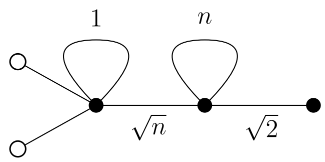

Figure 12: The quotient of $(K_{n+1} + I) \circ \overline{K}_2$ and its weighted adjacency matrix.

The eigenvalues of A are $\mu_1 = 0$, $\lambda_a = (-1)^a\sqrt{2}$, for $a = 0, 1$, and $\lambda_b = \frac{1}{2}(n + 1 + (-1)^b\sqrt{(n+1)^2 + 8})$, for $b = 2, 3$. Moreover, $E_{\mu_1}\mathbf{e}_u = -E_{\mu_1}\mathbf{e}_v$ and that $E_{\lambda_c}\mathbf{e}_u = E_{\lambda_c}\mathbf{e}_v$, for $c = 0, 1, 2, 3$, where u, v are the first two vertices. Suppose m_1 and ℓ_c, $c = 0, 1, 2, 3$, are integers for which $\sum_{a=0,1} \ell_a\lambda_a + \sum_{b=2,3} \ell_b\lambda_b = 0$ and $m_1 = -\sum_{a=0,1} \ell_a - \sum_{b=2,3} \ell_b$. Let $\beta = \frac{1}{2}\sqrt{(n+1)^2 + 8}$. We restate the first condition as

$$\sqrt{2} \sum_{a=0,1} (-1)^a\ell_a + \frac{n+1}{2} \sum_{b=2,3} \ell_b + \beta \sum_{b=2,3} (-1)^b\ell_b = 0. \tag{72}$$

Since $\sqrt{2}$ and β are linearly independent irrationals, we have $\sum_{a=0,1}(-1)^a\ell_a$, $\sum_{b=2,3}(-1)^b\ell_b$, and $\sum_{b=2,3} \ell_b$ are all zero. This shows that $m_1 = -\sum_{a=0,1} \ell_a$ is even. By Lemma 5.1, there is pretty good state transfer between u and v. □

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