Generating Functions for Inverted Semistandard Young Tableaux and Generalized Ballot Numbers

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Abstract

An inverted semistandard Young tableau is a row-standard tableau along with a collection of inversion pairs that quantify how far the tableau is from being column semistandard. Such a tableau with precisely k inversion pairs is said to be a k-inverted semistandard Young tableau. Building upon earlier work by Fresse and the author, this paper develops generating functions for the numbers of k-inverted semistandard Young tableaux of various shapes λ and contents μ. An easily-calculable generating function is given for the number of k-inverted semistandard Young tableaux that "standardize" to a fixed semistandard Young tableau. For m-row shapes λ and standard content μ, the total number of k-inverted standard Young tableaux of shape λ is then enumerated by relating such tableaux to m-dimensional generalizations of Dyck paths and counting the numbers of "returns to ground" in those paths. In the rectangular specialization of $\lambda = n^m$ this yields a generating function that involves m-dimensional analogues of the famed Ballot numbers. Our various results are then used to directly enumerate all k-inverted semistandard Young tableaux with arbitrary content and two-row shape $\lambda = a^1 b^1$, as well as all k-inverted standard Young tableaux with two-column shape $\lambda = 2^n$.

Keywords: Young tableaux, inversions of Young tableaux, lattice paths, ballot numbers

1 Introduction

For any non-increasing strong partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ of the positive integer N, a Young diagram Y of shape λ is a left-justified array of N boxes such that there are precisely λ_i boxes in the ith row of Y. A filling of a Young diagram Y is an assignment of positive integers (possibly repeated) to the boxes of Y such that no integer is skipped.

If a filling uses precisely μ_i copies of i for each integer $1 \leqslant i \leqslant M$, where $M \leqslant N$ and $\mu_1 + \ldots \mu_M = N$, we say that the filling has content $\mu = (\mu_1, \mu_2, \ldots, \mu_M)$. We will oftentimes use the shorthand notation $\mu = 1^{\mu_1} 2^{\mu_2} \ldots M^{\mu_M}$.

This paper will utilize several distinct types of fillings. A filling is said to be row-standard if entries increase from left-to-right across each row, and to be column-standard (resp. column-semistandard) if entries increase (resp. weakly increase) from top-to-bottom down each column. If a filling is both row-standard and column-standard, as well as if $\mu = 1^1 2^1 \ldots N^1$, the resulting array T is said to be a **standard Young tableau**. If a filling is merely row-standard and column-semistandard, the resulting array is said to be a **semistandard Young tableau**. We denote the set of all standard Young tableaux of shape λ by $S(\lambda)$, and the set of all semistandard Young tableaux of shape λ and content μ by $S(\lambda, \mu)$. For a comprehensive introduction to Young tableaux, see Fulton [7].

Our primary focus are inversions of Young tableaux, as first introduced by Fresse [5] to calculate the Betti numbers of Springer fibers in type A.[1] Adopting the terminology of Beagley and the author [2, 3], let τ be a row-standard filling and let i, j be a pair of entries from the same column of τ. Let i_k denote the entry precisely k boxes to the right of i in τ, and let j_k denote the entry precisely k boxes to the right of j in τ. Then (i, j) form an **inversion pair** of τ if $i < j$ and one of the following holds:

1. Either i_1 or j_1 doesn't exist, and i appears below j.

2. $i_1 > j_1$.

3. $i_k = j_k$ for all $1 \leqslant k \leqslant n$, either i_{n+1} or j_{n+1} doesn't exist, and i appears below j.

4. $i_k = j_k$ for all $1 \leqslant k \leqslant n$, and $i_{n+1} > j_{n+1}$.

Taken together, the four conditions above identify an instance where the column containing i, j is not in the correct (non-decreasing) order relative to the first column on its right where an appropriate order may be discerned. As such, inversion pairs quantify how far a row-standard tableau is from also being column semistandard. Notice that the absolute vertical placement of i and j within τ doesn't necessarily determine whether (i, j) constitutes an inversion pair; this is an entirely "local" phenomenon that only concerns the ordering of a column relative to the ordering of more rightward columns. See Figure 1 for examples of row-standard fillings along with their inversion pairs.

4	7	8	9
2	3	6	
1	5		

3	5	6	7
1	5	8	9
2	4	8	

Figure 1: A row-standard filling (left) with inversion pairs $(1, 2)$, $(5, 7)$, $(6, 8)$, and a row-standard filling (right) with inversion pairs $(1, 2)$, $(1, 3)$, $(4, 5)$, $(4, 5)$.

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[1]An entirely distinct notion also referred to as "tableaux inversions" has been presented by Shynar [10].

Observe that if τ possesses repeated entries, then it is possible for a single inversion pair to appear multiple times within τ. The existence of repeated inversion pairs only requires that the entries involved in the repeated inversion pairs (i, j) aren't the same instances of i and j. As such, we will always list the inversion pairs of τ with multiplicity. If it ever becomes necessary to specify that an inversion pair (i, j) involves entries from the kth column of τ, we adopt the notation $(i, j)^k$. Furthermore, if we need to specify which instances of a repeated entry are involved in an inversion pair, we will use alphabetic subscripts i_a, i_b, etc. that have been indexed from top-to-bottom in the kth column of τ.

If the row-standard filling τ contains precisely k inversion pairs, we write $\mathrm{n_{inv}}(\tau) = k$ and say that τ is a **k-inverted semistandard Young tableau**. More precisely, if $\mathrm{n_{inv}}(\tau) = k$ and τ lacks repeated entries we say that τ is a **k-inverted standard Young tableau**. We denote the set of all k-inverted standard Young tableaux of shape λ by $S_k(\lambda)$, and the set of all inverted standard Young tableaux of shape λ by $I(\lambda) = \bigcup_k S_k(\lambda)$. Similarly, we denote the set of all k-inverted semistandard Young tableaux with given λ and μ by $S_k(\lambda, \mu)$, and the set of all inverted semistandard Young tableaux of given λ and μ by $I(\lambda, \mu) = \bigcup_k S_k(\lambda, \mu)$. Observe that the traditional notions of standard and semistandard Young tableaux correspond to $S_0(\lambda) = S(\lambda)$ and $S_0(\lambda, \mu) = S(\lambda, \mu)$, as a row-standard filling is also column-(semi)standard if and only if it has exactly zero inversion pairs.

Fresse [5] argued that the columns of any inverted standard Young tableau τ may be independently reordered to produce a unique column-standard Young tableau that we refer to as the **standardization** $\mathrm{st}(\tau)$ of τ. Fresse's work [5] also allows us to conclude that every inverted standard Young tableau is uniquely determined by its standardization and its collection of inversion pairs. Fixing a standardization $T \in S(\lambda)$, we define $S_k^T(\lambda) = \{\tau \in S_k(\lambda) \mid \mathrm{st}(\tau) = T\}$ and $I^T(\lambda) = \{\tau \in I(\lambda) \mid \mathrm{st}(\tau) = T\}$. The author [3] later argued that unique standardizations $\mathrm{st}(\tau)$ also exist for inverted semistandard Young tableaux. Fixing semistandard $T \in S(\lambda, \mu)$, we adopt analogous notations $S_k^T(\lambda, \mu) = \{\tau \in S_k(\lambda, \mu) \mid \mathrm{st}(\tau) = T\}$ and $I^T(\lambda, \mu) = \{\tau \in I(\lambda, \mu) \mid \mathrm{st}(\tau) = T\}$.

One final piece of terminology that we will repeatedly use is the **height order** on entries in an inverted Young tableau. As originally introduced by the author [3], you may define a complete order $\blacktriangleleft$ on the entries $\{a_i\}$ in each column of an inverted Young tableau τ by working through τ one column at a time, from right to left, according to the following rules:

- If either a_i or a_j lacks an entry directly to its right and a_i appears above a_j, then $a_i \blacktriangleleft a_j$.

- If b_i is directly to the right of a_i, b_j is directly right of a_j, and $b_i < b_j$, then $a_i \blacktriangleleft a_j$.

- If b_i is directly to the right of a_i, b_j is directly right of a_j, and $b_i = b_j$ with $b_i \blacktriangleleft b_j$, then $a_i \blacktriangleleft a_j$.

If $c \in \tau$ is the kth smallest element in its column relative to the height order, we say that c has **height** k and write $\mathrm{ht}(c) = k$. The height order is constructed to tell us how

the entries in a column need to be ordered (relative to the columns on its right) if one wants to completely avoid inversion pairs in that column. This observation allows us to succinctly recast the definition of inversion pair:

Proposition 1.1 (Proposition 5 of [3]). *Let τ be an inverted Young tableau and let i, j be distinct entries from the same column of τ. Then (i, j) is an inversion pair of τ if and only if $i < j$ and $j \blacktriangleleft i$.*

1.1 Outline of Results

This paper is composed of three distinct yet interrelated pieces. In Section 2 we present a series of foundational results that formalize the theory of inverted semistandard Young tableaux beyond what was already presented by Fresse [5] and the author [3]. This includes propositions describing what information is necessary to uniquely identify an inverted semistandard tableau from its standardization (Proposition 2.1), and how any inverted semistandard tableau may be obtained from its standardization via a finite series of "partial row transpositions" (Proposition 2.2). Section 2 culminates with Theorem 2.4, which gives a generating function $\chi^T(q)$ for the numbers $|S_k^T(\lambda, \mu)|$ of k-inverted semistandard Young tableaux with fixed standardization $T \in S(\lambda, \mu)$. The dp and dp* operators below represent positive integers known as "inversions depths" that are easily read from T.

Theorem 1.2 (Theorem 2.4). *Let $T \in S(\lambda, \mu)$ be a semistandard Young tableau. Then:*

$$\chi^T(q) \;=\; \sum_k |S_k^T(\lambda, \mu)| q^k \;=\; \frac{\prod_{i,j}[\mathrm{dp}^*(a_{ij}) + 1]_q}{\prod_{i,j}[\mathrm{dp}^*(a_{ij}) - \mathrm{dp}(a_{ij}) + 1]_q}$$

Section 3 explores the relationship between inverted standard Young tableaux and various properties of higher-dimensional Dyck paths, expanding upon the well-known bijection between standard Young tableau $S(\lambda)$ of shape λ and Dyck paths $\mathcal{D}_\lambda$ of the same shape. In the two-row rectangular case, this allows us to give a non-T-specific generating function $\xi(q)$ for the $|S_k(\lambda)|$ in terms of the celebrated Ballot numbers $B(a, b)$, a result that eventually reappears as Corollary 3.3:

Corollary 1.3 (Corollary 3.3). *Let $\lambda = (n, n)$. Then the $S_k(\lambda)$ have generating function:*

$$\xi(q) \;=\; \sum_k |S_k(\lambda)| q^k \;=\; \sum_i B(n-1, n-i)(1+q)^i \;=\; \sum_i \frac{i}{n}\binom{2n-i-1}{n-i}(1+q)^i$$

That result is generalized to arbitrary m-row tableaux shapes in Theorem 3.7 by relating our generating function $\xi(q)$ to "generalized ballot numbers" $|\mathcal{D}_\lambda(k_1, \ldots, k_{m-1})|$, which represent the number of Dyck paths in $\mathcal{D}_\lambda$ possessing various numbers of "higher-dimensional returns-to-ground":

Theorem 1.4 (Theorem 3.7). *Let $\lambda = (\lambda_1, \ldots, \lambda_m)$. Then the $S_k(\lambda)$ have generating function:*

$$\xi(q) \;=\; \sum_k |S_k(\lambda)| q^k \;=\; \sum_{(i_1, \ldots, i_{m-1})} \left(|\mathcal{D}_\lambda(i_1, \ldots, i_{m-1})| \prod_{j=1}^{m-1} [j+1]_q^{i_j} \right)$$

In Section 4 we utilize the results of earlier sections to directly enumerate k-inverted tableaux in several cases that have not previously been attempted. Theorem 4.2 explicitly enumerates k-inverted semistandard Young tableaux for an arbitrary two-row shape $\lambda = a^1 b^1$, generalizing earlier enumerations [5, 3] of two-row inverted tableaux in the standard specialization. Theorem 4.5 then provides a generating function for the $|S_k(\lambda)|$ in the case of two-column rectangular shapes $\lambda = 2^n$.

As a whole, notice that this paper follows the purely combinatorial approach of Beagley and the author [2, 3], and will only briefly mention several potential geometric implications for Springer varieties (and Spaltenstein varieties) that may follow from the original work of Fresse [5].

2 Generating Functions for Inverted Semistandard Young Tableaux

Before proceeding to our central enumerative results, we pause to lay the necessary combinatorial groundwork. We begin this foundational section with a thorough treatment of "formal" results that were not previously addressed by the author [3]. This formalism accomplishes for inverted semistandard tableaux what Fresse has already established for inverted standard tableaux [5]. More importantly, it provides us with the vocabulary needed to precisely define an explicit generating function $\chi^T(q) = \sum_k |S_k^T(\lambda, \mu)| q^k$ for the number of k-inverted semistandard Young tableaux with a fixed standardization $T \in S(\lambda, \mu)$. This section closes with a series of corollaries demonstrating the usefulness of our generating function.

It should be noted that, although generating functions for the $|S_k^T(\lambda, \mu)|$ have yet to appear anywhere in the literature, the specialization of our $\chi^T(q)$ to the case of standard Young tableaux recovers Fresse's generating function for the $|S_k^T(\lambda)|$ (see Propositions 2.3b and 4.2 of [5]). Yet even in the case of non-repeated entries, our terminology is more direct (avoiding the reference of appropriately defined sub-tableaux) and thus allows for more quickly calculable generating functions.

As another brief aside, for the rest of this section we will often need to specify which instance of a repeated entry we are dealing with in a tableau. We reserve the term "entry" if we want to refer to a (potentially repeated) integer in a specific cell, and will use the notation a_{ij} if we want to emphasize that the entry is located in the ith row and j^{th} column of our inverted tableau. If a_{ij} is an instance of the integer k, we will call it an "entry of value k" and write $a_{ij} = k$.

Our first foundational result follows most directly from the work of Fresse [5]. It should be noted that the subsequent proposition may be independently derived via a careful

reworking of the "admissible transposition" procedure that appears in Proposition 2.2.

Proposition 2.1. *Take any inverted semistandard Young tableau $\tau \in I(\lambda, \mu)$. τ is uniquely identified by its standardization $st(\tau) = T$ alongside a collection of inversion pairs (a_{ij}, b_{ij}) that specify the entries of T involved in each inversion.*

Proof. Take any inverted tableau $\tau \in I^T(\lambda, \mu)$ with standardization $st(\tau) = T$, and assume $\sum_k \mu_k = N$. For each value α, we simultaneously place a complete order $\prec$ on the μ_α copies of α in both T and τ. Let $\prec$ be the unique complete order such that $\alpha_k \prec \alpha_l$ if α_k appears in a more leftward column than α_l, and $\alpha_k \prec \alpha_l$ if α_k, α_l appear in the same column and $\alpha_k \blacktriangleleft \alpha_l$. Label the copies of α in both T and τ according to this complete order, so that $\alpha_1 \prec \ldots \prec \alpha_{\mu_\alpha}$. Then re-index all N entries in both T and τ according to the map $\phi(\alpha_k) = \mu_1 + \ldots + \mu_{(\alpha-1)} + k$.

This re-indexing replaces T with a standard Young tableau $\widetilde{T} \in S(\lambda)$ and replaces τ with an inverted standard Young tableau $\widetilde{\tau} \in I^T(\lambda)$. By construction, $\tau \mapsto \widetilde{\tau}$ represents a bijection between $I^T(\lambda, \mu)$ and all inverted tableau in $I^T(\lambda)$ that do not possess an inversion pair of the form (a, b) with $\phi^{-1}(a) = \phi^{-1}(b)$. In particular, $n_{\text{inv}}(\tau) = n_{\text{inv}}(\widetilde{\tau})$ and $(a, b)^j$ is an inversion pair of τ if and only if $(\phi(a), \phi(b))^j$ is an inversion pair of $\widetilde{\tau}$. Citing Proposition 2.3a of Fresse [5], we know that $\widetilde{\tau}$ is uniquely identified from $\widetilde{T}$ via its collection of inversion pairs. If we specify which entries with a repeated value are involved in each inversion pair, the aforementioned bijection allows us to conclude that τ is uniquely identified from T via a collection of entry-specific inversion pairs. $\square$

Recall that, with standard tableaux, uniquely identifying an inverted tableau required only a standardization and a list of the values involved in each inversion pairs. In the semistandard case, as repeated inversion pairs (a, b) are possible it was obviously necessary to be column-specific and to count inversion pairs with multiplicity. Less obvious is the requirement that we must identify the exact location of entries involved in repeated inversion pairs. See Figure 2 for an example showing that a specific standardization as well as a collection of inversion pairs, even with column-specific information and accounting for multiplicity, is not enough to specify a unique inverted tableau.

2		1
1		2
1		2
2		1

Figure 2: A pair of distinct tableaux in the same set $I(\lambda, \mu)$ with identical standardizations T and identical inversion pairs $(1, 2)^1, (1, 2)^1$. The two tableaux are uniquely identified via the entry-specific inversion pairs sets $(1_a, 2_a)^1, (1_b, 2_a)^1$ and $(1_b, 2_a)^1, (1_b, 2_b)^1$

We now define an operation on inverted tableaux whose repeated application may be used to obtain any inverted tableau from its standardization. So take $\tau \in I^T(\lambda, \mu)$, and let $a_{i_1 j}, a_{i_2 j}$ be entries from the jth column of τ with $a_{i_1 j} < a_{i_2 j}$ and $|\text{ht}(a_{i_1 j}) - \text{ht}(a_{i_2 j})| = 1$.

A **partial row transposition** at $(a_{i_1 j}, a_{i_2 j})$, denoted $a_{i_1 j} \leftrightarrow a_{i_2 j}$, is a transposition of the i_1^{th} and i_2^{th} rows of τ from the jth column leftward. Now let b_1 denote the entry directly to the right of a_1 and let b_2 denote the entry directly to the right of a_2, if those entries in fact exist. If $a_{i_1 j}$ and $a_{i_2 j}$ are both smaller than b_1 and b_2 (if they exist), the partial row transposition $a_{i_1 j} \leftrightarrow a_{i_2 j}$ preserves row-standardness and we refer to the operation as an **admissible (partial row) transposition**.

If $\tau \in I^T(\lambda, \mu)$, the fact that an admissible transposition $a_{i_1 j} \leftrightarrow a_{i_2 j}$ doesn't permute entries between columns means that the resulting tableau τ' is also an element of $I^T(\lambda, \mu)$. τ' is related to τ in that it flips $\mathrm{ht}(a_{i_1 j})$ and $\mathrm{ht}(a_{i_2 j})$ while fixing the height of every other element. As $a_{i_1 j}$ and $a_{i_2 j}$ are consecutive elements in the height order on the jth column of τ, it follows that the admissible transposition $a_{i_1 j} \leftrightarrow a_{i_2 j}$ either adds or removes the inversion pair $(a_{i_1 j}, a_{i_2 j})$ while leaving other inversion pairs unchanged. These observations allow us to assert the following:

Proposition 2.2. *Every inverted tableau $\tau \in I^T(\lambda, \mu)$ may be obtained from its standardization $\mathrm{st}(\tau) = T$ via a finite sequence of admissible transpositions. Furthermore, the minimum number of admissible transpositions needed to obtain τ from T is $\mathrm{n}_{\mathrm{inv}}(\tau)$.*

Proof. Taking $\tau \in I^T(\lambda, \mu)$, we utilize Proposition 2.1 to uniquely identify τ via its collection of entry-specific inversion pairs $S = \{(a_\alpha, b_\alpha)^{j_\alpha}\}$. As each admissible transposition adds precisely one inversion pair, obtaining τ from T with fewer than $\mathrm{n}_{\mathrm{inv}}(\tau)$ transpositions is clearly impossible. To obtain τ with precisely $\mathrm{n}_{\mathrm{inv}}(\tau)$ transpositions, we add inversion pairs one column at a time, from right to left. Within each column, we work through entries from top-to-bottom via their placement in T, assuming that the relative ordering of repeated instances with a fixed value in the jth column is unchanged as we pass from T through various elements of $I^T(\lambda, \mu)$.

So assume that we have followed this procedure up to the jth column of T, whose entries we denote $a_{1j} \leqslant a_{2j} \leqslant \ldots a_{hj}$ from top-to-bottom, and that we are ready to add inversions whose larger element is the entry a_{ij}. Let $\widetilde{\tau} \in I^T(\lambda, \mu)$ denote the intermediate tableau that immediately precedes the addition of these inversion pairs. Notice that, as a_{ij} has yet to be the site of a partial row transposition, a_{ij} still possesses its original height of $\mathrm{ht}(a_{ij}) = i$ in $\widetilde{\tau}$. Furthermore, the entries of lower height in the jth column of $\widetilde{\tau}$ are $a_{1j}, \ldots, a_{(i-1)j}$, in some (possibly permuted) order. Reverse index those entries according to their height order in $\widetilde{\tau}$ as $c_{i-1} \blacktriangleleft \ldots \blacktriangleleft c_1$. If a_{ij} is the larger entry in precisely m inversion pairs in τ, notice that those inversion pairs must be $(c_1, a_{ij})^j, \ldots, (c_m, a_{ij})^j$. Otherwise, τ would have possessed an additional inversion pair of the form (c_β, c_γ) that we had failed to account for at an earlier step. Then perform the m admissible transpositions $c_1 \leftrightarrow a_{ij}, \ldots, c_m \leftrightarrow a_{ij}$ in the stated order. Continuing this procedure through all entries a_{ij} of T yields the desired inverted tableau τ. $\qquad \square$

It should be noted that the proof of Proposition 2.2 could have been accomplished much more directly via an induction of inversion number. Our method of proof was chosen because it provides an explicit algorithm for obtaining arbitrary $\tau \in I^T(\lambda, \mu)$ from its standardization. See Figure 3 for an example of our specific procedure. Also observe

that the algorithm from our proof provides only one minimal sequence of admissible transpositions yielding τ. In general, there are many such sequences, even if one performs only the minimum possible number of $n_{\text{inv}}(\tau)$ transpositions. The author suspects that there exists a rich theory of composition patterns for admissible row transpositions of row-standard tableaux. As ordinary permutations $\sigma \in S_n$ correspond to inverted tableaux of shape $\lambda = 1^n$, this topic would directly generalize the existing theory of reduced word decompositions for permutations.
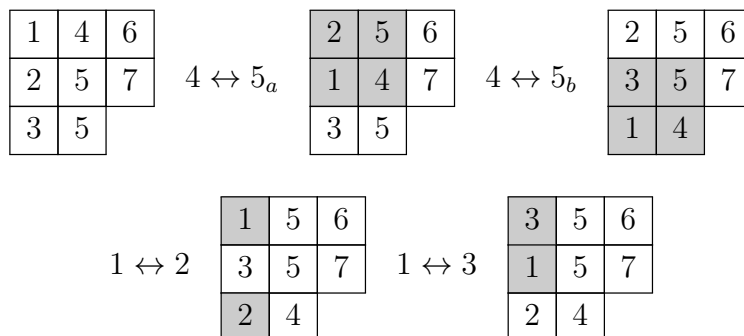

Figure 3: Obtaining τ with inversion pairs $(1,2)^1, (1,3)^1, (4,5_a)^2, (4,5_b)^2$ from $\text{st}(\tau) = T$ via a sequence of admissible transpositions. 5_a and 5_b denote the two entries of value 5 in second column, with $5_a \blacktriangleleft 5_b$.

We are now set to develop the language needed for our generating function $\chi^T(q) = \sum_k |S_k^T(\lambda, \mu)| q^k$. So take a semistandard Young tableau $T \in S(\lambda, \mu)$, and let a_{ij} be an entry of value k from the jth column of T. We define the **(standard) inversion depth** of a_{ij}, denoted $\text{dp}(a_{ij})$, to equal the number of entries b in the jth column of T such that $b < k$ minus the number of entries c in the $(j+1)^{st}$ column of T such that $c < k$ (and this second number is taken to be zero if T has no $(j+1)^{st}$ column). We similarly define the **modified inversion depth** of a_{ij}, denoted $\text{dp}^*(a_{ij})$ to equal the total number of entries above a_{ij} in the jth column of T minus the number of entries c in the $(j+1)^{st}$ column of T such that $c < k$ (once again taken to be zero if T has no $(j+1)^{st}$ column).

For an example of a semistandard Young tableau alongside the inversion depths of each of its entries, standard and modified, see Figure 4. Notice that, if a_{ij} is the only entry of value k in the jth column of of T, then $\text{dp}(a_{ij}) = \text{dp}^*(a_{ij})$. In particular, if T is a standard Young tableau, then $\text{dp}(a_{ij}) = \text{dp}^*(a_{ij})$ for all entries a_{ij}. In the case of T a standard Young tableau, also notice that our $\text{dp}(a_{ij})$ correspond with the p_k defined by Fresse [5] as the number of appropriately long rows in the sub-tableau $T[1, \ldots, k]$.

Standard inversion depth $\text{dp}(a_{ij})$ has been defined to equal the number of distinct admissible partial row-transpositions possible in the jth column of T where a_{ij} is the larger entry being transposed. This is equivalent to saying that $\tau \in I^T(\lambda, \mu)$ may possess at most $\text{dp}(a_{ij})$ distinct inversion pairs of the form $(a_{i'j}, a_{ij})^j$.

Despite placing a quick upper bound on $n_{\text{inv}}(\tau)$ for $\tau \in I^T(\lambda, \mu)$, knowing the standard inversion depth of each element in T will not be sufficient if one wishes to determine the generating function $\chi^T(q)$. This is because standard inversion depth doesn't easily account

1	2	6	10
2	5	8	10
3	5	9	
4	7		

$\Longrightarrow$

(0)	(0)	(0)	(0)
(0)	(1)	(1)	(0)
(1)	(1)	(2)	
(2)	(2)		

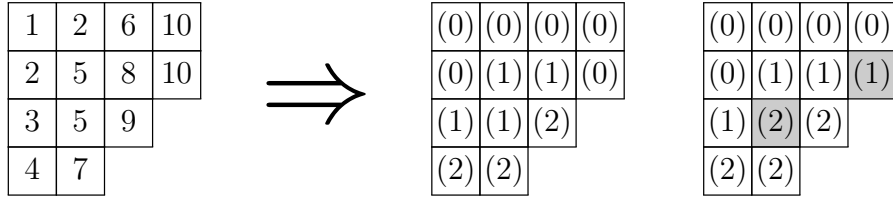| (0)	(0)	(0)	(0)
(0)	(1)	(1)	(1)
(1)	(2)	(2)	
(2)	(2)		

Figure 4: A semistandard Young tableau with the inversion depth (center) and modified inversion depth (right) of each of its entries. Entries where $\mathrm{dp}(a_{ij}) \neq \mathrm{dp}^*(a_{ij})$ have been highlighted.

for inversion pairs when the jth column of T contains multiple entries of a fixed value k. Luckily, modified inversion depth is perfectly suited to this task and immediately proves useful in the development of generating functions $\chi^T(q)$ for one-column semistandard Young tableaux. As presented below, Lemma 2.3 actually represents little more than a rewording of Theorem 13 from [3], which gave a generating function for the $|S_k(\lambda, \mu)|$ when $\lambda = 1^n$.

In what follows we use the standard notation for the q-number $[p]_q = 1 + q + \ldots + q^{p-1}$, the q-factorial $[p]_q! = [1]_q[2]_q \ldots [p]_q$, and the q-binomial coefficients $\binom{a}{b}_q = \frac{[a]_q!}{[b]_q![a-b]_q!}$.

Proposition 2.3. *Let $T \in S(\lambda, \mu)$ be a one-column semistandard Young tableau of shape $\lambda = 1^n$. Then:*

$$\chi^T(q) = \sum_k |S_k^T(\lambda, \mu)|q^k = \frac{\prod_i [\mathrm{dp}^*(a_{i1}) + 1]_q}{\prod_i [\mu_i]_q!}$$

Proof. Notice that if $\lambda = 1^n$ then $S(\lambda, \mu)$ consists of a single column-semistandard tableau, which we label T. Theorem 13 of [3] showed that $\sum_k |S_k^T(\lambda, \mu)|q^k = \sum_k |S_k(\lambda, \mu)|q^k = \frac{[n]_q!}{\prod_i [\mu_i]_q!}$. For a one-column tableau, the lack of rightward entries ensures $\mathrm{dp}^*(a_{i1}) = i - 1$ for all a_{i1}. Thus $[n]_q! = \prod_i [\mathrm{dp}^*(a_{i1}) + 1]_q$ and the result directly follows. $\square$

Before generalizing Lemma 2.3 to the multi-column case we pause to observe that, if a_{ij} is the mth occurrence of value k in the jth column of T, then $\mathrm{dp}(a_{ij}) + m - 1 = \mathrm{dp}^*(a_{ij})$. It follows that $\mathrm{dp}^*(a_{ij}) - \mathrm{dp}(a_{ij})$ measures the number of instances of k that lie above a_{ij} in the jth column of T.

Theorem 2.4. *Let $T \in S(\lambda, \mu)$ be a semistandard Young tableaux. Then:*

$$\chi^T(q) = \sum_k |S_k^T(\lambda, \mu)|q^k = \frac{\prod_{i,j} [\mathrm{dp}^*(a_{ij}) + 1]_q}{\prod_{i,j} [\mathrm{dp}^*(a_{ij}) - \mathrm{dp}(a_{ij}) + 1]_q}$$

Proof. Proposition 2.1 has already shown that every inverted semistandard Young tableau is uniquely determined by its standardization and a collection of entry-specific inversion pairs. Fixing $T \in S(\lambda, \mu)$, we merely need to determine which collections of inversion pairs are valid in the sense that they actually describe a tableau $\tau \in I^T(\lambda, \mu)$ that is row-standard.

When reordering the columns of T to produce an arbitrary inverted tableau $\tau \in I^T(\lambda, \mu)$, we proceed through the columns of T from right to left. Notice that a particular ordering of entries in a leftward column fixes inversion pairs in more rightward columns and has no effect upon whether those rightward columns are row-standard (when taken on their own). These observation allow us to independently determine valid collection of inversion pairs for $I^T(\lambda, \mu)$ one column at a time, given that we work through the columns of T from right to left.

So assume that we have reached the jth column of T, and that the jth column contains precisely c_k entries of value k. Modifying the technique used in Theorem 13 of [3], we consider the number of different ways to "build up" a valid height order on the jth column by recursively inserting all c_k copies of k into the height order of an "intermediate" column with content $1^{c_1} 2^{c_2} \ldots (k-1)^{c_{k-1}}$. This yields a sequence $\{\rho_1, \rho_2, \ldots, \rho_m\}$ of partial columns such that ρ_m describes the height order of the entire j^{th} column of a row-standard tableau.

Notice that distinct insertions at any step in this procedure result in distinct height orders on the jth column. Also notice that a valid placement of the c_k copies of k is not subsequently made invalid upon the insertion of larger entries, as the placement of later entries may only increase the height order of a specific entry. Lastly, note that the number of inversion pairs in the jth column whose larger element if k is completely determined by the step in our procedure where we insert the c_k copies of k into ρ_{k-1} to produce ρ_k.

So assume that we have a valid intermediate version ρ_{k-1} of the jth column and that we are prepared to insert the c_k copies of k in a way that preserves row-standardness. By the definition of standard inversion depth, each instance a_{ij} of k may be involved in up to (a fixed number) of $\mathrm{dp}(a_{ij}) = \mathrm{dp}(k)$ inversion pairs where it is the larger element. The number of such inversion pairs involving a specific instance a_{ij} is determined by the number of entries in ρ_{k-1} that it is moved ahead of in the height order. If we want the resulting column ρ_k to possess precisely l inversion pairs whose larger entry is some instance k, it follows that such arrangements are in bijection with partitions of l into at most c_k parts where each part has size at most $\mathrm{dp}(k)$.

Recall that the coefficient of q^l in $\binom{\alpha+\beta}{\alpha}_q$ equals the number of partitions of l into at most α parts, each of which has size at most β. If $k_1, \ldots, k_{c_k}$ denote the c_k instances of k in the jth column (read from top to bottom), this means that the number of different ways to produce a row-standard intermediate column ρ_k with various numbers of inversions whose larger entry is k has generating function:

$$f_{j,k}(q) = \binom{\mathrm{dp}(k)+c_k}{c_k}_q = \frac{[\mathrm{dp}(k)+1]_q \ldots [\mathrm{dp}(k)+c_k]_q}{[c_k]_q!} = \frac{\prod_{i=1}^{c_k} [\mathrm{dp}^*(k_i)+1]_q}{\prod_{i=1}^{c_k} [\mathrm{dp}^*(k_i)-\mathrm{dp}(k_i)+1]_q}$$

The last equality above uses the aforementioned fact that the mth instance k_m of value k in the jth column satisfies $\mathrm{dp}(k_m)+m-1 = \mathrm{dp}^*(k_m)$. Ranging over all values in the jth column of T, and then over all columns in T, gives the formula from the theorem. $\square$

Example 2.5. *If T is the tableau of Figure 4, Theorem 2.4 gives generating function:*

$$\chi^T(q) = \sum_k |S_k^T(\lambda, \mu)| q^k = \frac{(1+q)^4(1+q+q^2)^4}{(1+q)^2}$$

$$= 1 + 6q + 19q^2 + 40q^3 + 61q^4 + 70q^5 + 61q^6 + 40q^7 + 19q^8 + 6q^9 + q^{10}$$

We close this section by drawing a number of quick corollaries from Theorem 2.4. First notice that, if T lacks repeated entries, then $\mathrm{dp}^*(a_{ij}) = \mathrm{dp}(a_{ij})$ for all a_{ij} and the expression of Theorem 2.4 may be reduced to a generating function that is equivalent to the one introduced by Fresse [5]:

Corollary 2.6. *Let $T \in S(\lambda)$ be a standard Young tableaux. Then:*

$$\chi^T(q) = \sum_k |S_k^T(\lambda)|q^k = \prod_{i,j}[\mathrm{dp}(a_{ij}) + 1]_q$$

Now notice that our generating function $\chi^T(q)$ is always monic, meaning that there is precisely one "maximally inverted" tableau $\tau \in I^T(\lambda, \mu)$ for each choice of $T \in S(\lambda, \mu)$. This maximally inverted tableau has degree $m_T = \sum_{i,j} \mathrm{dp}^*(a_{ij}) - \sum_{i,j}(\mathrm{dp}^*(a_{ij}) - \mathrm{dp}(a_{ij})) = \sum_{i,j} \mathrm{dp}(a_{ij})$.

Corollary 2.7. *Let $T \in S(\lambda, \mu)$ be any semistandard Young tableau, and define $m_T = \sum_{i,j} \mathrm{dp}(a_{ij})$ as above. Then $|S_k^T(\lambda, \mu)| = 0$ for all $k > m_T$ and $|S_{m_T}^T(\lambda, \mu)| = 1$.*

Compare Corollary 2.7 to Theorem 7 of [3], in which the author determined a maximal inversion number $M_{\lambda,\mu}$ when ranging over all standardizations $T \in S(\lambda, \mu)$ and showed that precisely one $\tau_{max} \in I(\lambda, \mu)$ actually realized that number of inversions.[2] Applying Corollary 2.7 to the language of Theorem 7 from [3] allows us to conclude that the standardization $T^* = \mathrm{st}(\tau_{max})$ is the unique element of $S(\lambda, \mu)$ whose generating function $\chi^T(q)$ obtains the maximal possible degree of $M_{\lambda,\mu}$.

In order to draw our final corollaries, notice that $\chi^T(q)$ is always a product of palindromic unimodal polynomials of the form $\binom{\alpha}{\beta}_q$. As the product of two palindromic unimodal polynomial is itself palindromic and unimodal (see Proposition 1 of Stanley [13]), we have both of the following:

Corollary 2.8. *Take any $T \in S(\lambda, \mu)$. Then $\chi^T(q)$ is a palindromic polynomial. In particular, if $m_T = \sum_{i,j} \mathrm{dp}(a_{ij})$ then $|S_k^T(\lambda, \mu)| = |S_{m_T-k}^T(\lambda, \mu)|$ for all k.*

Corollary 2.9. *Let $T \in S(\lambda, \mu)$ be any semistandard Young tableau. Then $\chi^T(q)$ is unimodal.*

Pause to observe that the notion of unimodality addressed in Corollary 2.9 is distinct from the unimodality proven by Fresse, Mansour and Melnikov [6]. Corollary 2.9 proves the unimodality of $\chi^T(q) = \sum_k |S_k^T(\lambda, \mu)|q^k$ for fixed standardization T yet for arbitrary λ, μ. Fresse, Mansour and Melnikov restrict themselves to standard tableaux and prove the unimodality of $\xi(q) = \sum_T \chi^T(q) = \sum_k |S_k(\lambda)|$ for specific (relatively) easy choices of λ, such as two-row and "hook-shaped" cases.

---

[2]In [3], it is shown that $M_{\lambda,\mu} = \sum_j T_{(h_j-1)} - \sum_i T_{(\mu_i-1)}$, where T_k is the triangle number and h_j is the height of the jth column in any tableau of shape λ. The unique element of $I(\lambda, \mu)$ obtaining $M_{\lambda,\mu}$ inversions was labelled τ_{max}, and its standardization was denoted $\mathrm{st}(\tau_{max}) = T^*$.

Although the unimodality of $\chi^T(q)$ is nearly immediate, log-convexity of $\chi^T(q)$ does not necessarily follow because the q-binomial coefficients that constitute $\chi^T(q)$ need not be log-concave. For example, take the sole semistandard tableau T of $S(\lambda, \mu)$ for $\lambda = 1^4$ and $\mu = 1^2 2^2$ (i.e.- the standardization of both tableaux from Figure 2). This T has $\chi^T(q) = \binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4$, which is palindromic and unimodal but not log-concave.

If however we restrict our attention to standard Young tableaux, from Corollary 2.6 we see that $\chi^T(q)$ is a product of log-concave polynomials of the form $[\alpha]_q$ and hence is itself log-concave:

Corollary 2.10. *Let $T \in S(\lambda)$ be any standard Young tableau. Then $\chi^T(q)$ is log-concave.*

3 Inverted Young Tableaux & Generalized Ballot Numbers

In Section 2 we fixed a semistandard Young tableau $T \in S(\lambda, \mu)$ and developed a generating function for the numbers of k-inverted tableaux $|S_k^T(\lambda, \mu)|$ whose standardization was T. It is natural to ask whether our results for the $|S_k^T(\lambda, \mu)|$ aid in the enumeration of the entire sets $S_k(\lambda, \mu)$ or in the development of a non-T-specific generating function $\xi(q) = \sum_T \chi^T(q) = \sum_k |S_k(\lambda, \mu)| q^k$? As noted by Beagley and the author [2, 3], this widening of scope is extremely difficult because it requires specific knowledge of every $T \in S(\lambda, \mu)$. In particular, a direct generalization would require knowledge of how many $T \in S(\lambda, \mu)$ possess a fixed generating function $\chi^T(q)$.

It is because of these difficulties that previous attempts at enumerating the $S_k(\lambda, \mu)$ have disregarded the $S_k^T(\lambda, \mu)$ and utilized more direct techniques. Even in the standard case, such techniques have only been previously applied to calculate $S_k(\lambda)$ for specific easy choices of k and λ. See Fresse, Mansour and Melnikov [6] for calculations of $|S_k(\lambda)|$ when λ is a two-row or "hook" shape, as well as Beagley and the author [2] for an independent verification of the two-row case along with the $k = 1$ case for an arbitrary shape λ.

The situation for semistandard tableaux is even more daunting, as there does not even exist a generalized "hook-length formula" to enumerate semistandard Young tableaux $|S(\lambda, \mu)|$ with fixed shape and content. The author [3] was still able to determine $|S_k(\lambda, \mu)|$ for arbitrary μ when λ was a two-row or one-column shape, conveniently corresponding to the cases where calculating $|S(\lambda, \mu)|$ is relatively straightforward. For an arbitrary shape λ, the author [3] also placed $S_1(\lambda, \mu)$ in bijection with $\bigcup_{\lambda_i} S_0(\lambda_i, \mu)$ for a certain finite collection of related shapes λ_i.

For the remainder of this paper, we present two new approaches for enumerating the $S_k(\lambda, \mu)$. In this section, we restrict ourselves to the standard case and utilize the bijection between m-row standard Young tableaux and m-dimensional Dyck paths to explicitly determine how many $T \in S(\lambda)$ have a fixed generating function $\chi^T(q)$. This is done first in the familiar two-dimensional case, which motivates the general m-row case.

In all that follows, let $S(\lambda, \mu)|_\chi \subset S(\lambda, \mu)$ denote the set of semistandard Young tableaux that have a fixed generating function $\chi^T(q)$, a notation that we predictably adapt to the standard case as $S(\lambda)|_\chi$. If λ and μ are understood, we significantly shorten notation for the cardinalities of these sets to $\phi(\chi) = |S(\lambda, \mu)|_\chi|$. Clearly $\xi(q) = \sum_\chi \phi(\chi) \chi^T(q)$.

3.1 2-Dimensional Dyck Paths and Two-Row Inverted Young Tableaux

We begin by recapping basic results about traditional (two-dimensional) Dyck paths and how they relate to standard Young tableaux. Let $\lambda = (a, b)$, where a, b are positive integers. A **Dyck path of shape** λ is an integer lattice path from $(0,0)$ to (a, b) utilizing only "East" $E = (1, 0)$ and "North" $N = (0, 1)$ steps such that every point (x, y) along the path satisfies $y \leqslant x$. We denote the set of all Dyck paths of shape λ by $\mathcal{D}_\lambda = \mathcal{D}_{(a,b)}$. A particular Dyck path $P \in \mathcal{D}_\lambda$ will often be specified by $P = \{v_0, \dots, v_{a+b}\}$, where the v_i are the integer lattice points along the path. Thus $v_0 = (0, 0)$, $v_{a+b} = (a, b)$, and $v_i - v_{i-1} = (1, 0)$ or $(0, 1)$ for all $1 \leqslant i \leqslant a + b$.

Obviously $\mathcal{D}_{(a,b)} = \emptyset$ if $a > b$. If $a = b = n$, it is well-known that the cardinality of $\mathcal{D}_{(n,n)} = \mathcal{D}_n$ equals the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. More generally, there is a well-studied bijection between $\mathcal{D}_\lambda$ and standard Young tableaux of shape λ. See Figure 5 for an example of this straightforward bijection, in which E steps of the Dyck path correspond to values in the first row of the associated tableau and N steps of the Dyck path correspond to values in the second row of the tableau.
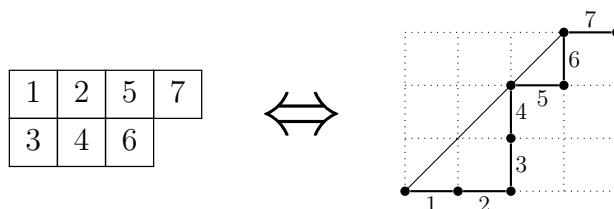

Figure 5: A standard Young tableau of shape $\lambda = (4, 3)$ and the corresponding Dyck path in $\mathcal{D}_\lambda$.

There are many important statistics on Dyck paths. The statistic that will prove useful here is a Dyck path's number of "returns to ground". Formally, a path $P = \{v_0, \dots, v_{a+b}\}$ in $\mathcal{D}_{(a,b)}$ has a **return to ground** at v_i if $v_i = (x, x)$ for some positive integer x. Observe that $v_0 = (0, 0)$ does not qualify as a return to ground. If $P \in \mathcal{D}_\lambda$ has precisely k returns to returns to ground, we write $\mathrm{ret}(P) = k$. We also establish the notation $\mathcal{D}_\lambda(k) = \{P \in \mathcal{D}_\lambda \mid \mathrm{ret}(P) = k\}$.

When $\lambda = (n, n)$, the sets $\mathcal{D}_n(k)$ are related to the famed ballot numbers $B(\alpha, \beta) = \frac{\alpha - \beta + 1}{\alpha + 1}\binom{\alpha + \beta}{\alpha}$ as $|\mathcal{D}_n(k)| = B(n-1, n-k) = \frac{k}{n}\binom{2n-k-1}{n-1}$. These ballot numbers are frequently presented as entries in the so-called Catalan triangle of Figure 6, whose rows sum to the Catalan numbers as $\sum_\beta B(\alpha, \beta) = C_\alpha$. This means that our $|\mathcal{D}_n(k)| = B(n-1, n-k)$ may be read by proceeding from right-to-left along the appropriate row of Catalan's triangle. See sequence A009766 on OEIS [11] for a thorough overview of Catalan triangle's, as well as Forder [4] or Barcucci and Verri [1] for earlier investigations of the Ballot numbers.

Yet how does all of this relate to inverted standard Young tableaux? In the example of Figure 5, notice that the associated tableau T has $\mathrm{dp}(4) = \mathrm{dp}(6) = 1$ and $\mathrm{dp}(i) = 0$ for all other entries $i \in T$. This suggests that a return to ground at v_i in $P \in \mathcal{D}_\lambda$ is related to the inversion depth of i in the associated standard Young tableau $T \in S(\lambda)$. This proves to be a general phenomenon for any standard Young tableau of any two-row shape λ:

$$
\begin{array}{cccccc}
1 & & & & & \\
1 & 1 & & & & \\
1 & 2 & 2 & & & \\
1 & 3 & 5 & 5 & & \\
1 & 4 & 9 & 14 & 14 & \\
1 & 5 & 14 & 28 & 42 & 42
\end{array}
$$

Figure 6: Rows 0 through 5 of the Catalan triangle, whose (α, β) entry is the Ballot number $B(\alpha, \beta) = \frac{\alpha - \beta + 1}{\alpha + 1} \binom{\alpha + \beta}{\alpha}$. Note that the leftmost column of the triangle corresponds to $\beta = 0$.

Proposition 3.1. *Take $T \in S(\lambda)$ for some two-row shape $\lambda = (a, b)$, and let $P = \{v_0, \ldots, v_{a+b}\}$ be the associated Dyck path in $\mathcal{D}_\lambda$. Then an entry i of T has $\mathrm{dp}(i) = 1$ if and only if P has a return to ground at v_i.*

Proof. Observe that an entry i of T has $\mathrm{dp}(i) = 1$ if and only if $i = 2m$ for some positive integer m, meaning that the first m columns of T coincide with some $\widetilde{T} \in S(\widetilde{\lambda})$ for $\widetilde{\lambda} = m^2$. This occurs if and only if the first $2m$ steps in the associated $P \in \mathcal{D}_\lambda$ correspond to a Dyck path from $(0, 0)$ to (m, m). Hence we may conclude that $\mathrm{dp}(i) = 1$ if and only if $v_i = (m, m)$ for some $m > 0$. $\qquad \square$

When λ is a two-row shape, $\mathrm{dp}(i) = 0$ and $\mathrm{dp}(i) = 1$ are the only options for each entry i of any $T \in S(\lambda)$. Proposition 3.1 then implies that an entry i of T has $\mathrm{dp}(i) = 1$ if the associated Dyck path $P \in \mathcal{D}_\lambda$ has a return to ground at v_i, while $\mathrm{dp}(i) = 0$ if P does not have a return to ground at v_i. This directly prompts our primary result of this subsection:

Theorem 3.2. *Take any two-row shape $\lambda = (a, b)$. Then the $S_k(\lambda)$ have generating function:*
$$
\xi(q) \;=\; \sum_k |S_k(\lambda)| q^k \;=\; \sum_i |\mathcal{D}_\lambda(i)| (1 + q)^i
$$

Proof. Take any $T \in S(\lambda)$ and let $P \in \mathcal{D}_\lambda$ be the corresponding Dyck path. By Corollary 2.6 and Proposition 3.1, $\chi^T(q) = [2]_q^i = (1 + q)^i$ if and only if $P \in \mathcal{D}_\lambda(i)$. Ranging over all $T \in S(\lambda)$, for each $i \geqslant 1$ we have precisely $|D_\lambda(i)|$ contributions of the form $(1 + q)^i$ to our overall generating function $\xi(q) = \sum_T \chi^T(q)$. The stated result immediately follows. $\quad \square$

It should be noted that Fresse, Mansour and Melnikov [6] have already derived a closed formula for each $|S_k(\lambda)|$ in the case of an arbitrary two-row shape $\lambda = (a, b)$. Skip forward to Subsection 4.1 to see our generalization of that formula to the semistandard case. The reason that we still take time to derive the generating function of Theorem 3.2, apart from the fact that it highlights an interesting relationship between inverted tableaux and Dyck paths, is that it easily generalizes to larger tableaux for which closed formulas do not currently exist. When $a = b = n$, Theorem 3.2 also admits the following specialization that directly relates the $|S_k(\lambda)|$ to the Ballot numbers:

Corollary 3.3. *Let* $\lambda = (n, n)$. *Then the* $S_k(\lambda)$ *have generating function:*

$$\xi(q) \;=\; \sum_k |S_k(\lambda)|q^k \;=\; \sum_i B(n-1, n-i)(1+q)^i \;=\; \sum_i \frac{i}{n}\binom{2n-i-1}{n-i}(1+q)^i$$

Example 3.4. *For* $\lambda = (4, 4)$ *the* $S_k(\lambda)$ *have generating function:*

$$\xi(q) \;=\; B(3,3)(1+q) + B(3,2)(1+q)^2 + B(3,1)(1+q)^3 + B(3,0)(1+q)^4$$

$$=\; 5(1+q) + 5(1+q)^2 + 3(1+q)^3 + 1(1+q)^4 \;=\; 14 + 28q + 20q^2 + 7q^3 + q^4$$

As one final comment about the two-row rectangular case, observe that the $|S_k(n,n)|$ resulting from Corollary 3.3 may be arranged into an integer triangle of their own. This triangle is already known under a variety of different contexts as OEIS sequence A039599 [11]. The relationship between the $|S_k(n,n)|$ and the other combinatorial interpretations of A039599 is an interesting topic for future investigation.

3.2 m-Dimensional Dyck Paths and Inverted Young Tableaux of Shape $\lambda = n^m$

Generalizing the methodology of Subsection 4.1 to m-row standard Young tableaux requires m-dimensional analogues of Dyck paths. So let $\lambda = (\lambda_1, \ldots, \lambda_m)$, where the λ_i are positive integers, and set $N = \lambda_1 + \ldots \lambda_m$. We adopt the standard notation of $\hat{e}_j$ as the unit vector in the jth coordinate of $\mathbb{R}^m$. By an **m-dimensional Dyck path of shape** λ we mean a lattice path $P = \{v_1, \ldots, v_N\}$ with vertices $v_i \in \mathbb{Z}^m$ such that:

1. $v_0 = (0, \ldots, 0)$ and $v_N = (\lambda_1, \ldots, \lambda_m)$.

2. For each $1 \leqslant i \leqslant N$ we have $v_i - v_{i-1} = \hat{e}_j$ for some $1 \leqslant j \leqslant m$.
   NOTE: If $v_i - v_{i-1} = \hat{e}_j$ we say that P has a "j-step" at v_i.

3. $v_i = (x_1, \ldots, x_m)$ satisfies $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_m$ for all $1 \leqslant i \leqslant N$.

By analogy with two-dimensional Dyck paths, we denote the set of all m-dimensional Dyck paths of shape λ by $\mathcal{D}_\lambda = \mathcal{D}_{(\lambda_1, \ldots, \lambda_m)}$. Also in direct analogy with two-dimensional Dyck paths, there is a well-known bijection between $\mathcal{D}_\lambda$ and m-row standard Young tableaux of shape λ. Under this bijection, $P \in \mathcal{D}_\lambda$ has a j-step at v_i if and only if i appears in the jth row of the corresponding tableau T. The fact that every point $v_i = (x_1, \ldots, x_m)$ along P satisfies $x_1 \geqslant \ldots \geqslant x_m$ ensures that the corresponding tableau T is in fact column standard. Notice that, if $\lambda = n^m$ is a rectangular two-row shape, this bijection implies that $|\mathcal{D}_\lambda|$ is the so-called m-dimensional Catalan number $C_{m,n} = \left(\prod_{i=0}^{m-1}\frac{i!}{(n+i)!}\right)(mn)!$. See OEIS sequence A060854 [11] or Gorska and Penson [8] for treatments of these numbers.

Before focusing upon the necessary properties of m-dimensional Dyck paths, we pause to establish a fact that follows quickly from the aforementioned bijection between $\mathcal{D}_\lambda$ and $S(\lambda)$. Here we use the standard notation of $T[1, \ldots, i]$ to denote the sub-tableau of $T \in S(\lambda)$ that retains the cells of T with values $1, \ldots, i$.

Proposition 3.5. *Take an m-row tableau $T \in S(\lambda)$ and let $P = \{v_0, v_1, \ldots\}$ be the corresponding m-dimensional Dyck path in $\mathcal{D}_\lambda$. Then $T[1, \ldots, i]$ has shape $\lambda_i = (x_1, \ldots, x_m)$ if and only if $v_i = (x_1, \ldots, x_m)$.*

Proof. $v_i = (x_1, \ldots, x_m)$ if and only if the first i steps of P contain x_1 1-steps, x_2 2-steps, etc. This corresponds to the situation where precisely x_1 elements of $[i] = \{1, \ldots, i\}$ have been assigned to the first row of T, precisely x_2 elements of $[i]$ have been assigned to the second row of T, etc. $\square$

The relevant statistic on m-dimensional Dyck paths is a higher-dimensional analogue of our two-dimensional "returns to ground". These "returns" will be points $v_i = (x_1, \ldots, x_m)$ along $P \in \mathcal{D}_\lambda$ at which we have some sort of equality within $x_1 \geqslant \ldots \geqslant x_m$. The difficulty here is that, even in the three-dimensional case, different lengths of equalities are possible within $x_1 \geqslant \ldots \geqslant x_m$ and some of these lengths are possible with different collections of coordinates.

Take an m-dimensional Dyck path $P = \{v_0, \ldots v_N\}$ in $\mathcal{D}_\lambda$, and let $v_{i-1} = (x_1, \ldots, x_m)$, $v_i = (x'_1, \ldots, x'_m)$ be two consecutive lattice points along P. By definition, $x'_\gamma = x_\gamma + 1$ for precisely one $1 \leqslant \gamma \leqslant m$ while $x'_i = x_i$ for all other $1 \leqslant i \leqslant m$. We say that P has a **d-degree return to ground (in the γ coordinate)** at v_i if $x'_{\gamma-d} = x'_{\gamma-d+1} = \ldots = x'_\gamma$ yet $x_{\gamma-d-1} \neq x_\gamma$. If P has an $(m-1)$-degree return to ground in the mth coordinate at v_i, meaning that $v_i = (x, x, \ldots, x)$ for some positive integer x, we say that P has a **full return to ground** at v_i.

As the situation described above ensures $x'_{\gamma+1} < x'_\gamma$, the degree d is one less than the length of the longest string of equalities in the coordinates of v_i that did not already exist in the coordinates of v_{i-1}. Clearly, d-degree returns to ground are only possible in the γ coordinate if $\gamma > d$. In the case of a two-dimensional Dyck path, the only valid returns to ground are 1-degree returns to ground in the second coordinate, which qualify as full returns to ground. In this sense, our definition is a natural extension of the pre-existing notion of returns to ground for two-dimensional Dyck paths.

If $P \in \mathcal{D}_\lambda$ has precisely k_i i-degree returns to ground for each $i \geqslant 1$ (disregarding the coordinates in which those returns appear), we write $\overrightarrow{\text{Ret}}(P) = (k_1, k_2, \ldots) = \vec{k}$. If $P \in \mathcal{D}_\lambda$ possesses precisely k full returns to ground, we write $\text{ret}(P) = k$ in direct analogy with the two-dimensional case. Define $\mathcal{D}_\lambda(\vec{k}) = \{P \in \mathcal{D}_\lambda \mid \overrightarrow{\text{Ret}}(P) = \vec{k}\}$ and $\mathcal{D}_\lambda(k) = \{P \in \mathcal{D}_\lambda \mid \text{ret}(P) = k\}$.

Even in the case of $\lambda = n^m$, there has been no previous attempt to enumerate the $\mathcal{D}_\lambda(\vec{k})$ or the $\mathcal{D}_\lambda(k)$. Such an attempt is clearly outside the scope of this paper. We do pause to note that, if $\lambda = n^m$, ranging over $n \geqslant 1$ allows one to compile both the $|\mathcal{D}_\lambda(\vec{k})|$ and the $|\mathcal{D}_\lambda(k)|$ into analogues of the Catalan triangle. For the $|\mathcal{D}_\lambda(\vec{k})|$ this would take the form an $(m+1)$-dimensional array such that the set of all integers in each m-dimensional "tier" sums to the m-dimensional Catalan number $C_{d,n}$. For the $|\mathcal{D}_\lambda(k)|$ we have a two-dimensional array that presumably has much more in common with the original Catalan triangle, with each row in the array summing to $C_{d,n}$. One could define the entries in this second array to be "m-dimensional ballot numbers", namely $B^m(n-1, n-k) = |\mathcal{D}_\lambda(k)|$ when $\lambda = n^m$.

As in the two-dimensional case of Subsection 3.1, the interest of this paper lies not in the inherent enumerative properties of the $\mathcal{D}_\lambda(\vec{k})$ but in how the sizes of these sets determine generating functions for inverted standard Young tableaux. The most general result, which holds for any m-row shape, is the following generalization of Proposition 3.1:

Theorem 3.6. *For an m-row shape $\lambda = (\lambda_1, \ldots \lambda_m)$, take $T \in S(\lambda)$ and let $P = \{v_0, v_1, \ldots\}$ be the associated m-dimensional Dyck path in $\mathcal{D}_\lambda$. Then the entry $a_{ij} = k$ of T has $\mathrm{dp}(k) = d$ if and only if P has a d-degree return to ground at v_k.*

Proof. Take an entry $a_{ij} = k$ in the ith row and j^{th} column of T, and let $v_k = (x_1, \ldots, x_m)$ be the associated lattice point in $P \in \mathcal{D}_\lambda$. By definition, $\mathrm{dp}(k) = i - 1 - \beta_k$, where β_k is the number of entries in the $(j+1)^{st}$ column of T that are smaller than k. By Proposition 3.5, β_k corresponds to the number of rows in $T[1, \ldots, k]$ whose length is strictly greater than j. It follows that $\mathrm{dp}(k) = \gamma_k - 1$, where γ_k equals the number of rows in $T[1, \ldots, k]$ of length precisely j. Once again citing Proposition 3.5, γ_j equals the longest string of equalities in the coordinates of v_k whose final coordinate is x_i. By definition, we have that P possesses a $(\gamma_k - 1)$-degree return to ground at v_k. Both directions of the theorem statement immediately follow. $\qquad\square$

Those familiar with the work of Fresse will notice that the notation of the preceding proof falls far closer to that of [5] than our own notation of Section 2. Our usage of Fresse's subtableau $T[1, \ldots, k]$ actually doesn't extend beyond Theorem 3.6, as the approach of Section 2 once again becomes much more convenient in applying that result toward a generating function $\xi(q)$:

Theorem 3.7. *Let $\lambda = (\lambda_1, \ldots, \lambda_m)$. Then the $S_k(\lambda)$ have generating function:*

$$\xi(q) \;=\; \sum_k |S_k(\lambda)| q^k \;=\; \sum_{(i_1, \ldots, i_{m-1})} \left(|\mathcal{D}_\lambda(i_1, \ldots, i_{m-1})| \prod_{j=1}^{m-1} [j+1]_q^{i_j} \right)$$

Proof. Take $T \in S(\lambda)$ and let $P \in \mathcal{D}_\lambda$ be the associated Dyck path. Corollary 2.6 and Theorem 3.6 together imply that $\chi^T(q) = \prod_{j=1}^{m-1}[j+1]_q^{i_j}$ if and only if $P \in \mathcal{D}_\lambda(i_1, \ldots, i_{m-1})$. Ranging over all $T \in S(\lambda)$, there are precisely $|\mathcal{D}_\lambda(i_1, \ldots, i_{m-1})|$ contributions of the form $\prod_{j=1}^{m-1}[j+1]_q^{i_j}$ to $\xi(q) = \sum_T \chi^T(q)$. The theorem immediately follows. $\qquad\square$

4 Direct Enumerations of Inverted Young Tableaux

In this final section, we present direct enumerations of the $|S_k(\lambda, \mu)|$ whose methodologies are completely distinct from the generalized Dyck path approach of Section 3. All of these methods make direct use of our generating function $\chi^T(q)$ from Theorem 2.4 or our notions of inversion depth. This allows us to tackle a somewhat different collection of cases than previously attempted, and to generalize some pre-existing results to the semistandard case for the very first time. In particular, we develop an explicit formula for the $|S_k(\lambda, \mu)|$ in

the general two-row case of $\lambda = a^1 b^1$. Opening up a wider range of tractable shapes by specializing to standard tableau, we close with an explicit enumeration of $|S_k(\lambda)|$ for the two-column rectangular case of $\lambda = 2^n$.

4.1 Enumeration of $S_k(\lambda, \mu)$ for $\lambda = a^1 b^1$

In the standard specialization, Fresse, Mansour and Melnikov [6] have already explicitly enumerated the $S_k(\lambda)$ for a general two-row shape $\lambda = (a, b)$. Theorem 2.1 of [6] gives:

$$|S_k(\lambda)| \ = \ \beta^\lambda_{b-k} \ = \ \frac{a - b + 1 + 2k}{a + 1 + k} \binom{a + b}{b - k} \tag{1}$$

In this equation, β^λ_j denotes the Betti number of the associated Springer variety F_λ, an algebraic interpretation that we do not make use of here. The shift in subscript follows from the fact that $\dim(F_\lambda) = b$, a quantity that equals the maximal inversion number M_λ for $\tau \in I(\lambda)$.

The $S_k(\lambda, \mu)$ were independently enumerated in the semistandard generalization by the author [3], but only in the rectangular case of $a = b = n$. Theorem 14 of [3] equated $|S_k(\lambda, \mu)|$ with a product of various Catalan numbers whose subscripts strongly partition n. One can verify that this enumeration agrees with Equation 1 as $|S_k(\lambda)| = \frac{1+2k}{n+1+k} \binom{2n}{n-k}$ for standard content μ if one rewrites the Catalan products of the author [3] as an appropriate two-parameter Fuss-Catalan number (Raney number) via a modification of the results from Hilton and Pedersen [9].

Here we directly extend Equation 1 to the semistandard case in a way that recovers the formula of the author [3] when $a = b = n$. Essential to our technique is the invariance of $|S_k(\lambda, \mu)|$ under permutation of content, as originally presented by the author [3]:

Theorem 4.1 (Theorem 11 of [3]). *For any λ, compatible content $\mu = 1^{\mu_1} 2^{\mu_2} \ldots M^{\mu_M}$, and permutation $\sigma \in S_M$, $|S_k(\lambda, \mu)| = |S_k(\lambda, \sigma(\mu))|$ for all $k \geqslant 0$.*

For any two-row shape $\lambda = (a, b)$, clearly the only contents μ that allow for row-standard tableaux are those where each value appears at most twice. Using Theorem 4.1, we are justified in restricting our attention to μ where $1, 2, \ldots, m$ each appear twice and $m + 1, m + 2, \ldots, a + b - 2m$ each appear once. This allows for the following:

Theorem 4.2. *Let $\lambda = (a, b)$ for any $a \geqslant b \geqslant 1$, and let $\mu = 1^{\mu_1} 2^{\mu_2} \ldots M^{\mu_M}$ be any content with $\sum_i \mu_i = a + b$. If $\mu_i = 2$ for m choices of i and $\mu_i = 1$ for the remaining $a + b - 2m$ choices of i, then:*

$$|S_k(\lambda, \mu)| \ = \ \frac{a - b + 1 + 2k}{a + 1 + k - m} \binom{a + b - 2m}{b - k - m}$$

Proof. As previously argued, Theorem 4.1 allows us to restrict our attention to the specific content μ where $1, 2, \ldots, m$ each appear twice and $m+1, m+2, \ldots, a+b-2m$ each appear once. Any tableau with this λ and μ must take the form shown in Figure 7:

Such tableaux are clearly in bijection with standard Young tableaux of shape $\widetilde{\lambda} = (a - m, b - m)$, under the map that re-indexes entries in the final $b - m$ columns by

$x \mapsto x - 2m$. For each $T \in S(\lambda, \mu)$, this bijection preserves the associated generating function $\chi^T(q)$. This is due to the fact that all entries a_{ij} in the first m columns of T have $\mathrm{dp}(a_{ij}) = \mathrm{dp}^*(a_{ij}) = 0$ and hence do not contribute to $\chi^T(q)$ by Theorem 2.4. It follows that $\xi(q) = \sum_k |S_k(\lambda, \mu)| q^k = \sum_k |S_k(\widetilde{\lambda})| q^k$. For any $k \geqslant 0$, directly applying Equation 1 then gives:

$$|S_k(\lambda, \mu)| \;=\; |S_k(\widetilde{\lambda})| \;=\; \frac{(a-m)-(b-m)+1+2k}{(a-m)+1+k}\binom{(a-m)+(b-m)}{(b-m)-k} \qquad \square$$

1	2	...	m		...		...	
1	2	...	m		...			

Figure 7: An arbitrary $T \in S(\lambda, \mu)$ with $\lambda = (a, b)$ and $\mu = 1^2 \ldots m^2 (m+1)^1 \ldots (a + b - 2m)^1$. Here the shaded cells correspond to some standard Young tableau under the re-indexing $x \mapsto x - 2m$.

4.2 Enumeration of $S_k(\lambda)$ for $\lambda = 2^n$

Enumeration of the $|S_k(\lambda, \mu)|$ in the two-column rectangular case of $\lambda = 2^n$ has yet to appear anywhere in the literature, even for standard tableaux. In this final subsection, we tackle the two-column case in the standard specialization.

Pause to observe that this entire subsection is made necessary because inversion depths, and hence the $|S_k(\lambda)|$, are not preserved under transposition. If $\bar{\lambda}$ denotes the conjugate partition of λ (obtained by transposing of the underlying Young diagram), it is well known that for standard Young tableaux we have $|S(\lambda)| = |S(\bar{\lambda})|$. However, it is typically not true that $|S_k(\lambda)| = |S_k(\bar{\lambda})|$ for $k \geqslant 1$ if $\bar{\lambda}$ is in fact distinct from λ. For a basic example of this fact, consider any one-row shape $\lambda = m^1$, so that $\bar{\lambda} = 1^m$. Clearly $|S_k(\lambda)| = 0$ for all $k > 0$, yet $|S_k(\bar{\lambda})| \neq 0$ for all $k \leqslant \binom{m}{2}$ via an application of Corollary 2.6.

To obtain a generating function $\xi(q)$ for the $|S_k(\lambda)|$, we determine which T-specific generating functions $\chi^T(q)$ are possible for $T \in S(\lambda)$, and then develop a closed formula for the number $\phi(\chi)$ of tableaux with a fixed $\chi^T(q)$. As in Section 3, this allows us to conclude that $\xi(q) = \sum_\chi \phi(\chi) \chi^T(q)$. The first step in this process is a recognition of the fact that, for two-column tableaux, there is at most one $T \in S(\lambda)$ with a particular arrangement of inversion depths across its entries:

Lemma 4.3. *Let $\lambda = 2^m 1^{n-m}$, and take $T \in S(\lambda)$. If $a_1, \ldots, a_n$ are the entries in the first column of T, read from top to bottom, then T is uniquely identified by the collection of inversion depths $\mathrm{dp}(a_i) = c_i$.*

Proof. Pause to note that, if $b_1, \ldots, b_m$ are the entries in the second (rightmost) column of T, read from top to bottom, then $\mathrm{dp}(b_i) = i - 1$ no matter our choice of T. This is why we require only the inversion depths of the a_i to distinguish tableaux in $S(\lambda)$.

So fix $T \in S(\lambda)$ and let $a_1, \ldots, a_n$ be the entries in the first column of T. By definition, $\mathrm{dp}(a_i) = i - 1 - \beta_i$, where β_i is the number of entries in the second column of T that are smaller than a_i. As T possesses only two-columns, we also know that $a_i = i + \beta_i$ and hence that $\mathrm{dp}(a_i) = i - 1 - (a_i - i) = 2i - 1 - a_i$. Given this direct relationship between the a_i and the $\mathrm{dp}(a_i) = c_i$, it is clear that distinct tableaux in $S(\lambda)$ must possess distinct values of c_i for at least one i. $\qquad \square$

Using Lemma 4.3 still requires us to characterize which sequences of inversion depths $c_1, c_2, \ldots$ are possible. The necessary conditions on the $c_1, c_2, \ldots$ actually prove to be fairly straightforward:

Lemma 4.4. *Set $\lambda = 2^n$, and let $c_1, \ldots, c_n$ be a sequence of non-negative integers. Then there exists (unique) $T \in S(\lambda)$ whose first column entries $a_1, \ldots, a_n$ satisfy $\mathrm{dp}(a_i) = c_i$ for all $1 \leqslant i \leqslant n$ if and only if $c_1 = 0$ and $c_i \leqslant c_{i-1} + 1$ for all $i > 1$.*

Proof. ($\Rightarrow$) Take any $T \in S(\lambda)$. Clearly, the top entry a_1 in the first column of T must have an inversion depth of 0, requiring $c_1 = 0$. By the definition of inversion depth, for any $i > 1$ there exist $\beta_{i-1} = (i - 1) - 1 - c_{i-1}$ entries in the second column of T that are smaller than a_{i-1} and $\beta_i = i - 1 - c_i$ entries in the second column of T smaller than a_i. As $a_{i-1} < a_i$ we have $\beta_{i-1} \leqslant \beta_i$ and thus $c_i \leqslant c_{i-1} + 1$.

($\Leftarrow$) Now take a sequence of non-negative integers $c_1, \ldots, c_n$ satisfying the stated conditions. We use that sequence to construct the tableau T below, where $b_1, \ldots, b_n$ are the n remaining integers not used in the first column (arranged in increasing order).

$1 - c_1$	b_1
$3 - c_2$	b_2
$\vdots$	$\vdots$
$2n - 1 - c_n$	b_n

Via equivalent reasoning to the proof of Lemma 4.3, if $a_i = 2i - 1 - c_i$ then $\mathrm{dp}(a_i) = 2i - 1 - (2i - 1 - c_i) = c_i$. The assumptions that $c_1 = 0$ and $c_i \leqslant c_{i-1} + 1$ imply $c_i \leqslant i - 1$ and hence $a_i \geqslant i$. This ensures that the entries in the first column of T are always positive integers between 1 and $2n - 1$. By construction, $a_i \leqslant 2i - 1$ and there are always enough larger entries leftover for the b_i to ensure that T is row-standard. The fact that $c_i \leqslant c_{i-1} + 1$ also guarantees that $a_i \geqslant a_{i-1}$, allowing us to conclude that T is column-standard. Hence $T \in S(\lambda)$ is our standard Young tableau with the required inversion depths down its first column. $\qquad \square$

Combining Lemmas 4.3 and 4.4, we may conclude that there is precisely one $T \in S(\lambda)$ for every sequence of non-negative integers $c_1, \ldots, c_n$ such that $c_1 = 0$ and $c_i \leqslant c_{i-1} + 1$ for all $1 \leqslant i \leqslant n$. One may use a generating tree to verify that the total number of such sequences is the Catalan number C_n, as expected from the hook-length formula applied to $\lambda = 2^n$.

Notice that if $T \in S(\lambda)$ is associated with the sequence $c_1, \ldots, c_n$, then Corollary 2.6 gives generating function $\chi^T(q) = [n]_q! \prod_i [c_i + 1]_q$, where the leading $[n]_q!$ comes from the

inversion depths of entries in the second column of T. To identify the number $\phi(\chi)$ of $T \in S(\lambda)$ with a fixed generating function $\chi^T(q)$, we thus only need to determine how many valid sequences $c_1, \ldots, c_n$ have a particular number of zeroes, ones, twos, etc.

Once again pause to observe that the conditions on our sequences $c_1, \ldots, c_n$ guarantee that $c_i \leqslant i - 1$ and hence that we only need to count occurrences of $0, 1, \ldots, n - 1$. Henceforth let $\psi(\alpha_0, \ldots, \alpha_{n-1})$ denote the number of non-negative integer sequences $c_1, \ldots, c_n$ satisfying both $c_1 = 0$ and $c_i \leqslant c_{i-1} + 1$ that possess precisely α_k instances of k. Clearly we must have $\sum_i \alpha_i = n$ in order for $\psi(\alpha_0, \ldots, \alpha_{n-1}) \neq 0$. To achieve $\psi(\alpha_0, \ldots, \alpha_{n-1}) \neq 0$, the $c_i \leqslant c_{i-1} + 1$ condition also requires that a zero value $\alpha_k = 0$ never be followed by a nonzero value $\alpha_{k+1} \neq 0$. One can show that these are the only two conditions needed to guarantee that $\psi(\alpha_0, \ldots, \alpha_{n-1}) \neq 0$, although we do not require this fact in the subsequent proof.

Theorem 4.5. *Let* $\lambda = 2^n$. *Then the* $|S_k(\lambda)|$ *have generating function:*

$$\xi(q) \;=\; \sum_k |S_k(\lambda)| q^k \;=\; [n]_q! \left(\sum_{\alpha_0 + \ldots + \alpha_{n-1} = n} \left(\prod_{i=1}^{n-1} \binom{\alpha_i + \alpha_{i-1} - 1}{\alpha_i} [i+1]_q^{\alpha_i} \right) \right)$$

Proof. Our primary goal is to show that $\psi(\alpha_0, \ldots, \alpha_{n-1}) = \prod_{i=1}^{n-1} \binom{\alpha_i + \alpha_{i-1} - 1}{\alpha_i}$. So assume that we want to construct an arbitrary sequence of non-negative integers $c_1, \ldots, c_n$ satisfying both $c_1 = 0$ and $c_i \leqslant c_{i-1} + 1$ that possesses precisely α_k instances of k for each $k \geqslant 0$. We recursively build up this sequence by simultaneously inserting all α_k copies of k into a partial sequence that contains α_j copies of j for all $0 \leqslant j \leqslant k - 1$, making sure that our conditions on the c_i are preserved after each step.

We begin by inserting the α_0 copies of 0 into an empty sequence, noticing that $c_1 = 0$ implies that we always have $\alpha_0 > 0$. There is obviously only one way to perform this insertion. Now assume that we have continued this process up to the insertion of the α_k copies of k. Preserving the $c_i \leqslant c_{i-1} + 1$ condition requires that these instances of k may only appear directly after an instance of $(k - 1)$ or another instance of k. This means that our previous placement of all instances of $0, 1, \ldots, (k - 2)$ is irrelevant to the validity of our placement at the step, and that we only need to consider the relative placement of the $\alpha_k + \alpha_{k-1}$ instances of k and $(k - 1)$. It follows that there are precisely $\binom{\alpha_k + \alpha_{k-1} - 1}{\alpha_k}$ valid ways to insert our α_k copies of k.

Ranging over all k, we may conclude that there are precisely $\psi(\alpha_0, \ldots, \alpha_{n-1}) = \prod_{i=1}^{n-1} \binom{\alpha_i + \alpha_{i-1} - 1}{\alpha_i}$ valid sequences $c_1, \ldots, c_n$ that contain α_0 copies of 0, α_1 copies of 1, etc. Notice that, if $\alpha_{k+1} \neq 0$ follows $\alpha_k = 0$, then $\binom{\alpha_{k+1} + \alpha_k - 1}{\alpha_{k+1}} = \binom{\alpha_{k+1} - 1}{\alpha_{k+1}} = 0$ and $\psi(\alpha_0, \ldots, \alpha_{n-1}) = 0$, as expected.

So assume that $c_1, \ldots, c_n$ possesses α_k copies of k for all $0 \leqslant k \leqslant (n - 1)$, and that $T \in S(\lambda)$ is any tableau associated to $c_1, \ldots, c_n$ via the bijection of Lemma 4.4. Corollary 2.6 then gives generating function $\chi^T(q) = [n]_q! \prod_{i=1}^{n-1} [i+1]_q^{\alpha_i}$. The $\phi(\alpha_0, \ldots, \alpha_{n-1})$ tableaux with that generating function contribute a total of $\phi(\alpha_0, \ldots, \alpha_{n-1}) [n]_q! \prod_{i=1}^{n-1} [i+1]_q^{\alpha_i} = [n]_q! \prod_{i=1}^{n-1} \binom{\alpha_i + \alpha_{i-1} - 1}{\alpha_i} [i+1]_q^{\alpha_i}$ to the overall generating function $\xi(q) = \sum_k |S_k(\lambda)| q^k$. Ranging over all combinations of the α_i such that $\alpha_0 + \ldots + \alpha_{n-1} = n$ then gives the desired result. $\square$

Example 4.6. *For $\lambda = 2^3$, we have $\mathrm{dp}(a_{ij}) \leqslant 2$ for all entries a_{ij} of $T \in S(\lambda)$. The only non-zero values for $\psi(\alpha_0, \alpha_1, \alpha_2)$ are shown below, followed by the generating function for the $|S_k(\lambda)|$:*

$$\psi(3,0,0) = 1 \qquad \psi(2,1,0) = 2 \qquad \psi(1,2,0) = 1 \qquad \psi(1,1,1) = 1$$

$$\begin{aligned} \xi(q) &= [3]_q! \left(1[1]_q^3 + 2[1]_q^2[2]_q + 1[1]_q[2]_q^2 + 1[1]_q[2]_q[3]_q \right) \\ &= 5 + 16x + 25x^2 + 24x^3 + 14x^4 + 5x^5 + x^6 \end{aligned}$$

A methodology similar to the one above may be applied to the non-rectangular two-column case of $\lambda = 2^m 1^{n-m}$, as tableaux of this shape are still uniquely identified by the inversion depths of entries in their first column. Sadly, a generalization of the generating function from Theorem 4.5 becomes so convoluted that we do not attempt a full derivation here. The difficulty in generalizing to $\lambda = 2^m 1^{n-m}$ is that a different (and more complicated) collection of inversion depths $c_1, \ldots, c_n$ are now possible. In particular, entries a_i in the one-column "tail" of $T \in S(\lambda)$ now have a nonzero lower bound on $\mathrm{dp}(a_i) = c_i$. One may quickly verify that inversion depths in the first column of $T \in S(\lambda)$ fall within the ranges shown in Figure 8.

(0)	(0)
$0 \leqslant c_2 \leqslant 1$	(1)
$\vdots$	$\vdots$
$0 \leqslant c_m \leqslant m-1$	$(m-1)$
$0 \leqslant c_{m+1} \leqslant m$	
$1 \leqslant c_{m+2} \leqslant m+1$	
$\vdots$	
$n-m-1 \leqslant c_{m+n} \leqslant n-1$	

Figure 8: Possible inversion depths for entries in an arbitrary $T \in S(\lambda)$ of shape $\lambda = 2^m 1^{n-m}$.

With the ranges of Figure 8 in mind, we close this paper with the following generalization of Lemma 4.4. The following proposition would be the first step in developing a closed generating function $\xi(q)$ for the $|S^k(\lambda)|$ when $\lambda = 2^m 1^{n-m}$.

Proposition 4.7. *Set $\lambda = 2^m 1^{n-m}$, and let $c_1, \ldots, c_n$ be a sequence of non-negative integers. Then there exists (unique) $T \in S(\lambda)$ whose first column entries $a_1, \ldots, a_n$ satisfy $\mathrm{dp}(a_i) = c_i$ for all $1 \leqslant i \leqslant n$ if and only if $c_1 = 0$, $c_i \leqslant c_{i-1} + 1$ for all $i > 1$, and $c_i > i - m - 1$ for all $i > m$.*

Proof. The method of proof is directly equivalent to that of Lemma 4.4. For the ($\Leftarrow$) direction, a valid sequence $c_1, \ldots, c_n$ of integers is now associated with the unique tableau shown in Figure 9, which is a minor modification of the tableau from Lemma 4.4. $\qquad\square$

$1 - c_1$	b_1
$\vdots$	$\vdots$
$2m - 1 - c_m$	b_m
$2m + 1 - c_{m+1}$	
$2m + 3 - c_{m+2}$	
$\vdots$	
$2n - 1 - c_n$	

Figure 9: Tableau for Proposition 4.7.

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