Circulant homogeneous factorisations of complete digraphs $\mathbf{K}_{p^d}$ with p an odd prime

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Abstract

Let $\mathcal{F} = (\mathbf{K}_n, \mathcal{P})$ be a circulant homogeneous factorisation of index k, that means $\mathcal{P}$ is a partition of the arc set of the complete digraph $\mathbf{K}_n$ into k circulant factor digraphs such that there exists $\sigma \in S_n$ permuting the factor circulants transitively amongst themselves. Suppose further such an element σ normalises the cyclic regular automorphism group of these circulant factor digraphs, we say $\mathcal{F}$ is normal. Let $\mathcal{F} = (\mathbf{K}_{p^d}, \mathcal{P})$ be a circulant homogeneous factorisation of index k where p^d, $(d \geqslant 1)$ is an odd prime power. It is shown in this paper that either $\mathcal{F}$ is normal or $\mathcal{F}$ is a lexicographic product of two smaller circulant homogeneous factorisations.

1 Introduction

Let $\Gamma = (V, A)$ be a *digraph* with vertex set $V = V(\Gamma)$ and arc set $A = A(\Gamma)$. A *factorisation* of Γ is a partition $\mathcal{P} = \{P_1, \ldots, P_k\}$ of A, and is denoted by the pair $(\Gamma, \mathcal{P})$. This gives rise to *factor digraphs*, $\Gamma_i = (V, P_i)$ for $i = 1, \ldots, k$. Moreover the integer $k = |\mathcal{P}|$ is called the *index* of the factorisation and $|V|$ is called the *order* of the factorisation.

An *automorphism* of a factorisation $(\Gamma, \mathcal{P})$ is an automorphism of the digraph Γ that preserves the partition $\mathcal{P}$. The automorphism group $\mathrm{Aut}(\Gamma, \mathcal{P})$ consists of all automorphisms of $(\Gamma, \mathcal{P})$. A factorisation $(\Gamma, \mathcal{P})$ is called *transitive* if $\mathrm{Aut}(\Gamma, \mathcal{P})$ induces a transitive action on $\mathcal{P}$; further $(\Gamma, \mathcal{P})$ is called *homogeneous* if it is transitive and in addition the kernel of $\mathrm{Aut}(\Gamma, \mathcal{P})$ acting on $\mathcal{P}$ is transitive on the vertex set $V(\Gamma)$. That is there exists a subgroup $G \leqslant \mathrm{Aut}(\Gamma, \mathcal{P})$ that permutes the parts P_i transitively, and the kernel

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$M(= G \cap \mathrm{Aut}(\Gamma_1) \cap \cdots \cap \mathrm{Aut}(\Gamma_k))$ of G acting on $\mathcal{P}$ is vertex transitive on each of the Γ_i. Hence G induces isomorphisms between each pair of the factor digraphs. To emphasise the groups M and G, we say that the factorisation $(\Gamma, \mathcal{P})$ is (M, G)-*homogeneous.* Such a factorisation is also denoted by $(\Gamma, \mathcal{P}, M, G)$. We say a homogeneous factorisation $(\Gamma, \mathcal{P}, M, G)$ is *cyclic* if the induced group $G^{\mathcal{P}}$ is cyclic.

A digraph is called a *circulant* if it has a cyclic group of automorphisms which is regular on the vertex set. (A permutation group is *regular* if it is transitive and the only element that fixes a point is the identity.) Let $(\Gamma, \mathcal{P})$ be (M, G)-homogeneous, suppose further that M contains a regular cyclic subgroup. Then all of the factor digraphs Γ_i are circulants relative to the same cyclic regular subgroup, and $(\Gamma, \mathcal{P})$ is called a *circulant* (M, G)-homogeneous factorisation.

We denote by $\mathbf{K}_n$ the complete digraph on n vertices in which each ordered pair of distinct vertices is an arc. Homogeneous factorisations of $\mathbf{K}_n$ with index 2 arise from pairs $\Gamma, \overline{\Gamma}$, where Γ is a vertex transitive digraph isomorphic to its complement $\overline{\Gamma}$, that is, Γ is a vertex transitive *self-complementary* directed graph. Suppose further that the factors $\Gamma, \overline{\Gamma}$ are undirected. Then Γ is a vertex transitive self-complementary undirected graph. Moreover it is easy to see that self-complementary circulants $(\Gamma, \overline{\Gamma})$ correspond to a circulant homogeneous factorisation arising from this pair. A better understanding of vertex transitive self-complementary undirected or directed graphs is a principal motivation for studying homogeneous factorisations of $\mathbf{K}_n$, see for example [7, 8]. In [9], self-complementary circulants of prime power order have been classified. Our main purpose of this paper is to classify circulant homogeneous factorisations of complete digraphs with prime power order, which can be viewed as a generalization of the result of [9].

Let $\mathcal{F} = (\mathbf{K}_n, \mathcal{P})$ be a circulant homogeneous factorisation. An automorphism $\tau(\in \mathrm{Aut}(\mathcal{F}))$ is called a *cyclic isomorphism* of $\mathcal{F}$ if τ is transitive on $\mathcal{P}$. Suppose further that $\mathcal{F}$ is (M, G)-homogeneous. It is proved in [8, Theorem 4.1] that $\mathcal{F}$ must be cyclic, that is there exists $\sigma \in G \setminus M$ such that $G = \langle M, \sigma \rangle$ and $G^{\mathcal{P}} = \langle \sigma^{\mathcal{P}} \rangle$. Such an element σ is a cyclic isomorphism of $\mathcal{F}$, and we often refer to $(\mathbf{K}_n, \mathcal{P}, M, \sigma)$ as this factorisation when we wish to emphasise this cyclic isomorphism. Moreover, let $Z_n(\leqslant M)$ be the regular subgroup on $V(\mathbf{K}_n)$ and we identify the vertex set with this regular group Z_n. Suppose further that there exists a cyclic isomorphism σ fixing point 1 (the identity element of Z_n) and normalising the regular cyclic subgroup Z_n. We say such a circulant homogeneous factorisation is *normal*. In this case, σ can be viewed as an automorphism of the cyclic group Z_n, and we can construct such factorisations easily, see Construction 4.3 for more details. We also give examples of non-normal circulant homogeneous factorisations in Proposition 4.8.

We next present a lexicographic product construction of two circulant homogeneous factorisations which is analogous to the lexicographic product construction of two digraphs. For more information see [1, 6].

For two digraphs $\Gamma_i = (V_i, A_i)$ with $i = 1, 2$, we denote by $A_1[A_2]$ the set of all pairs $((u_1, u_2), (v_1, v_2))$ such that either $(u_1, v_1) \in A_1$, or $u_1 = v_1$ and $(u_2, v_2) \in A_2$. Then the *lexicographic product* $\Gamma_1[\Gamma_2]$ is defined as the digraph with vertex set $V_1 \times V_2$ and arc set $A_1[A_2]$.

For $i = 1, 2$, let $\mathcal{F}_i = (\mathbf{K}_{n_i}, \mathcal{P}_i, M_i, \sigma_i)$ be a circulant homogeneous factorisation of index k where σ_i is a cyclic isomorphism of $\mathcal{F}_i$. Let $\mathcal{P}_i = \{P_{i,1}, \ldots, P_{i,k}\}$ and suppose that $\sigma_i : P_{i,j} \mapsto P_{i,j+1}$ (reading the second subscript modulo k). Write the vertex set $\mathbf{K}_{n_1 n_2}$ as $V(\mathbf{K}_{n_1}) \times V(\mathbf{K}_{n_2})$ and let $\mathcal{F}_1[\mathcal{F}_2] = (\mathbf{K}_{n_1 n_2}, \mathcal{P})$ where $\mathcal{P} = \{P_{1,j}[P_{2,j}] | 1 \leqslant j \leqslant k\}$. It is proved in Lemma 3.1 that $\mathcal{F}_1[\mathcal{F}_2]$ is a circulant homogeneous factorisation of index k and $\mathcal{F}_1[\mathcal{F}_2]$ is called the *lexicographic product* of $\mathcal{F}_1$ and $\mathcal{F}_2$. Let $\Gamma_{i,j} (i = 1, 2, j = 1, \ldots, k)$ be the factors of $\mathcal{F}_i$. Then $\Gamma_{1,j}[\Gamma_{2,j}]$ are factors of $\mathcal{F}_1[\mathcal{F}_2]$.

In this paper, we will classify circulant homogeneous factorisations of a complete digraph $\mathbf{K}_{p^d}$ where p is an odd prime and $d \geqslant 1$. The main theorem is the following.

Theorem 1.1. *Let $\mathcal{F} = (\mathbf{K}_{p^d}, \mathcal{P})$ be a circulant homogeneous factorisation of index k where $p^d (d \geqslant 1)$ is an odd prime power. Then either $\mathcal{F}$ is normal or $\mathcal{F} = \mathcal{F}_1[\mathcal{F}_2]$ is a lexicographic product where $\mathcal{F}_i$ is a circulant homogeneous factorisation of index k and order p^{d_i} for $i = 1, 2$ and $d = d_1 + d_2$ $(d_1, d_2 \geqslant 1)$.*

2 Preliminary results

From now on, we always assume that $p^d (d \geqslant 1)$ is an odd prime power.

A finite permutation group is called a *c-group* if it contains a cyclic regular subgroup. A precise list of primitive c-groups is given in the following lemma.

Lemma 2.1. ([8, Theorem 1.2]) *Suppose that X is a primitive permutation group on Ω and X contains a cyclic regular subgroup. Then one of the following holds:*

(i) $|\Omega| = q$, *and* $X \leqslant \mathrm{AGL}(1, q)$, *where q is a prime;*

(ii) $|\Omega| = 4$, *and* $X = Sym(\Omega)$;

(iii) *X is almost simple and 2-transitive on Ω.*

Corollary 2.2. *Suppose that X is a solvable primitive c-group on Ω where $|\Omega| = p^d$ is an odd prime power. Then $|\Omega| = p$ and $\mathbb{Z}_p \leqslant X \leqslant \mathrm{AGL}(1, p)$.*

Let Z_n be a cyclic group of order n, considered in its action (by multiplication) as a subgroup of the symmetric group $\mathrm{Sym}(Z_n)$. A *Z_n-circulant* is a Cayley digraph $\Gamma = \mathrm{Cay}(Z_n, S)$ with vertex set Z_n and arc set $A(\Gamma) = \{(g, sg) \mid g \in Z_n, s \in S\}$, for some non-empty subset S of $Z_n \setminus \{1\}$. We also denote by $\hat{Z}_n$ the right regular representation of the group Z_n. Then each Z_n-circulant Γ admits $\hat{Z}_n$ as a subgroup of automorphisms. Consider also $\mathrm{Aut}(Z_n)$ as a subgroup of $\mathrm{Sym}(Z_n)$ in its natural action. Then $\mathrm{Aut}(Z_n)$ normalises $\hat{Z}_n$ in $\mathrm{Sym}(Z_n)$, and $\mathrm{Aut}(Z_n) \cap \mathrm{Aut}(\Gamma)$ is equal to $\mathrm{Aut}(Z_n, S) := \{\sigma \in \mathrm{Aut}(Z_n) \mid S^\sigma = S\}$. In fact the normaliser $N_{\mathrm{Aut}(\Gamma)}(\hat{Z}_n) = \hat{Z}_n \rtimes \mathrm{Aut}(Z_n, S)$, see for example [2, 11]. The Cayley digraph $\Gamma = \mathrm{Cay}(Z_n, S)$ is said to be a *normal circulant* if $\hat{Z}_n$ is normal in $\mathrm{Aut}(\Gamma)$, or equivalently, if $\mathrm{Aut}(\Gamma) = \hat{Z}_n \rtimes \mathrm{Aut}(Z_n, S)$.

A circulant is called *arc-transitive* if its automorphism group is transitive on the arc set. The finite arc-transitive circulants were classified independently by István Kovács [3] and Cai Heng Li [5] in 2004. The following result concerning arc-transitive circulants of order p^d is an immediate corollary of Theorem 1.3 in [5], (just note that in [5, Theorem 1.3], the orders of the deleted lexicographic product type digraphs cannot be a prime power).

Theorem 2.3. *Let $\Gamma = \mathrm{Cay}(Z_{p^d}, S)$ be a connected arc transitive directed circulant of order p^d where p is an odd prime and $d \geqslant 1$ is an integer. Then one of the following holds:*

(i) Γ is a complete digraph.

(ii) Γ is a normal circulant.

(iii) There exists an arc-transitive circulant Σ of order p^{d-i} such that $\Gamma = \Sigma[\overline{K}_{p^i}]$ where $1 \leqslant i < d$. Let $Z_{p^i} \leqslant Z_{p^d}$ be the subgroup of order p^i. Then $sZ_{p^i} \subseteq S$ for any $s \in S$.

Let Ω be the vertex set of $\mathbf{K}_n$. Then a factorisation $(\mathbf{K}_n, \mathcal{P})$ is also simply denoted by $(\Omega, \mathcal{P})$.

Let $\mathcal{F} = (\mathbf{K}_n, \mathcal{P})$ be a (M, G)-homogeneous factorisation of index k where $\mathcal{P} = \{P_1, \ldots, P_k\}$, and assume that G is imprimitive on the vertex set of $\mathbf{K}_n$. Let B be a block of G. Let $P_i^B = P_i \cap (B \times B)$ and $\mathcal{P}_B = \{P_1^B, P_2^B, \ldots, P_k^B\}$. Then the factorisation $(B, \mathcal{P}_B)$ is called *the induced sub-factorisation* of $\mathcal{F}$ on the block B.

Lemma 2.4. ([7, Lemma 4.1]) *Let $(\mathbf{K}_n, \mathcal{P}, M, G)$ be a homogeneous factorisation of index k and let B be a nontrivial block of G. Then the induced sub-factorisation $(B, \mathcal{P}_B)$ is an (M_B^B, G_B^B)-homogeneous factorisation of index k. Further, $G^{\mathcal{P}}$ is permutationally isomorphic to $G_B^{\mathcal{P}_B}$.*

Let $(\mathbf{K}_n, \mathcal{P})$ be a circulant (M, G)-homogeneous factorisation of index k where $\mathcal{P} = \{P_1, \ldots, P_k\}$ and identify the vertex set with the regular group Z_n. Choose the point $1 \in Z_n$, let $P_i(1) = \{\alpha \in Z_n | (1, \alpha) \in P_i\}$ and let $\mathcal{P}(1) = \{P_1(1), \ldots, P_k(1)\}$. Then $\mathcal{P}(1)$ is a partition of $Z_n \setminus \{1\}$, and the factor digraphs $\Gamma_i = \mathrm{Cay}(Z_n, P_i(1))$. The lemma below gives a relation between the partition $\mathcal{P}$ and $\mathcal{P}(1)$, it can be derived from [7, Lemma 2.3] easily.

Lemma 2.5. ([7, Lemma 2.3]) *1. Let $(\mathbf{K}_n, \mathcal{P})$ be a circulant (M, G)-homogeneous factorisation. Then $\mathcal{P}(1)$ is a partition of $Z_n \setminus \{1\}$ and each factor digraph $\Gamma_i = \mathrm{Cay}(Z_n, P_i(1))$. Let G_1 be the point stabilizer. Then G_1 induces a transitive action on $\mathcal{P}(1)$, $G_1^{\mathcal{P}} = G^{\mathcal{P}}$ and the G_1-actions on $\mathcal{P}$ and $\mathcal{P}(1)$ are equivalent.*

2. Let $\mathcal{P}(1) = \{P_1(1), \ldots, P_k(1)\}$ be a partition of $Z_n \setminus \{1\}$. Define $\Gamma_i = \mathrm{Cay}(Z_n, P_i(1))$ and let P_i be the set of arcs of Γ_i. Then $\mathcal{P} = \{P_1, \ldots, P_k\}$ is a partition of the arc set of $\mathbf{K}_n$. Let G be such that $\hat{Z}_n \leqslant G \leqslant \mathrm{Sym}(Z_n)$. Suppose the point stabilizer G_1 leaves $\mathcal{P}(1)$ invariant, and acts transitively on $\mathcal{P}(1)$. Then $G = \hat{Z}_n G_1$ leaves $\mathcal{P}$ invariant, and acts transitively on $\mathcal{P}$. Let M be the kernel of the action of G on $\mathcal{P}$. Then $(\mathbf{K}_n, \mathcal{P})$ is a circulant (M, G)-homogeneous factorisation.

In the papers [8, 10], circulant homogeneous factorisations of complete digraphs $\mathbf{K}_n$ have been studied. The following theorem gives some basic properties of such factorisations.

Theorem 2.6. ([8, Theorem 4.1]) *Let $(\mathbf{K}_n, \mathcal{P})$ be a circulant (M, G)-homogeneous factorisation of index k. Then the following statements hold:*

(i) G is soluble and $G^{\mathcal{P}} \cong \mathbb{Z}_k$.

(ii) for each prime divisor r of n, $k|(r-1)$.

We list some results concerning the induced sub-factorisations in the following lemma, the proofs of these results can be found in [8, 9].

Lemma 2.7. *Let $(\mathbf{K}_n, \mathcal{P})$ be a circulant (M, G)-homogeneous factorisation of index k where $\mathcal{P} = \{P_1, \ldots, P_k\}$ and identify the vertex set of $\mathbf{K}_n$ with the regular group Z_n.*

(1) Let Δ be a block of G with order m where $m|n$. Then $\Delta = xZ_m$ is a coset of the subgroup Z_m, and the induced sub-factorisation $(\Delta, \mathcal{P}_\Delta)$ is a circulant $(M_\Delta^\Delta, G_\Delta^\Delta)$-homogeneous factorisation of index k.

(2) Let $\sigma \in G$ be a cyclic isomorphism which fixes the point 1 and maps $P_i \to P_{i+1}$ (reading the subscript modulo k). Let $\Delta = Z_m$ be a block of G. Then σ fixes Z_m setwise and $\sigma|_{Z_m}$ is a cyclic isomorphism of the induced sub-factorisation $(\Delta, \mathcal{P}_\Delta)$ where $\mathcal{P}_\Delta = \{P_1^\Delta, P_2^\Delta, \ldots, P_k^\Delta\}$. Moreover $\sigma|_{Z_m} : P_i^\Delta \to P_{i+1}^\Delta$.

(3) Let B be a minimal block of G such that $1 \in B$. Then $B = Z_p$ for some prime p and $\hat{Z}_p \lhd G_{Z_p}^{Z_p} \leqslant \mathrm{AGL}(1, p)$.

(4) Suppose $n = p^d (d \geqslant 2)$ is an odd prime power. Then $Z_p \subset Z_{p^2} \subset \cdots \subset Z_{p^{d-1}}$ is a block chain of G. Let G_1 be the point stabilizer of 1. Then G_1 maps elements of order p^i to elements of order p^i for $i = 1, \ldots, d$.

Proof. (1) Let B be a block of G such that $1 \in B$ and $|B| = m$. Consider the multiplications by the elements in B, we have $BB = B$. Thus $B = Z_m$ is the subgroup of order m of Z_n. Suppose next that Δ is a block of order m, and $x \in \Delta$. Then $\Delta = xZ_m$ as required. Hence M_Δ^Δ contains the regular subgroup Z_m. It follows from Lemma 2.4 that the induced sub-factorisation $(\Delta, \mathcal{P}_\Delta)$ is a circulant $(M_\Delta^\Delta, G_\Delta^\Delta)$-homogeneous factorisation of index k.

(2) Since σ fixes 1 and $1 \in \Delta$, σ fixes the block Δ. Since $P_i^\Delta = P_i \cap (\Delta \times \Delta)$, $\sigma|_\Delta : P_i^\Delta \to P_{i+1}^\Delta$. Therefore $\sigma|_{Z_m}$ is a cyclic isomorphism of the induced sub-factorisation $(\Delta, \mathcal{P}_\Delta)$.

(3) This is [8, Lemma 4.4].

(4) It follows from [8, Lemma 2.4] and Corollary 2.2 that $Z_p \subset Z_{p^2} \subset \cdots \subset Z_{p^{d-1}}$ is a block chain of G. For $i = 1, \ldots, d$, G_1 fixes block Z_{p^i} and so maps elements of order p^i to elements of order p^i. (See also [9, Lemma 4.3, Corollary 4.4].) $\square$

We also need the following useful lemma. Since the circulant homogeneous factorisation must be cyclic by Theorem 2.6 (1), the following lemma is a direct corollary of [7, Lemma 5.2].

Lemma 2.8. ([7, Lemma 5.2]) *Let $(\mathbf{K}_n, \mathcal{P})$ be a circulant (M, G)-homogeneous factorisation of index k. Let K be the kernel of the G-action on $\mathcal{P}$, and let $\mathcal{B}$ be a nontrivial G-invariant partition of $V(\mathbf{K}_n)$. Then each element of $G \setminus K$ fixes exactly one block of $\mathcal{B}$.*

We finish this section by giving the following easy proposition.

Proposition 2.9. *Suppose that $\mathcal{F} = (\mathbf{K}_p, \mathcal{P})$ is a circulant homogeneous factorisation of order p where p is an odd prime. Then $\mathcal{F}$ is normal.*

Proof. Let $G = \mathrm{Aut}(\mathcal{F})$ and let M be the kernel of G acting on $\mathcal{P}$. By Theorem 2.6 (1) G is solvable, and hence G is a primitive solvable c-group. By Corollary 2.2, $\hat{Z}_p \lhd G \leqslant \hat{Z}_p \rtimes Z_{p-1}$. Any element $\sigma \in G \setminus M$ is a cyclic isomorphism of $\mathcal{F}$ and hence $\mathcal{F}$ is normal by definition. □

3 Lexicographic product constructions

Suppose $(\mathbf{K}_n, \mathcal{P}, X, Y)$ is a circulant homogeneous factorisation of index k and let $\sigma \in Y$ be a cyclic isomorphism. Recall that we also refer to $(\mathbf{K}_n, \mathcal{P}, X, \sigma)$ as this factorisation.

Lemma 3.1. *Suppose that $\mathcal{F}_i = (\mathbf{K}_{n_i}, \mathcal{P}_i, M_i, \sigma_i)(i = 1, 2)$ are two circulant homogeneous factorisations of index k where σ_i are cyclic isomorphisms of $\mathcal{F}_i$. Let $\mathcal{P}_i = \{P_{i,1}, \ldots, P_{i,k}\}$ and suppose that $\sigma_i : P_{i,j} \mapsto P_{i,j+1}$ (reading the second subscript modulo k). Let $\mathcal{F}_1[\mathcal{F}_2] = (\mathbf{K}_{n_1 n_2}, \mathcal{P}, M, \sigma_1 \times \sigma_2)$ where $\mathcal{P} = \{P_{1,j}[P_{2,j}] | 1 \leqslant j \leqslant k\}$ and $M = M_2 \wr M_1$. Then $\mathcal{F}_1[\mathcal{F}_2]$ is a circulant homogeneous factorisation of index k and Z_{n_2} is a block of $\langle M, \sigma_1 \times \sigma_2\rangle$. Moreover, for any $j \in \{1, \ldots, k\}$, let $\mathrm{Cay}(Z_{n_1 n_2}, S_j)$ be the factor digraphs of $\mathcal{F}_1[\mathcal{F}_2]$. Then $sZ_{n_2} \subseteq S_j$ for any $s \in S_j \setminus Z_{n_2}$.*

Proof. Write the vertex set $\mathbf{K}_{n_1 n_2}$ as $V(\mathbf{K}_{n_1}) \times V(\mathbf{K}_{n_2})$. Then it is easy to see that $\mathcal{P} = \{P_{1,j}[P_{2,j}] | 1 \leqslant j \leqslant k\}$ is a partition of the arc set of $\mathbf{K}_{n_1 n_2}$.

Let $\Gamma_{i,j} (i = 1, 2, j = 1, \ldots, k)$ be the factors of $\mathcal{F}_i$. Then $\Gamma_{1,j}[\Gamma_{2,j}]$ are factors of $(\mathbf{K}_{n_1 n_2}, \mathcal{P})$. For each $j = 1, \ldots, k$, since $M_1 \leqslant \mathrm{Aut}(\Gamma_{1,j}), M_2 \leqslant \mathrm{Aut}(\Gamma_{2,j})$, the automorphism group of $\Gamma_{1,j}[\Gamma_{2,j}]$ contains $M_2 \wr M_1$ which is transitive on the vertex set. In addition, let $\langle x_i \rangle \subseteq M_i$ be the corresponding regular cyclic group on $V(\mathbf{K}_{n_i})$. Then $(1, \cdots, 1, x_2)x_1 (\in M_2 \wr M_1)$ generates a regular cyclic group on $V(\mathbf{K}_{n_1 n_2})$. Lastly, it is easy to check that the natural action of $\sigma_1 \times \sigma_2$ on $V(\mathbf{K}_{n_1}) \times V(\mathbf{K}_{n_2})$ maps $P_{1,j}[P_{2,j}]$ to $P_{1,j+1}[P_{2,j+1}]$. Therefore $(\mathbf{K}_{n_1 n_2}, \mathcal{P})$ is a circulant $(M, \sigma_1 \times \sigma_2)$ homogeneous factorisation of index k.

We may assume that the factor digraph $\Gamma_{1,j}[\Gamma_{2,j}]$ is a circulant $\mathrm{Cay}(Z_{n_1 n_2}, S_j)$ for $j = 1, \ldots, k$. For any $u \in V(\mathbf{K}_{n_1})$, let $B_u = \{(u, v) | v \in V(\mathbf{K}_{n_2})\}$. Then $\{B_u | u \in V(\mathbf{K}_{n_1})\}$ is a block system of $\langle M_2 \wr M_1, \sigma_1 \times \sigma_2\rangle$ acting on the vertex set $V(\mathbf{K}_{n_1}) \times V(\mathbf{K}_{n_2})$. By Lemma 2.7 (1), $\{B_u | u \in V(\mathbf{K}_{n_1})\} = \{xZ_{n_2} | x \in Z_{n_1 n_2}\}$, hence it is easy to deduce that $sZ_{n_2} \subseteq S_j$ for any $s \in S_j \setminus Z_{n_2}$. □

Lemma 3.2. *Let n, n_1, n_2 be positive integers such that $n = n_1 n_2$ and $n_1, n_2 \geqslant 2$. Let $\mathcal{F} = (\mathbf{K}_n, \mathcal{P}, X, \sigma)$ be a circulant homogeneous factorisation of index k where σ is a cyclic* *isomorphism fixing point 1 and let* $\mathcal{P} = \{P_1, \ldots, P_k\}$. *Let* $\Sigma_j = \mathrm{Cay}(Z_n, P_j(1))(j = 1, \ldots, k)$ *be the corresponding factor digraphs. Suppose that* Z_{n_2} *is a block of* $Y = \langle X, \sigma\rangle$ *and for any* $j \in \{1, \ldots, k\}$ *and any* $s \in P_j(1) \setminus Z_{n_2}$, $sZ_{n_2} \subseteq P_j(1)$. *Then there exists two circulant homogeneous factorisations* $\mathcal{F}_i$ *of order* $n_i, (i = 1, 2)$ *such that* $\mathcal{F} = \mathcal{F}_1[\mathcal{F}_2]$.

Proof. Let $\mathcal{F}_2 = (\mathbf{K}_{n_2}, \mathcal{P}_2, X^{Z_{n_2}}_{Z_{n_2}}, Y^{Z_{n_2}}_{Z_{n_2}})$ be the induced sub-factorisation on the block Z_{n_2} by $\mathcal{F}$ where $\mathcal{P}_2 = \{P_{2,j} | j = 1, \ldots, k\}$ and $P_{2,j} = P_j \cap (Z_{n_2} \times Z_{n_2})$. By Lemma 2.7 (1) (2), $\mathcal{F}_2$ is a circulant homogeneous factorisation of order n_2 and $\sigma|_{Z_{n_2}}$ is a cyclic isomorphism of $\mathcal{F}_2$.

Let $\bar{X}$ and $\langle\bar{\sigma}\rangle$ be the induced permutation groups of X and $\langle\sigma\rangle$ on the block system $Z_n/Z_{n_2} = \{gZ_{n_2} | g \in Z_n\}$ respectively. Then $\widehat{Z_n/Z_{n_2}} \leqslant \bar{X}$. Next we define an induced quotient factorisation $\mathcal{F}_1$ on this block system Z_n/Z_{n_2}. For $j = 1, \ldots, k$, let

$$P_{1,j}(1) = P_j(1) \setminus Z_{n_2} \text{ and } P_{2,j}(1) = P_j(1) \cap Z_{n_2}.$$

Then $P_j(1) = P_{1,j}(1) \cup P_{2,j}(1)$. First note that $\Gamma_{2,j} := \mathrm{Cay}(Z_{n_2}, P_{2,j}(1))(j = 1, \ldots, k)$ are the factor digraphs of $\mathcal{F}_2$. Suppose that $\sigma : P_j \to P_{j+1}$. Then $\sigma|_{Z_{n_2}} : P_{2,j}(1) \to P_{2,j+1}(1)$ and hence $\sigma|_{Z_{n_2}} : \Gamma_{2,j} \to \Gamma_{2,j+1}$.

On the other hand, it is proved in [4, Lemma 2.2] that $\mathrm{Cay}(Z_n, P_{1,j}(1)) = \Gamma_{1,j}[\overline{\mathbf{K}_{n_2}}]$ where $\Gamma_{1,j} = \mathrm{Cay}(Z_n/Z_{n_2}, \overline{P_{1,j}(1)})$ and $\overline{P_{1,j}(1)} = \{\bar{s} = sZ_{n_2} \mid s \in P_{1,j}(1)\}$. Then $\bar{\sigma} : \overline{P_{1,j}(1)} \to \overline{P_{1,j+1}(1)}$ and so $\bar{\sigma} : \Gamma_{1,j} \to \Gamma_{1,j+1}$. Moreover, $\Sigma_j = \Gamma_{1,j}[\Gamma_{2,j}]$ is a lexicographic product graph. Let $\mathcal{P}_1 = \{P_{1,j} | j = 1, \ldots, k\}$ where $P_{1,j}$ is the arc set of $\Gamma_{1,j}$. Write the vertex set of $\mathbf{K}_{n_1}$ as the quotient group Z_n/Z_{n_2} and let $\mathcal{F}_1 = (\mathbf{K}_{n_1}, \mathcal{P}_1)$. It is easy to deduce that $\mathcal{F}_1$ is a circulant $(\bar{X}, \bar{\sigma})$-homogeneous factorisation of order n_1 and $\mathcal{F} = \mathcal{F}_1[\mathcal{F}_2]$ as required. □

Corollary 3.3. *Let* $\mathcal{F} = (\mathbf{K}_{p^d}, \mathcal{P})$ *be a circulant homogeneous factorisation of index* k *where* p^d *is an odd prime power and let* $m < d$ *be a positive integer. For any* $j \in \{1, \ldots, k\}$, *let* $\mathrm{Cay}(Z_{p^d}, S_j)$ *be the factor digraphs of* $\mathcal{F}$. *Then the following two statements are equivalent.*

1. *There exist two circulant homogeneous factorisations* $\mathcal{F}_i (i = 1, 2)$, *such that* $\mathcal{F} = \mathcal{F}_1[\mathcal{F}_2]$ *where* $\mathcal{F}_2$ *is of order* p^m.
2. *For any* $j \in \{1, \ldots, k\}$ *and any* $s \in S_j$ *with* $o(s) > p^m$, $sZ_{p^m} \subseteq S_j$.

Proof. Just note that by Lemma 2.7 (4), Z_{p^m} is a block of $\mathrm{Aut}(\mathcal{F})$, the result follows from Lemma 3.1 and 3.2. □

Moreover we have the following useful remark.

Remark 3.4. Let $\mathcal{F} = (\mathbf{K}_n, \mathcal{P}, X, \sigma)$ be a circulant homogeneous factorisation of index k where σ is a cyclic isomorphism fixing point 1 and let $\mathcal{P} = \{P_1, \ldots, P_k\}$. Suppose further that $\mathcal{F} = \mathcal{F}_1[\mathcal{F}_2]$ is of lexicographic product form where $\mathcal{F}_2$ is of order n_2, and Z_{n_2} is a block of $\langle X, \sigma\rangle$. Then $\mathcal{F}_2$ is the sub-factorisation induced on the block Z_{n_2} by $\mathcal{F}$, and $\mathcal{F}_1$ is the quotient factorisation induced on the block system $\{gZ_{n_2} | g \in Z_n\}$ defined as in the proof of Lemma 3.2. In particular, with the notation in Lemma 3.2, the factor digraphs of $\mathcal{F}_1$ are the quotient cayley digraphs $\Gamma_{1,j} = \mathrm{Cay}(Z_n/Z_{n_2}, \overline{P_{1,j}(1)})$ and $\overline{P_{1,j}(1)} = \{\bar{s} = sZ_{n_2} \mid s \in P_{1,j}(1)\}$. Suppose that $\sigma : P_j \to P_{j+1}$. Then $\bar{\sigma} : \Gamma_{1,j} \to \Gamma_{1,j+1}$.

4 Cyclic isomorphisms and normal circulant homogeneous factorisations of order p^d

Let $(\mathbf{K}_{p^d}, \mathcal{P})$ be a circulant (X,Y)-homogeneous factorisation of index k, and suppose that X contains a regular cyclic subgroup Z_{p^d}. We always identify the vertex set of $\mathbf{K}_{p^d}$ with the group Z_{p^d}. The following lemma discusses the orders of the cyclic isomorphisms.

Lemma 4.1. *Let $\mathcal{F} = (\mathbf{K}_{p^d}, \mathcal{P})$ be a circulant (X,Y)-homogeneous factorisation of index k and let τ be a cyclic isomorphism. Then $k|(p-1)$ and $k|o(\tau)$. Moreover there exists a cyclic isomorphism $\sigma \in Y$ such that σ fixes 1 and $r|k$ for each prime divisor r of the order $o(\sigma)$.*

Proof. By Theorem 2.6, $k|(p-1)$. Since τ is transitive on $\mathcal{P}$ and $|\mathcal{P}| = k$, $k|o(\tau)$. By Theorem 2.6 again, there exists a cyclic isomorphism $\sigma \in Y \setminus X$ such that $Y = \langle X, \sigma\rangle$ and $Y^{\mathcal{P}} = \langle \sigma^{\mathcal{P}}\rangle$ is a cyclic group of order k. Since $\hat{Z}_{p^d} \leqslant X$ is vertex transitive, we may assume that σ fixes the point 1. Let q be a prime such that $(q,k) = 1$. For any positive integer m, $\sigma^{q^m} (\in Y)$ is still transitive on $\mathcal{P}$ and so is also a cyclic isomorphism of $\mathcal{F}$. Therefore replacing σ by some power of σ if necessary, we may assume $r|k$ for each prime divisor r of the order $o(\sigma)$. $\square$

Remark 4.2. Let $\mathcal{F} = (\mathbf{K}_{p^d}, \mathcal{P})$ be a circulant homogeneous factorisation of index k. For convenience we will assume from now on that, for a cyclic isomorphism σ of $\mathcal{F}$, σ fixes the point 1. Moreover if $r|k$ for each prime divisor r of the order $o(\sigma)$, then we say the cyclic isomorphism σ *satisfies Lemma 4.1.*

Suppose further that the circulant homogeneous factorisation $(\mathbf{K}_{p^d}, \mathcal{P})$ is normal, that is there exists a cyclic isomorphism σ normalising the regular subgroup Z_{p^d}. Hence σ can be viewed as an automorphism of the group Z_{p^d}, that is $\sigma \in \mathrm{Aut}(Z_{p^d}) \cong Z_{p^{d-1}(p-1)}$.

Construction 4.3. *Let p^d be an odd prime power such that $d \geqslant 1$ and $k \geqslant 2$ a positive integer such that $k|(p-1)$. Suppose that $\sigma \in \mathrm{Aut}(Z_{p^d})$ such that each prime divisor of $o(\sigma)$ divides k and $k|o(\sigma)$. In particular $(o(\sigma), p) = 1$. Since $\mathrm{Aut}(Z_{p^d}) \cong Z_{p^{d-1}} \times Z_{p-1}$, σ belongs to the cyclic subgroup of order $p-1$ and hence σ acts semiregularly on $Z_{p^d} \setminus \{1\}$. Let $\Delta_1, \ldots, \Delta_t$ be the orbits of σ on $Z_{p^d} \setminus \{1\}$. As $k|o(\sigma)$, σ^k divides each Δ_i into k orbits, say $\Delta_{i,1}, \Delta_{i,2}, \ldots, \Delta_{i,k}$. Relabeling if necessary, we may assume that $\Delta_{i,j}^{\sigma} = \Delta_{i,j+1}$ where $i \in \{1,\ldots,t\}, j \in \{1,\ldots,k\}$ (reading the second subscript modulo k). For any $j_1,\ldots,j_t \in \{1,\ldots,k\}$, let*

$$P_1(1) = \Delta_{1,j_1} \cup \Delta_{2,j_2} \cup \cdots \cup \Delta_{t,j_t}.$$

For $i = 2,\ldots,k$, let

$$P_i(1) = P_1(1)^{\sigma^{i-1}} = \Delta_{1,j_1+(i-1)} \cup \Delta_{2,j_2+(i-1)} \cup \cdots \cup \Delta_{t,j_t+i-1}.$$

Then $\mathcal{P}(1) = \{P_1(1), \ldots, P_k(1)\}$ is a σ-invariant partition of $Z_{p^d} \setminus \{1\}$. For each i, let $\Gamma_i = \mathrm{Cay}(Z_{p^d}, P_i(1))$, let P_i be the set of arcs of the circulant Γ_i, and let $\mathcal{P} = \{P_1, \ldots, P_k\}$.

By Lemma 2.5 (2) (taking G as $\langle Z_{p^d},\sigma\rangle$), $\mathcal{F}=(\mathbf{K}_{p^d},\mathcal{P})$ is a normal circulant homogeneous factorisation of index k and σ is a cyclic isomorphism of this factorisation. Denote $J=\{j_1,\ldots,j_t\}$. Since this construction $\mathcal{F}$ depends on the choice of σ and J, we also denote this construction by $\mathcal{F}_{\sigma,J}$.

Note that there do exist p,d,k,σ satisfying the conditions of Construction 4.3. Conversely, suppose that $(\mathbf{K}_{p^d},\mathcal{P})$ is a normal circulant homogeneous factorisation of index k. Then each factor digraph Γ_i is a circulant $\mathrm{Cay}(Z_{p^d},P_i(1))$ where $P_i(1)=\{\alpha\in Z_n|(1,\alpha)\in P_i\}$. Let $\sigma\in\mathrm{Aut}(Z_{p^d})$ be a cyclic isomorphism. As proved in Lemma 4.1, we may assume that each prime divisor of the order $o(\sigma)$ divides k. Therefore σ induces a transitive action on $\mathcal{P}(1)=\{P_1(1),\ldots,P_k(1)\}$ and σ^k fixes each $P_i(1)$ for $i=1,\ldots,k$. As defined above in Construction 4.3, let $\Delta_1,\ldots,\Delta_t$ be the orbits of σ on $Z_{p^d}\setminus\{1\}$. And for each i, σ^k divides Δ_i into k orbits, say $\Delta_{i,1},\Delta_{i,2},\ldots,\Delta_{i,k}$ such that $\Delta_{i,j}^{\sigma}=\Delta_{i,j+1}$. It is easy to show that there exists $j_1,\ldots,j_t\in\{1,\ldots,k\}$ such that $P_1(1)=\{\Delta_{1,j_1}\cup\Delta_{2,j_2}\cup\cdots\cup\Delta_{t,j_t}\}$ and $P_i(1)=P_1(1)^{\sigma^{i-1}}$ as in Construction 4.3. Therefore Construction 4.3 provides us with a method for constructing all normal circulant homogeneous factorisations of order p^d, we write this result in the following proposition.

Proposition 4.4. *Let $\mathcal{F}$ be a normal circulant homogeneous factorisations of order p^d and index k. And let $\sigma\in\mathrm{Aut}(Z_{p^d})$ be a cyclic isomorphism of $\mathcal{F}$ which satisfies Lemma 4.1. With above notation, there exists $J=\{j_1,\ldots,j_t\}$ such that $\mathcal{F}=\mathcal{F}_{\sigma,J}$ which is defined in Construction 4.3.*

We will need the following easy lemma for the proof of the main theorem.

Lemma 4.5. *Let p^d be an odd prime power such that $d\geqslant 2$, and let $\sigma\in\mathrm{Aut}(Z_{p^d})$ such that $o(\sigma)|(p-1)$. Let $Z_{p^i}\leqslant Z_{p^d}$ be a subgroup. Then*

(1) $o(\sigma|_{Z_{p^i}})=o(\sigma)$.

(2) Let $\bar\sigma\in\mathrm{Aut}(Z_{p^d}/Z_{p^i})$ be the automorphism of the quotient group induced by σ. Then $o(\sigma)=o(\bar\sigma)$.

Proof. (1) Suppose $\mathrm{Aut}(Z_{p^d})=\langle\mu\rangle\times\langle\gamma\rangle=Z_{p-1}\times Z_{p^{d-1}}$. Then μ is a product of $\frac{p^d-1}{p-1}$ disjoint $(p-1)$-cycles acting on $Z_{p^d}\setminus\{1\}$. By assumption $\sigma\in\langle\mu\rangle$ and hence $o(\sigma|_{Z_{p^i}})=o(\sigma)$.

(2) It is easy to check that $\mathrm{Aut}(Z_{p^d}/Z_{p^i})=\mathrm{Aut}(Z_{p^d})/\langle\gamma^{p^{d-1-i}}\rangle$ and so $o(\sigma)=o(\bar\sigma)$. □

Lemma 4.6. *Suppose that $(\mathbf{K}_{p^d},\mathcal{P})$ is a normal circulant homogeneous factorisation of index k and $\sigma\in\mathrm{Aut}(Z_{p^d})$ is a cyclic isomorphism satisfying Lemma 4.1. Let $\tau\in Aut(Z_{p^d})$ such that $k|o(\tau)$ and each prime divisor of $o(\tau)$ divides k. Then τ is a cyclic isomorphism of $(\mathbf{K}_{p^d},\mathcal{P})$ if and only if $o(\sigma)=o(\tau)$.*

Proof. By assumption we may assume that $o(\sigma)=mk,o(\tau)=nk$ where m,n are positive integers and each prime divisor of m and n divides k.

Suppose first that $o(\sigma)=o(\tau)$. Since $\mathrm{Aut}(Z_{p^d})$ is cyclic, $\tau=\sigma^l$ for some integer l such that $(l,o(\tau))=1$. This implies $(l,k)=1$. Since σ is transitive on $\mathcal{P}$ and $|\mathcal{P}|=k$, so is τ. Therefore τ is a cyclic isomorphism of $(\mathbf{K}_{p^d},\mathcal{P})$.

Conversely, suppose that τ is a cyclic isomorphism. Then τ is transitive on $\mathcal{P}$. Let $H = \langle \sigma, \tau \rangle$. Then H is transitive on $\mathcal{P}$. Note that $\sigma, \tau \in \mathrm{Aut}(Z_{p^d}) \cong Z_{(p-1)p^{d-1}}$, we have that $H = \langle \alpha \rangle$ where α generates H and $o(\alpha) = \frac{mk \cdot nk}{(m,n)k} = \frac{mn}{(m,n)}k$. Suppose that $m \neq n$, without loss of generality, we may assume $\frac{m}{(m,n)} \neq 1$ and $\tau = \alpha^{\frac{m}{(m,n)}}$. Since both α and τ are transitive on $\mathcal{P}$, $(k, \frac{m}{(m,n)}) = 1$. However by assumption each prime divisor of m divides k, a contradiction. Therefore $m = n$ and $o(\sigma) = o(\tau)$. $\square$

Lemma 4.7. *Suppose that $\mathcal{F} = (\mathbf{K}_{p^d}, \mathcal{P})$ is a normal circulant homogeneous factorisation of index k where $\mathcal{P} = \{P_1, .., P_k\}$. Let σ be a cyclic isomorphism (not required to be a group automorphism) such that $\sigma : P_i \to P_{i+1}$ (reading the subscript modulo k). Then there exists a cyclic isomorphism $\tau \in \mathrm{Aut}(Z_{p^d})$ satisfying Lemma 4.1 and $\tau : P_i \to P_{i+1}$ too.*

Proof. Since $\mathcal{F}$ is normal, by definition there exists a cyclic isomorphism $\tau \in \mathrm{Aut}(Z_{p^d})$ satisfying Lemma 4.1. Consider $Y = \langle \hat{Z}_{p^d}, \sigma, \tau \rangle$ and let X be the kernel of Y acting on $\mathcal{P}$. Let $\bar{\sigma} = \sigma^{\mathcal{P}}$ and $\bar{\tau} = \tau^{\mathcal{P}}$. Then $Y/X = \langle \bar{\sigma} \rangle = \langle \bar{\tau} \rangle$, and hence $\bar{\sigma} = \bar{\tau}^l$ for some positive integer. Replacing τ by τ^l, we have that $\tau : P_i \to P_{i+1}$ as required. $\square$

Lastly in this section we give examples of non-normal circulant homogeneous factorisations of order p^2. Note that there do exist p, k, σ_1, σ_2 satisfying the conditions of the following proposition.

Proposition 4.8. *Let p be an odd prime and $k(\geqslant 2)$ a positive integer such that $k|(p-1)$. Suppose further that $\sigma_1, \sigma_2 \in \mathrm{Aut}(Z_p)$ such that $o(\sigma_1) = k$ and $o(\sigma_2) = mk$ where $m \geqslant 2$ and each prime divisor of m divides k. By Construction 4.3, there exist circulant homogeneous factorisations $\mathcal{F}_i$ of index k and order p such that σ_i is a cyclic isomorphism of $\mathcal{F}_i$ respectively. Take $\mathcal{F} = \mathcal{F}_1[\mathcal{F}_2]$. Then $\mathcal{F}$ is a non-normal circulant homogenous factorisation of order p^2.*

Proof. By Lemma 3.1 $\mathcal{F}$ is a circulant homogeneous factorisation of order p^2. Suppose conversely that $\mathcal{F}$ is normal and $\sigma \in \mathrm{Aut}(Z_{p^2})$ is a cyclic isomorphism of $\mathcal{F}$ satisfying Lemma 4.1. By Remark 3.4, $\mathcal{F}_2$ is the sub-factorisation induced on the block Z_p. By Lemma 2.7 (2), $\sigma|_{Z_p} \in \mathrm{Aut}(Z_p)$ is also a cyclic isomorphism of $\mathcal{F}_2$. By Lemma 4.6, $o(\sigma|_{Z_p}) = o(\sigma_2)$. It follows from Lemma 4.5 that $o(\sigma_2) = o(\sigma)$. On the other hand, $\mathcal{F}_1$ is the quotient factorisation of the block system $\{gZ_p | g \in Z_{p^2}\}$ by Remark 3.4, and the induced $\bar{\sigma} \in \mathrm{Aut}(Z_{p^2}/Z_p)$ is also a cyclic isomorphism of $\mathcal{F}_1$. Still by Lemma 4.6 and Lemma 4.5, we have $o(\sigma_1) = o(\bar{\sigma}) = o(\sigma)$ which contradicts the fact that $o(\sigma_1) \neq o(\sigma_2)$. Therefore $\mathcal{F}$ is not normal. $\square$

5 Proof of Theorem 1.1

We first study the structures of circulant homogeneous factorisations of order p^d.

Lemma 5.1. *Let $(\mathbf{K}_{p^d}, \mathcal{P}, X, Y)$ be a circulant homogeneous factorisation of index k, and let $\sigma \in Y$ be a cyclic isomorphism. Let $\Gamma_i = \mathrm{Cay}(Z_{p^d}, S_i)$ be the factor digraphs where*

$1 \leqslant i \leqslant k$. *Then either* $\hat{Z}_{p^d} \lhd Y$ *and* $\sigma \in \mathrm{Aut}(Z_{p^d})$, *or for any* $i \in \{1, \ldots, k\}$ *and any* $s \in S_i$ *with* $o(s) = p^d$, $sZ_p \subset S_i$.

Proof. If $d = 1$ then Y is a soluble primitive c-group. It follows from Corollary 2.2 that $\hat{Z}_p \lhd Y \leqslant \mathrm{AGL}(1, p)$ and $\sigma \in \mathrm{Aut}(Z_p)$.

Next suppose that $d \geqslant 2$. Let X_1 and Y_1 be the point stabilizers of 1 in X and Y respectively. Then $Y_1 = \langle X_1, \sigma\rangle$ and suppose that $\sigma : S_i \to S_{i+1}(1 \leqslant i \leqslant k)$, reading the subscript modulo k.

First take any $g \in S_1$ with $o(g) = p^d$. Consider the orbital digraph Γ_g of Y with arc set $\Delta = (1, g)^Y$. Then $\Gamma_g = \mathrm{Cay}(Z_{p^d}, S_{1g} \cup S_{2g} \cup \cdots \cup S_{kg})$ where $S_{1g} = g^{X_1}$ is a subset of S_1 and $S_{ig} = (S_{1g})^{\sigma^{i-1}} \subseteq S_i$ for $i = 2, \ldots, k$. Since $o(g) = p^d$, Γ_g is a connected Y-arc-transitive circulant of order p^d. By Lemma 2.7 (4), $o(h) = p^d$ for any $h \in S_{1g} \cup S_{2g} \cup \cdots \cup S_{kg}$. In particular, Γ_g is not the complete graph $\mathbf{K}_{p^d}$. Applying Theorem 2.3, we have either $\hat{Z_{p^d}} \lhd Y$ and $\sigma \in \mathrm{Aut}(Z_{p^d})$ or $gZ_p \subset S_{1g} \cup S_{2g} \cup \cdots \cup S_{kg}$.

In the latter case, we claim that $gZ_p \subset S_{1g} \subset S_1$. Note that $\mathcal{B} = \{xZ_p \mid x \in Z_{p^d}\}$ is a block system of Y. Suppose $gZ_p \cap S_{ig} \neq \varnothing$ where $k \geqslant i \geqslant 2$. Then $g^{\sigma^{1-i}} Z_p \cap S_{1g} \neq \varnothing$. Since $S_{1g} = g^{X_1}$, there exists $x \in X_1$ such that $g^x \in g^{\sigma^{1-i}} Z_p$. It follows that $(gZ_p)^{\sigma^{1-i}} = (gZ_p)^x$ and so gZ_p is fixed by $x\sigma^{i-1}$. Since $gZ_p \neq Z_p$, $x\sigma^{i-1}$ fixes at least two blocks gZ_p and Z_p, contradicting Lemma 2.8. Hence $gZ_p \subset S_{1g}$.

For any $g_i \in S_i (i \geqslant 2)$ with $o(g_i) = p^d$, we consider the orbital digraph Γ_{g_i} of Y with arc set $\Delta = (1, g_i)^Y$ as well. Applying the same argument as above repeatedly, it is easy to deduce that either $\hat{Z}_{p^d} \lhd Y$ and $\sigma \in \mathrm{Aut}(Z_{p^d})$, or for any $i \in \{1, \ldots, k\}$, $sZ_p \subset S_i$ where $s \in S_i$ with $o(s) = p^d$. □

Proposition 5.2. *Let* $(\mathbf{K}_{p^d}, \mathcal{P}, X, Y)$ *be a circulant homogeneous factorisation of index* k, *and let* $\sigma \in Y$ *be a cyclic isomorphism. Let* $\Gamma_i = \mathrm{Cay}(Z_{p^d}, S_i)$ *be the factor digraphs. Then there exists* $n \in \{1, \ldots, d\}$ *such that* $\hat{Z_{p^n}} \lhd Y_{Z_{p^n}}^{Z_{p^n}}$ *and* $\sigma|_{Z_{p^n}} \in \mathrm{Aut}(Z_{p^n})$. *Moreover, for each* $i \in \{1, \ldots, k\}$ *and each* $s \in S_i$ *with* $o(s) > p^n$, *we have* $sZ_p \subset S_i$.

Proof. If $d = 1$, then take $n = 1$, the result follows from Corollary 2.2. Suppose next that $d \geqslant 2$. By Lemma 5.1, we may assume that for any $i \in \{1, \ldots, k\}$, $sZ_p \subset S_i$ where $s \in S_i$ with $o(s) = p^d$. By Lemma 2.7 (4), the subgroup $Z_{p^{d-1}}$ is a block of Y and the induced factorisation $\mathcal{F}_{d-1} = (\mathbf{K}_{p^{d-1}}, \mathcal{P}_{Z_{p^{d-1}}}, X_{Z_{p^{d-1}}}^{Z_{p^{d-1}}}, Y_{Z_{p^{d-1}}}^{Z_{p^{d-1}}})$ is a circulant homogeneous factorisation where $Y_{Z_{p^{d-1}}}^{Z_{p^{d-1}}} = \langle X_{Z_{p^{d-1}}}^{Z_{p^{d-1}}}, \sigma|_{Z_{p^{d-1}}}\rangle$. For any $i \in \{1, \ldots, k\}$, let $\Gamma_i(d-1) = \mathrm{Cay}(Z_{p^{d-1}}, S_i(d-1))$ be the corresponding factor digraphs of $\mathcal{F}_{d-1}$. Then $S_i(d-1) = \{s \in S_i | o(s) \leqslant p^{d-1}\}$. Applying Lemma 5.1 to $\mathcal{F}_{d-1}$, we deduce that either $\hat{Z_{p^{d-1}}} \lhd Y_{Z_{p^{d-1}}}^{Z_{p^{d-1}}}$ and $\sigma|_{Z_{p^{d-1}}} \in \mathrm{Aut}(Z_{p^{d-1}})$, or for any $i \in \{1, \ldots, k\}$, $sZ_p \subset S_i$ where $s \in S_i$ with $o(s) = p^{d-1}$.

Continuing in this fashion (note that $\hat{Z}_p \lhd Y_{Z_p}^{Z_p}$ by Lemma 2.7 (3)) we have that an integer n exists such that $n \in \{1, \ldots, d\}$, $\hat{Z_{p^n}} \lhd Y_{Z_{p^n}}^{Z_{p^n}}$, $\sigma|_{Z_{p^n}} \in \mathrm{Aut}(Z_{p^n})$ and for any $i \in \{1, \ldots, k\}$, $sZ_p \subset S_i$ where $s \in S_i$ with $o(s) > p^n$. □

Suppose $(\mathbf{K}_{p^d}, \mathcal{P}, X, Y)$ is a circulant homogeneous factorisation and denote the factor digraphs by $\Gamma_i = \mathrm{Cay}(Z_{p^d}, S_i)$. By Proposition 5.2, there exists $n \in \{1, \dots, d\}$ such that $\hat{Z_{p^n}} \lhd Y_{Z_{p^n}}^{Z_{p^n}}$ and for each $i \in \{1, \dots, k\}$ and each $s \in S_i$ with $o(s) > p^n$, $sZ_p \subset S_i$. Thus for any $i \in \{1, \dots, k\}$, we divide the Cayley subset S_i into the following two parts. Let

$$S_i^1 = \{s \in S_i \mid o(s) \leqslant p^n\} \text{ and } S_i^2 = \{s \in S_i \mid p^d \geqslant o(s) \geqslant p^{n+1}\}. \tag{1}$$

Then $S_i = S_i^1 \cup S_i^2$. Note that we set $S_i^2 = \varnothing$ if $n = d$.

Lemma 5.3. *Suppose that $\mathcal{F} = (\mathbf{K}_{p^d}, \mathcal{P}, X, Y)$ is a circulant homogeneous factorisation of index k with $d \geqslant 2$, and let $\sigma \in Y$ be a cyclic isomorphism. For any $i \in \{1, \dots, k\}$, let $\Gamma_i = \mathrm{Cay}(Z_{p^d}, S_i)$ be the factor digraphs, and suppose that $\sigma : \Gamma_i \to \Gamma_{i+1}$ (reading the subscript modulo k). Suppose further that n, S_i^1, S_i^2 are defined as above in (1). Then there exists a factorisation $\mathcal{F}' = (\mathbf{K}_{p^d}, \mathcal{P}')$ satisfying the following conditions.*

(i) $\mathcal{F}'$ is also a circulant (X, Y)-homogeneous factorisation of index k. Let $\Sigma_i = \mathrm{Cay}(Z_{p^d}, T_i)$ be the corresponding factor digraphs of $\mathcal{F}'$. Then $\sigma : \Sigma_i \to \Sigma_{i+1}$ is a cyclic isomorphism of $\mathcal{F}'$.

(ii) For any $i \in \{1, \dots, k\}$ and any $r \in T_i$ such that $o(r) > p$, we have $rZ_p \subseteq T_i$. (This implies $\mathcal{F}'$ is of lexicographic product form by Corollary 3.3.)

(iii) For any $i \in \{1, \dots, k\}$, let $T_i^2 = \{r \in T_i | o(r) > p^n\}$. Then $T_i^2 = S_i^2$.

(iv) For any $i \in \{1, \dots, k\}$, let $S_i^0 = \{s \in S_i \mid o(s) = p\}$ and $T_i^0 = \{r \in T_i \mid o(r) = p\}$. Then $T_i^0 = S_i^0$.

Proof. If $n = 1$, take $\mathcal{F}' = \mathcal{F}$. We next assume that $n \geqslant 2$. Let $X_n = X_{Z_{p^n}}^{Z_{p^n}}, Y_n = Y_{Z_{p^n}}^{Z_{p^n}}$ and let $\tau = \sigma|_{Z_{p^n}}$. Then $Y_n = \langle X_n, \tau \rangle$. By Lemma 2.7 the induced sub-factorisation (on the block Z_{p^n}) $\mathcal{F}_n = (Z_{p^n}, \mathcal{P}_{Z_{p^n}})$ is (X_n, Y_n)-homogeneous and τ is a cyclic isomorphism of $\mathcal{F}_n$.

By Proposition 5.2, $X_n = \hat{Z}_{p^n} \rtimes L$ where $L \leqslant \mathrm{Aut}(Z_{p^n})$ and fixes each $S_i^1 (1 \leqslant i \leqslant k)$ setwise. Moreover $\tau \in \mathrm{Aut}(Z_{p^n})$ and $\tau^k \in L$. Applying Lemma 2.7 to $\mathcal{F}_n$, $\mathcal{B}_1 = \{gZ_p \mid g \in Z_{p^n}\}$ forms a complete block system of Y_n on Z_{p^n}. As τ normalises L, $(gZ_p)^{L\tau} = (gZ_p)^{\tau L}$. For any $gZ_p \in \mathcal{B}_1$ such that $gZ_p \neq Z_p$, it follows from Lemma 2.8 that $g^{\tau^i} Z_p \neq (gZ_p)^x$ for any $x \in L$ and $1 \leqslant i < k$. Thus we may suppose that L has km orbits (for some integer $m \geqslant 1$) on $\mathcal{B}_1 \setminus \{Z_p\}$ and after relabelling if necessary, we may suppose that $\Delta_{j,1}, \Delta_{j,2}, \dots, \Delta_{j,k}$ are L-orbits on $\mathcal{B}_1 \setminus \{Z_p\}$ such that $\Delta_{j,i}^\tau = \Delta_{j,i+1}$ where $j \in \{1, \dots, m\}, i \in \{1, \dots, k\}$.

Let

$$R_i = \bigcup_{j=1}^{m} \Delta_{j,i} \text{ and } S_i^0 = \{s \in S_i^1 \mid o(s) = p\}.$$

Take $T_i^1 = S_i^0 \cup R_i$ and take $T_i^2 = S_i^2$. Let $T_i = T_i^1 \cup T_i^2$. Then $\{T_1, \dots, T_k\}$ is a partition of $Z_{p^d} \setminus \{1\}$. By Lemma 2.7 (4) and our construction it is easy to see that the vertex stabilizer

X_1 of X fixes each $T_i (i = 1, \dots, k)$ and $\sigma : T_i \mapsto T_{i+1}$. Define $\Sigma_i = \mathrm{Cay}(Z_{p^d}, T_i), (i = 1, \dots, k)$ and let P'_i be the set of arcs of Σ_i and let $\mathcal{P}' = \{P'_1, \dots, P'_k\}$. By Lemma 2.5 (2) $\mathcal{F}' = (\mathbf{K}_{p^d}, \mathcal{P}')$ is a circulant (X, Y)-homogeneous factorisation of index k, and σ is a cyclic isomorphism of $\mathcal{F}'$. The results (ii), (iii) and (iv) follow from the construction of $\mathcal{F}'$ obviously. □

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1: We proceed by induction on d. If $d = 1$, the result follows from Proposition 2.9. Assume inductively the result holds for circulant homogeneous factorisations of order $\leqslant p^{d-1}$ where $d \geqslant 2$.

Let $\mathcal{F} = (\mathbf{K}_{p^d}, \mathcal{P})$. Let $Y = \mathrm{Aut}(\mathcal{F})$ and X be the kernel of Y acting on $\mathcal{F}$. Then $\mathcal{F}$ is a circulant (X, Y)-homogeneous factorisation of index k. Let σ be a cyclic isomorphism satisfying Lemma 4.1 and let $\Gamma_i = \mathrm{Cay}(Z_{p^d}, S_i)$ be the factor digraphs $(1 \leqslant i \leqslant k)$. We assume that $\sigma : \Gamma_i \to \Gamma_{i+1}$ (reading the subscript modulo k). By Proposition 5.2, there exists $n \in \{1, \dots, d\}$ such that $\hat{Z_{p^n}} \lhd Y_{Z_{p^n}}^{Z_{p^n}}$ and $\sigma|_{Z_{p^n}} \in \mathrm{Aut}(Z_{p^n})$. If $n = d$ then $\mathcal{F}$ is normal as required. So we assume next that $n < d$.

By Proposition 5.2, for each $i \in \{1, \dots, k\}$ and each $s \in S_i$ with $o(s) > p^n$, $sZ_p \subset S_i$. Let $S_i^1 = \{s \in S_i \mid o(s) \leqslant p^n\}$ and $S_i^2 = \{s \in S_i \mid p^d \geqslant o(s) \geqslant p^{n+1}\}$. Then $S_i = S_i^1 \cup S_i^2$. By Lemma 5.3, there exists a circulant (X, Y) homogeneous factorisation $\mathcal{F}' = (\mathbf{K}_{p^d}, \mathcal{P}')$ such that σ is also a cyclic isomorphism of $\mathcal{F}'$. Let $\Sigma_i = \mathrm{Cay}(Z_{p^d}, T_i)$ be the corresponding factor digraphs of $\mathcal{F}'$. Then $\sigma : \Sigma_i \to \Sigma_{i+1}$, (reading the subscript modulo k). Setting $T_i^1 = \{s \in T_i \mid o(s) \leqslant p^n\}$ and $T_i^2 = \{s \in T_i \mid p^d \geqslant o(s) \geqslant p^{n+1}\}$, we have $T_i^2 = S_i^2$. Moreover, let $S_i^0 = \{s \in S_i \mid o(s) = p\}$ and $T_i^0 = \{r \in T_i \mid o(s) = p\}$. Then $T_i^0 = S_i^0$ by Lemma 5.3.

By Corollary 3.3 $\mathcal{F}'$ is of lexicographic product form. Suppose $l (\geqslant 1)$ is maximal such that $\mathcal{F}' = \mathcal{F}'_1[\mathcal{F}'_2]$ where $\mathcal{F}'_2$ is a circulant homogeneous factorisation of order p^l and $\mathcal{F}'_1$ is a circulant homogeneous factorisation of order p^{d-l}. Then $\mathcal{F}'_1$ can not be a lexicographic product of two smaller circulants homogeneous factorisations. By induction, $\mathcal{F}'_1$ is normal.

Suppose first that $l \geqslant n$. By Corollary 3.3, for each $i \in \{1, \dots, k\}$, $rZ_{p^l} \subseteq T_i$ for any $r \in T_i^2$ with $o(r) > p^l$. Since $T_i^2 = S_i^2 (1 \leqslant i \leqslant k)$, we have $sZ_{p^l} \subseteq S_i$ for any $s \in S_i^2$ with $o(s) > p^l$. By Corollary 3.3, $\mathcal{F} = \mathcal{F}_1[\mathcal{F}_2]$ where $\mathcal{F}_2$ is a circulant homogeneous factorisation of order p^l and $\mathcal{F}_1$ is a circulant homogeneous factorisation of order p^{d-l}.

Suppose next that $l < n$. We will show that $\mathcal{F}$ is normal in this case.

Consider first the induced sub-factorisation $\mathcal{A}'$ of $\mathcal{F}'$ on the block Z_{p^n}. Let $\tau = \sigma|_{Z_{p^n}} \in \mathrm{Aut}(Z_{p^n})$. Then it follows from Lemma 2.7 that $\tau : T_i^1 \to T_{i+1}^1$ is a cyclic isomorphism of $\mathcal{A}'$ and so $\mathcal{A}'$ is normal. Since σ satisfies Lemma 4.1 and $o(\tau) | o(\sigma)$, the cyclic isomorphism τ of $\mathcal{A}'$ satisfies Lemma 4.1, that is each prime divisor of the order $o(\tau)$ divides k.

On the other hand $\mathcal{F}' = \mathcal{F}'_1[\mathcal{F}'_2]$, and recall that for each $i \in \{1, \dots, k\}$, $rZ_{p^l} \subseteq T_i$ for any $r \in T_i$ with $o(r) > p^l$. By Remark 3.4 we may suppose the factor digraphs of $\mathcal{F}'_1$ are the quotient Cayley digraphs $\overline{\Sigma}_i = \mathrm{Cay}(Z_{p^d}/Z_{p^l}, \overline{T_i})$ and $\overline{T_i} = \{\overline{r} = rZ_{p^l} \mid r \in T_i, o(r) > p^l\}$ where $1 \leqslant i \leqslant k$. And the induced natural cyclic isomorphism $\bar{\sigma} : \overline{T}_i \to \overline{T}_{i+1}$.

By induction, we have seen that $\mathcal{F}'_1$ is normal, and so by Lemma 4.7 there exists a cyclic isomorphism $\bar{\psi} \in \mathrm{Aut}(Z_{p^d}/Z_{p^l})$ satisfying Lemma 4.1 such that $\bar{\psi} : \overline{T}_i \to \overline{T}_{i+1}$. By

Lemma 4.5, we may also assume that $\bar{\psi}$ is induced by $\psi \in \mathrm{Aut}(Z_{p^d})$ where $o(\psi) = o(\bar{\psi})$. In particular $\psi : T_i^2 \to T_{i+1}^2, (1 \leqslant i \leqslant k$, reading the subscript modulo k).

Since $l < n$, we have $rZ_{p^l} \subseteq T_i^1$ for any $r \in T_i^1$ with $o(r) > p^l$, and so the normal circulant homogeneous factorisation $\mathcal{A}' = \mathcal{A}'_1[\mathcal{A}'_2]$ is also of lexicographic product type where $\mathcal{A}'_1$ is of order p^{n-l} and $\mathcal{A}'_2$ is of order p^l. Again by Remark 3.4, $\mathcal{A}'_1$ is the quotient factorisation induced by $\mathcal{A}'$ on the block system $\{xZ_{p^l} | x \in Z_{p^n}\}$, and so the induced automorphism $\bar{\tau} \in \mathrm{Aut}(Z_{p^n}/Z_{p^l})$ is a cyclic isomorphism of $\mathcal{A}'_1$. As τ satisfies Lemma 4.1, by Lemma 4.5, $o(\bar{\tau}) = o(\tau)$. On the other hand, since $Z_{p^n}/Z_{p^l} < Z_{p^d}/Z_{p^l}$, we can also view $\mathcal{A}'_1$ as the induced sub-factorisation by $\mathcal{F}'_1$ on the block Z_{p^n}/Z_{p^l}, and so $\bar{\psi}|_{Z_{p^n}/Z_{p^l}}$ is also a cyclic isomorphism of $\mathcal{A}'_1$. This forces $o(\psi|_{Z_{p^n}}) = o(\bar{\psi}|_{Z_{p^n}/Z_{p^l}}) = o(\bar{\tau}) = o(\tau)$ by Lemma 4.5 and Lemma 4.6. Next we apply Lemma 4.6 to $\mathcal{A}'$, then $\psi|_{Z_{p^n}}$ is also a cyclic isomorphism of $\mathcal{A}'$ and hence $\psi|_{Z_{p^n}}$ induces a transitive action on $\{T_i^1 | i = 1, \dots, k\}$. Note that $\bar{\psi} : \overline{T}_i \to \overline{T}_{i+1} (1 \leqslant i \leqslant k)$ where $\overline{T_i} = \{\bar{r} = rZ_{p^l} \mid r \in T_i, o(r) > p^l\}$ and $l < n$. We deduce that $\psi|_{Z_{p^n}} : T_i^1 \to T_{i+1}^1$. Let $\mathcal{A}$ be the induced sub-factorisation on Z_{p^n} by $\mathcal{F}$. Applying Lemma 4.6 again to $\mathcal{A}$, we have $\psi|_{Z_{p^n}}$ is also a cyclic isomorphism of $\mathcal{A}$ and so induces a transitive action on $\{S_i^1 | i = 1, \dots, k\}$. By Lemma 5.3, $T_i^0 = S_i^0$. and so it is easy to deduce that $\psi|_{Z_{p^n}} : S_i^1 \to S_{i+1}^1$. It then follows from $S_i^2 = T_i^2$ that $\psi : S_i \to S_{i+1}$ and so $\psi \in \mathrm{Aut}(Z_{p^d})$ is a cyclic isomorphism of $\mathcal{F}$. Therefore $\mathcal{F}$ is normal. This completes the proof. □

References

[1] M. Giudici, C. H. Li, P. Potocnik and C. E. Praeger, Homogeneous factorisations of graph products, *Discrete Math.* 308 (2008) 3652–3667.

[2] C. D. Godsil, On the full automorphism group of a graph, *Combinatorica* 1 (1981), 243–256.

[3] I. Kovács, Classifying arc-transitive circulants, *J. Algebraic Combin.* 20 (2004), 353–358.

[4] C. H. Li, Finite abelian groups with the m-DCI property, *Ars Combin.* 51(1999), 77–88.

[5] C. H. Li, Permutation groups with a cyclic regular subgroup and arc transitive circulants, *J. Algebraic Combin.* 21 (2005), 131–136.

[6] C. H. Li and C. E. Praeger, Constructing homogeneous factorisations of complete graphs and digraphs, *Graphs Combin.* **18** (2002), no. 4, 757–761.

[7] C.H.Li and C.E.Praeger, On partitioning the orbitals of a transitive permutation group, *Trans. Amer. Math. Soc.* 355 (2003) 637–657.

[8] C. H. Li and C. E. Praeger, On finite permutation groups with a transitive cyclic subgroup, *J. Algebra* 349, (2012) 117–127.

[9] C. H. Li, S. Sun and J. Xu, Self-complementary circulants of prime-power order, *SIAM J. Discrete Math.* 28 (2014), no. 1, 8–17.

[10] C.E. Praeger, C.H.Li and L. Stringer, Common circulant cyclic homogeneous factorisations of complete graphs, *Discrete Math.* 309 (2009), 3006–3012.

[11] M. Y. Xu, Automorphism groups and isomorphisms of Cayley digraphs, *Discrete Math.* **182** (1998), 309–320.