Enumeration of Fuss-Schröder paths

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Abstract

In this paper we enumerate the number of (k, r)-Fuss-Schröder paths of type λ. Y. Park and S. Kim studied small Schröder paths with type λ. Generalizing the results to small (k, r)-Fuss-Schröder paths with type λ, we give a combinatorial interpretation for the number of small (k, r)-Fuss-Schröder paths of type λ by using Chung-Feller style. We also give two sets of sparse noncrossing partitions of $[2(k+1)n+1]$ and $[2(k+1)n+2]$ which are in bijection with the set of all small and large, respectively, (k, r)-Fuss-Schröder paths of type λ.

Keywords: Fuss-Schröder paths, type, sparse noncrossing partitions

1 Introduction

A *Dyck path* of length n is a lattice path from $(0, 0)$ to (n, n) using east steps $E = (1, 0)$ and north steps $N = (0, 1)$ such that it stays weakly above the diagonal line $y = x$. It is well-known that the number of all Dyck paths of length n is given by the famous Catalan numbers

$$\frac{1}{n+1}\binom{2n}{n}.$$

A *large Schröder path* of length n is a lattice path from $(0, 0)$ to (n, n) using east steps E, north steps N, and diagonal steps $D = (1, 1)$ staying weakly above the diagonal line $y = x$. The number of all large Schröder paths of length n is

$$\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k-1}\binom{n}{k}2^k.$$

A *small Schröder path* of length n is a large Schröder path of length n with no diagonal steps on the diagonal line. The number of all small Schröder paths of length n is the half of the number of all large Schröder paths of length n. Note that a Dyck path is a large Schröder path that does not use diagonal steps.

For a large Schröder path (and hence for a Dyck path), its *type* is the integer partition formed by the length of the maximal adjacent east steps. For example, the large Schröder path $NENNNNEEDNEEDNE$ has type $\lambda = (2, 2, 1, 1)$. The enumeration of Dyck paths by type was first done by Kreweras [4] in the context of noncrossing partitions while the enumeration of large Schröder paths by type was recently done by An, Eu, and Kim [1]. The number of small Schröder paths of given type is *not* the half of the number of large Schröder paths of the same type and it is enumerated by Park and Kim [6].

Now we introduce the Fuss analogue of Dyck and Schröder paths. Given a positive number k, a *k-Fuss-Catalan path* of length n is a path from $(0, 0)$ to (n, kn) using east steps E and north steps N such that it stays weakly above the line $y = kx$. The number of all k-Fuss-Catalan paths of length n is given by the Fuss-Catalan numbers

$$\frac{1}{kn+1}\binom{(k+1)n}{n}$$

and Armstrong [2] enumerates the number of k-Fuss-Catalan paths of given type.

For k, r $(1 \leqslant r \leqslant k)$, a *large (k, r)-Fuss-Schröder path* of length n is a path π from $(0, 0)$ to (n, kn) using east steps, north steps, and diagonal steps that satisfies the following two conditions:

(C1) the path π never passes below the line $y = kx$, and

(C2) the diagonal steps of π are only allowed to go from the line $y = kj + r - 1$ to the line $y = kj + r$, for some j.

The *type* of a large Fuss-Schröder path is determined by its east steps. A *small Fuss-Schröder path* is a large Fuss-Schröder path with no diagonal steps touching the line $y = kx$. The number of small (k, r)-Fuss-Schröder paths with fixed length and number of diagonal steps is independent of r and it is given by Eu and Fu [3].

We will provide the number of small (k, r)-Fuss-Schröder paths of given length and type. We also give two conjectures about Fuss-Schröder paths and sparse noncrossing partitions which might be useful for the formula for the number of large (k, r)-Fuss-Schröder paths of given type and length.

2 Dyck and Schröder paths by type

In this section, we introduce previous results about the numbers of Dyck and Schröder paths with given length and type.

Given an integer partition λ, we set $m_\lambda := m_1(\lambda)!m_2(\lambda)!m_3(\lambda)!\cdots$, where $m_i(\lambda)$ is the number of parts of λ equal to i. Note that m_λ here is not the monomial symmetric function. We use $|\lambda|$ for the sum of the parts of λ.

First, we begin with the number of Dyck paths of given type. Kreweras [4] shows the following theorem using recursions and Liaw et al. [5] give a bijective proof.

Theorem 1. *The number*

$$\frac{n(n-1)\cdots(n-\ell+2)}{m_\lambda}$$

is equal to the cardinality of:

1. *Dyck paths of length n with type $\lambda = (\lambda_1, \ldots, \lambda_\ell)$.*
2. *noncrossing partitions of $[n] := \{1, 2, \ldots, n\}$ with type λ.*

If diagonal steps are allowed, Theorem 1 is generalized to Schröder path cases. An, Eu, and Kim [1] enumerate the number of large Schröder paths of given length and type.

Theorem 2. *The number*

$$\frac{1}{|\lambda|+1}\binom{n}{|\lambda|}\binom{n+1}{\ell}\frac{\ell!}{m_\lambda}$$

is equal to the cardinality of:

1. *large Schröder paths of length n with type $\lambda = (\lambda_1, \ldots, \lambda_\ell)$.*
2. *sparse noncrossing set partitions of $[n + |\lambda| + 1]$ with arc type λ.*

It is well-known that the number of small Schröder paths of length n is the half of the number of large Schröder paths of length n. This is not the case when we count the number of small Schröder paths of fixed type. Park and Kim [6] provide the number of small Schröder paths of given length and type.

Theorem 3. *The number*

$$\frac{1}{n+1}\binom{n-1}{|\lambda|-1}\binom{n+1}{\ell}\frac{\ell!}{m_\lambda}$$

is equal to the cardinality of:

1. *small Schröder paths of length n with type $\lambda = (\lambda_1, \ldots, \lambda_\ell)$.*
2. *large Schröder paths of length n with type λ with no diagonal steps after the last north step.*
3. *connected sparse noncrossing set partitions of $[n + |\lambda| + 1]$ with arc type λ.*

3 Fuss-Schröder paths

In this section, we consider Fuss analogue of Dyck and Schröder paths of given length and type, i.e. Fuss-Catalan paths and Fuss-Schröder paths of fixed length n with type λ. Our goal is to enumerate the number of small Fuss-Schröder paths of length n with type λ.

First, we begin with the number of Fuss-Catalan paths of given type which is shown by Armstrong [2].

Theorem 4. *The number of k-Fuss-Catalan paths of type $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is*

$$\frac{(kn)!}{m_\lambda \cdot (kn + 1 - \ell)!}.$$

The following shows that the number of small (k, r)-Fuss-Schröder paths of given length and type is independent of r.

Lemma 5. *Let $\mathcal{A}_{n,\lambda}^{(k,r)}$ be the set of all small (k, r)-Fuss-Schröder paths of length n with type λ. Then there is a bijection between $\mathcal{A}_{n,\lambda}^{(k,i)}$ and $\mathcal{A}_{n,\lambda}^{(k,j)}$ for $1 \leqslant i < j \leqslant k$.*

Proof. Let π be a path in $\mathcal{A}_{n,\lambda}^{(k,i)}$. If π has a diagonal step D from the level $kt + i - 1$ to the level $kt + i$, then it can be decomposed into $\mu_1 D \mu_2 \omega N \mu_3$, where ω is the section of east steps on level $kt + j - 1$, N is the north step from the level $kt + j - 1$ to $kt + j$, μ_1 goes from $(0, 0)$ to the level $kt + i - 1$, μ_2 goes from the level $kt + i$ to the level $kt + j - 1$, and μ_3 goes from the level $kt + j$ to (n, kn). Let τ be the path $\mu_1 N \mu_2 \omega D \mu_3$. Then τ has a diagonal step from the level $kt + j - 1$ to the level $kt + j$ and its type is the same as the type of π. By applying the similar operations to all diagonal steps in π, we get a path in $\mathcal{A}_{n,\lambda}^{(k,j)}$. □

The following theorem provides the number of small (k, r)-Fuss-Schröder paths of fixed length and type.

Theorem 6. *The number of small (k, r)-Fuss-Schröder paths of type $\lambda = (\lambda_1, \dots, \lambda_\ell)$ for $1 \leqslant r \leqslant k$ is*

$$\binom{n-1}{|\lambda|-1}\binom{nk}{\ell-1}\frac{(\ell-1)!}{m_\lambda} = \frac{1}{nk+1}\binom{n-1}{|\lambda|-1}\binom{nk+1}{\ell}\frac{\ell!}{m_\lambda}.$$

Note that the number of small (k, r)-Fuss-Schröder paths of the case $k = 1$ is the number of small Schröder paths since paths staying above the line $y = x$ are considered. When $k = 1$, Theorem 6 gives

$$\binom{n-1}{|\lambda|-1}\binom{n}{\ell-1}\frac{(\ell-1)!}{m_\lambda} = \frac{1}{n+1}\binom{n-1}{|\lambda|-1}\binom{n+1}{\ell}\frac{\ell!}{m_\lambda}$$

which is Theorem 3. In the case of $|\lambda| = n$, the number of small (k, r)-Fuss-Schröder paths is the number of k-Fuss-Catalan paths since we count paths without using diagonal steps only. When $|\lambda| = n$, Theorem 6 gives

$$\binom{nk}{\ell-1}\frac{(\ell-1)!}{m_\lambda} = \frac{1}{nk+1}\binom{nk+1}{\ell}\frac{\ell!}{m_\lambda}.$$

which is the same as Theorem 4.

Proof of Theorem 6. Since the number of small (k, r)-Fuss-Schröder paths of length n with type λ $(1 \leqslant r \leqslant k)$ is independent of r (by Lemma 5), $r = k$ is assumed in this proof. To show the number of small (k, k)-Fuss-Schröder paths of type $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is $\frac{1}{nk+1}\binom{n-1}{|\lambda|-1}\binom{nk+1}{\ell}\frac{\ell!}{m_\lambda}$, we first consider all the paths from $(0, 0)$ to (n, kn) of type λ using east steps, north steps, and diagonal steps such that the diagonal steps are only allowed to go in the ikth rows for $2 \leqslant i \leqslant n$. Being such a lattice path, $n - |\lambda|$ rows are chosen from $2k, 3k, \ldots, nk$th rows for $n - |\lambda|$ diagonal steps, and ℓ lines are needed from $nk + 1$ horizontal lines for ℓ east runs (i.e. maximal consecutive east steps). Since the ℓ east runs are ordered in $\frac{\ell!}{m_\lambda}$ different ways, there are total $\binom{n-1}{|\lambda|-1}\binom{nk+1}{\ell}\frac{\ell!}{m_\lambda}$ lattice paths satisfying the above conditions.

We say a diagonal step of the lattice paths has a flaw if it is located below the line $y = kx + k$, and a north step has a flaw if it is below the line $y = kx$. North steps in $(ik - 1)$th, $(ik - 2)$th,..., $(ik - k + 1)$th rows between $y = kx$ and $y = kx + k$ are also considered as flawed steps if a flawed diagonal step is contained in the ikth row. Note that each lattice path has j flaws for some $j \in [1, kn]$ if and only if it is not a small (k, k)-Fuss-Schröder path. The following rules identify nk flawed paths corresponding to a small (k, k)-Fuss-Schröder path by increasing the number of flaws from the small (k, k)-Fuss-Schröder path one by one, and vice versa.

1. Increasing one flaw:

   (1a) See a given path as a sequence on $\{E, N, D\}^{(k+1)n}$, and take the leftmost east run which is located right before D, NE, any flawed step, or nothing (i.e. the east run is the last steps). Move the east run to the left on the sequence until the path get exactly one more flaw on condition that the east run does not pass D, NE, or any flawed step without an increase on the number of flaws. Example 7 is given for this basic case.

   (1b) If the leftmost east run passes D, NE, or any flawed step without an increase on the number of flaws as seen in Example 8, stop the moving right after passing the step, and do (1a) for the following east runs again.

   (1c) In the case that the leftmost east run in (1a) contains the first east step of the sequence, apply $(-1, -k)$-circular shifts (i.e. shifting each step of the path along a vector $(-1 \bmod n, -k \bmod kn)$ to get another path) repeatedly until obtaining a new path from $(0, 0)$ to (n, kn) such that the first step is not an east step and the kth row doesn't have a diagonal step. If there is no such path, the original path already has kn flaws and we may stop. See Example 9. To keep type λ for the new path, we consider two east runs of the original path are separately even if they are connected after circular shifts.

   (1d) If a new path is obtained in (1c), take the leftmost one (say the sth east run) and the rightmost one (say the $(s + t)$th east run) among east runs located right before D, NE, any flawed step, or on the same line with another east run.

Move the $(s+t-u)$th east run to the position of the $(s+t-u-1)$th east run $(0 \leqslant u \leqslant t-1)$ and, after that, apply (1a) to the sth east run. There are two examples for the circular shift in Example 10.

2. Decreasing one flaw:

   (2a) Note that there is always an east run right before the leftmost flawed step on a sequence. Move the east run to the right until it meets D, E, NE, or the second leftmost flawed step if those steps exist. Otherwise, the east run becomes the last steps of the sequence. Look at the second part in Example 7.

   (2b) If another east run is located right before flawless D or flawless NE which is prior the east run used in (2a) as Figure 2(b) in Example 8, move the another one to the right until meeting with next D, NE or a flawed step.

   (2c) If two east runs share the same horizontal line (i.e. east lines are back to back on a sequence) while any east run rearrangement is done, move the right east run to the right until it meets a next east run or a horizontal line $y = (w-1)k$ where wkth row is the highest row containing no diagonal step among ikth rows $(2 \leqslant i \leqslant n)$ as Example 10.

   (2d) If (2c) has ever been applied like both examples in Example 10, then we need a $(-w+1 \bmod n, (-w+1)k \bmod kn)$-circular shift. Otherwise, we may stop.

Therefore, there are $\frac{1}{nk+1}\binom{n-1}{|\lambda|-1}\binom{nk+1}{\ell}\frac{\ell!}{m_\lambda}$ small (k,k)-Fuss-Schröder paths of type λ. $\square$

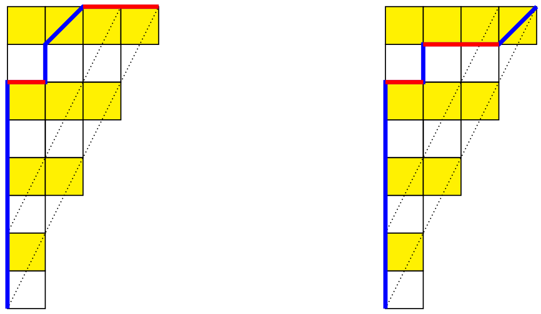
In the next four examples, we consider lattice paths from $(0,0)$ to (n,kn) of type λ using east steps, north steps, and diagonal steps such that the diagonal steps are only allowed to go in the ikth rows for $2 \leqslant i \leqslant n$ where $n=4$, $k=2$, and $\lambda=(2,1)$.

Example 7. The 2-east run in Figure 1(a) is the leftmost one among east runs located right before D, NE, any flaw step, or nothing. Move the 2-east run to the left one unit on the sequence so that the path gets one more flaw, a diagonal step below the line $y=2x+2$, as in Figure 1(b).

In the reverse, the leftmost (and unique) flawed step is a diagonal step, and there is a 2-east run right before the flawed step in Figure 1(b). Since there is no D, E, NE, or the second leftmost flawed step after the 2-east run, move the 2-east run to the right until it becomes the last steps of the sequence as in Figure 1(a).

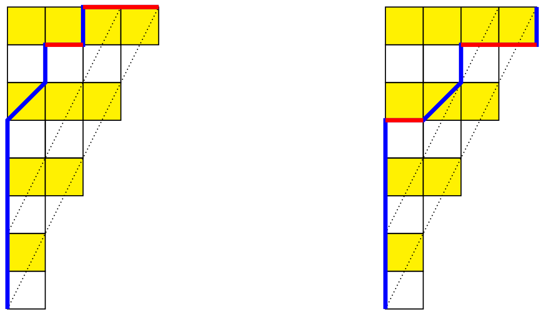
Example 8. The 1-east run in Figure 2(a) is the leftmost one among east runs right before D, NE, any flaw step, or nothing. Moving the 1-east run to the left, it passes D without increase on the number of flaws. Hence, stop the moving right after passing the diagonal step, and move the second leftmost one, the 2-east run, to the left one unit on the sequence so that the path gets one more flaw, a north step below the line $y=2x$, as in Figure 2(b).

Inversely, the leftmost flawed step is the highest north step in Figure 2(b), and move a 2-east run right before the flawed step until it becomes the last steps of the sequence.



(a) No flaws (b) 1 flaw

Figure 1: Rules for adding and subtracting flaws in Theorem 6(1a) and 6(2a).



(a) No flaws (b) 1 flaw

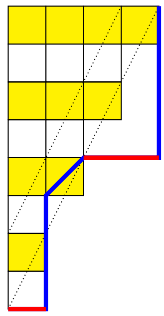
Figure 2: Rules for adding and subtracting flaws in Theorem 6(1b) and 6(2b).

After that, move the 1-east run, right before flawless D prior the 2-east run, to the right 2 units until meeting with NE as in Figure 2(a).

Example 9. If $(-1,-2)$-circular shift is applied to the path in Figure 3 only once, we get a path containing a diagonal step on the 2nd row. If $(-1,-2)$-circular shift is applied twice, a path having 2-east run as first steps is obtained. Three $(-1,-2)$-circular shifts generate a disconnected path not from $(0,0)$ to $(4,8)$. Hence, one more flaw cannot be added, and it is natural since the path in Figure 3 is already containing 8 flaws fully.

Example 10.

1. If $(-1,-2)$-circular shift is applied to the path in Figure 4(a) twice, we obtain a new path from $(0,0)$ to $(4,8)$ such that the first step is not an east step and the 2nd row doesn't have a diagonal step as in Figure 4(b). To keep type λ for the new path, 3 east steps on the same horizontal line after circular shifts are considered as two east runs, 2-east run and 1-east run in order, separately. Now in Figure 4(b),

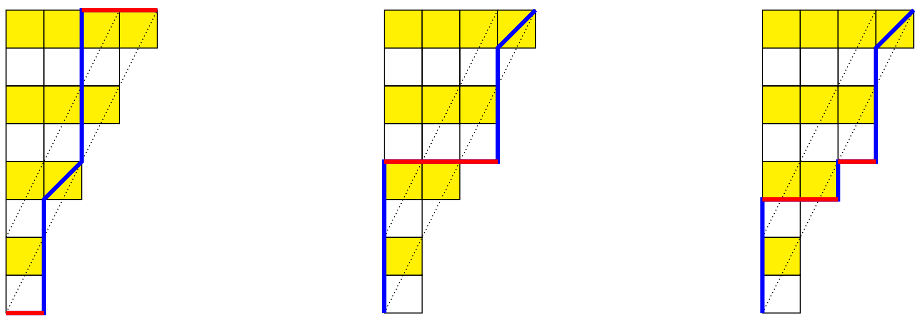


8 flaws

Figure 3: Rules for adding and subtracting flaws in Theorem 6(1c).

take the leftmost one (2-east run) and the rightmost one (1-east run) among east runs right before D, NE, any flawed step, or on the same line with another east run. The 1-east run is already in the (next) position of the 2-east run on the sequence, and the 2-east run is moved to the left one unit so that a path gets one more flawed north step as shown in Figure 4(c).

To the opposite direction, the 2-east run right before the leftmost flawed step in Figure 4(c) is moved to the right one unit on the sequence and meets another east run as in Figure 4(b). Since two east runs share the same horizontal line, and 6th ($w = 3$) row is the highest row containing no diagonal step among ikth rows ($2 \leqslant i \leqslant n$), and the right east run (1-east run) is already on the horizontal line $y = 4$, the only necessary thing to get a path in Figure 4(a) is a $(2, 4)$-circular shift.



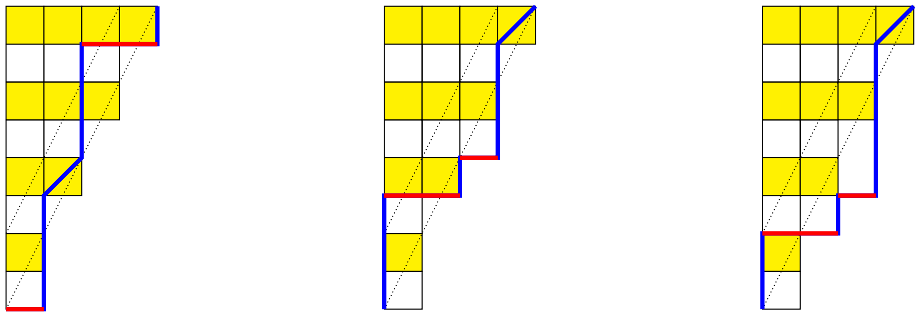
(a) 4 flaws (b) circular shifting (c) 5 flaws

Figure 4: Rules for adding and subtracting flaws in Theorem 6(1c) and 6(2c).

2. Similar to the previous case, $(-1, -2)$-circular shift is applied to the path in Figure 5(a) twice. Take the leftmost one (2-east run) and the rightmost one (1-east run) of the newly obtained path in Figure 5(b), and move the 1-east run to the position

of the 2-east run on the sequence. As the last step, the 2-east run is moved to the left one unit to get one more flawed north step as shown in Figure 5(c).

Conversely, the 2-east run right before the leftmost flawed step in Figure 5(c) is moved to the right one unit on the sequence and meets another east run. Then, move the right east run (1-east run) between two east runs sharing the same horizontal line to the right until it meets a horizontal line $y = 4$ as in Figure 5(b). Lastly, a $(2, 4)$-circular shift is applied to get a path in Figure 5(a).



(a) 5 flaws (b) circular shifting (c) 6 flaws

Figure 5: Rules for adding and subtracting flaws in Theorem 6(1d) and 6(2d).

4 Fuss-Schröder paths and noncrossing partitions

In this section, to extend the results of small (k, r)-Fuss-Schröder paths with type λ to large (k, r)-Fuss-Schröder paths with type λ, we introduce sparse noncrossing partitions which are in bijection with the set of (k, r)-Fuss-Schröder paths of type λ.

A *noncrossing partition* of $[n]$ is a pairwise disjoint subsets $B_1, B_2, \ldots, B_l$ of $[n]$ whose union is $[n]$ in which, if a and b belong to one block B_i and x and y to another block B_j, they are not arranged in the order $axby$. Note that, if the elements $1, 2, \ldots, n$ are equally-spaced dots on a horizontal line, and all the successive elements of the same block are connected by arc above the line, then no arches cross each other for a noncrossing partition of $[n]$. A noncrossing partition is called *sparse* if no two consecutive integers are in the same block. We give an order to the blocks by the order of the smallest element of each block. *Connected components* of a noncrossing partition of $[n]$ are $\{1, 2, \ldots, i_1\}, \{i_1 + 1, i_1 + 2, \ldots, i_2\}, \ldots, \{i_t + 1, i_t + 2, \ldots, n\}$ where i_1 is the greatest element of the block containing 1, i_2 is the greatest element of the block containing $i_1 + 1$, and so on. The *arc type* of a noncrossing partition is the integer partition obtained from the numbers of connected arcs.

In Figure 6, a noncrossing partition of $[10]$ is written by 4 ordered blocks. Two connected components are $\{1, 2, \ldots, 8\}$ and $\{9, 10\}$, and the arc type of the given noncrossing partition is $(3,2,1)$. Since 2 and 3 are in the same block, this partition is not a sparse noncrossing partition.

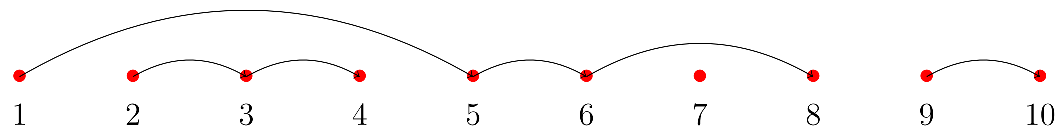


Figure 6: Noncrossing partition $(\{1,5,6,8\},\{2,3,4\},\{7\},\{9,10\})$ of $[10]$.

Before introducing a special noncrossing partition, we need more notations and labellings concerning large (k,k)-Fuss-Schröder paths. On a skew shape from $(0,0)$ to (n,kn), the $(n-i)k$th row is labeled by $i(k+1)+2$ for $i \geqslant 0$, and the horizontal line $y=(n-i)k+j$ is labeled by $i(k+1)+1-j$ if $0 \leqslant j < k$. See Figure 7 for example.

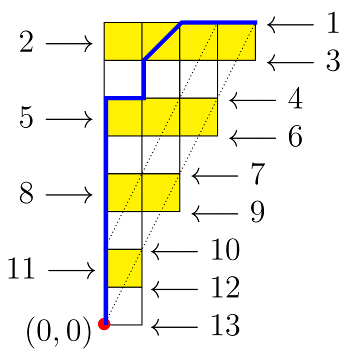


Figure 7: Labels when $n=4$ and $k=2$.

We can represent a large (k,k)-Fuss-Schröder path of length n as a sequence $s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n$ such that s_i's are the labels in which E and D steps located. In Figure 7, the sequence corresponding to the path is 1124. Then, we trace the sequence representation of a large (k,k)-Fuss-Schröder path as follows:

1. Start with two numbers 1 and 2 on a horizontal line, and read a sequence from s_1 to s_n.
2. For m consecutive east steps labeled by $s_j = s_{j+1} = \cdots = s_{j+m-1}$,
   (a) replace each number $i(> s_j)$ with $i + m(k+1)$, and
   (b) replace s_j with $s_j, s_j+1, s_j, s_j+2, s_j, \ldots, s_j, s_j+m(k+1), s_j$.
3. For a diagonal step labeled by s_j,
   (a) replace each number $i(> s_j)$ with $i+k+1$, and
   (b) replace s_j with $s_j, s_j+1, s_j, s_j+2, s_j, \ldots, s_j, s_j+(k+1), s_j$.

4. For a north step, do nothing.

Then the resulting sequence gives a sparse noncrossing partition of [2(k+1)n + 2] in which the elements at the number i positions are the elements of ith block. See Example 11.

Example 11. The sparse noncrossing partition corresponding to the large (k,k)-Fuss-Schröder path in Figure 7 is given in Figure 8.

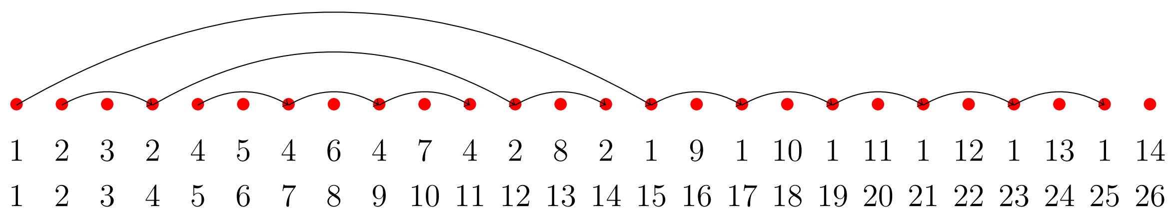


Figure 8: Sparse noncrossing partition $(\{1,15,17,19,21,23,25\},\{2,4,12,14\},\{3\},\{5,7,9,11\},\{6\},\{8\},\{10\},\{13\},\{16\},\{18\},\{20\},\{22\},\{24\},\{26\})$ of [26] with 14 blocks.

See the following conjecture about characteristics of sparse noncrossing partitions from large (k,k)-Fuss-Schröder paths.

Conjecture 12. The set of large (k,k)-Fuss-Schröder paths of type $\lambda = (\lambda_1,\dots,\lambda_\ell)$ of length n is in bijection with the set of sparse noncrossing partitions of $[2(k+1)n+2]$ with 2 connected components such that

1. the arc type is $((k+1)\lambda_1,(k+1)\lambda_2,\dots,(k+1)\lambda_\ell,(k+1)^{n-|\lambda|})$,
2. the set of $(i(k+1)+2)$th blocks consists of $n-|\lambda|$ blocks of arc type $k+1$ and $|\lambda|$ singleton blocks where $0 \leqslant i \leqslant n-1$, and
3. the set of last $t(k+1)$ blocks has at least $t(k-1)+1$ singleton blocks for $t \geqslant 1$.

Note that, if the second connected component is a singleton block, the corresponding large (k,k)-Fuss-Schröder path is a small (k,k)-Fuss-Schröder path. Hence, we have a next conjecture.

Conjecture 13. The set of small (k,k)-Fuss-Schröder paths of type $\lambda = (\lambda_1,\dots,\lambda_\ell)$ of length n is in bijection with the set of connected sparse noncrossing partitions of $[2(k+1)n+1]$ such that

1. the arc type is $((k+1)\lambda_1,(k+1)\lambda_2,\dots,(k+1)\lambda_\ell,(k+1)^{n-|\lambda|})$,
2. the set of $(i(k+1)+2)$th blocks consists of $n-|\lambda|$ blocks of arc type $k+1$ and $|\lambda|$ singleton blocks where $0 \leqslant i \leqslant n-1$, and
3. the set of last $t(k+1)$ blocks has at least $t(k-1)+1$ singleton blocks for $t \geqslant 1$.

However, it has a different correspondence when the first connected component is a singleton block. In this case, all the partitions count large (k, k)-Fuss-Schröder paths starting with a diagonal step.

For the future work, if Conjecture 12 is confirmed, it would be useful to find the number of large Fuss-Schröder paths of given type and length from the sparse noncrossing partitions bijectively. Furthermore, we want to enumerate the number of large Fuss-Schröder paths of given type and length directly from small Fuss-Schröder paths by using the connection between Conjecture 12 and Conjecture 13.

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