# Erdős-Ko-Rado type theorems for simplicial complexes

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**Abstract**

It is shown that every shifted simplicial complex $\Delta$ is EKR of type $(r, s)$, provided that the size of every facet of $\Delta$ is at least $(2s+1)r-s$. It is moreover proven that every $i$-near-cone simplicial complex is EKR of type $(r, i)$ if $\mathrm{depth}_{\mathbb{K}}\Delta \geqslant (2i+1)r-i-1$, for some field $\mathbb{K}$. Furthermore, we prove that if $G$ is a graph having at least $(2i+1)r-i$ connected components, including $i$ isolated vertices, then its independence simplicial complex $\Delta_G$ is EKR of type $(r, i)$. The results of this paper, generalize the main result of Frankl (2013).

**Keywords:** Erdős-Ko-Rado theorem, Simplicial complex, Matching number, Algebraic shifting, $i$-Near-Cone

## 1 Introduction and preliminaries

Throughout this paper, the set of positive integers $\{1, 2, \ldots\}$ is denoted by $\mathbb{N}$. For $m, n \in \mathbb{N}$ with $m \leqslant n$, the set $\{i \in \mathbb{N} : m \leqslant i \leqslant n\}$ is denoted by $[m, n]$; for $m = 1$, we also write $[n]$.

A family $\mathcal{A}$ of sets is *intersecting* if $A \cap B \neq \emptyset$, for every pair of sets $A, B \in \mathcal{A}$. A classical result in extremal set theory is the famous theorem of Erdős, Ko, and Rado [7]. It asserts that the maximum size of an intersecting family of $r$-subsets (i.e., subsets of size $r$) of $[n]$ is $\binom{n-1}{r-1}$, provided that $n \geqslant 2r$. In other words, the largest possible intersecting families of $r$-subsets of $[n]$ are the families of all $r$-subsets containing some fixed element of $[n]$, whenever $n \geqslant 2r$. For a nice survey on this topic we refer to [4].

Let $\mathcal{A}$ be a family of subsets of $[n]$. A subfamily $\mathcal{M}$ of $\mathcal{A}$ is called a *matching* if the elements of $\mathcal{M}$ are pairwise disjoint. The *matching number* of $\mathcal{A}$, denoted by $\nu(\mathcal{A})$ is defined to be the size of the largest matching of $\mathcal{A}$. Therefore, a nonempty family $\mathcal{A}$ is intersecting if and only if $\nu(\mathcal{A}) = 1$. In [6], Erdős proposed the following conjecture concerning the maximum size of a family of subsets of $[n]$ which has a given matching number.

**Conjecture 1.1.** *Assume that $\mathcal{A}$ is a family of $r$-subsets of $[n]$ with $\nu(\mathcal{A}) \leqslant s$. If $n \geqslant (s+1)r - 1$, then*

$$|\mathcal{A}| \leqslant \max\left\{\binom{(s+1)r-1}{r}, \binom{n}{r} - \binom{n-s}{r}\right\}. \tag{1}$$

For $s = 1$, Conjecture 1.1 is the same as Erdős-Ko-Rado Theorem (note that Conjecture 1.1 is trivially true for $s = 1$ and $n = r(s+1) - 1$). Also, for $r = 1$, the conjecture clearly holds. Erdős [6] proved that there exists an integer $n_0(r, s)$ such that the inequality (1) is valid for every family of $r$-subsets of $[n]$ with $\nu(\mathcal{A}) \leqslant s$, provided that $n \geqslant n_0(r, s)$. Frankl [8] proved that $n_0(r, s)$ can be chosen to be $(2s+1)r - s$. More explicit, he proved the following result.

**Theorem 1.2.** *[8, Theorem 1.1] Assume that $\mathcal{A}$ is a family $r$-subsets of $[n]$ with $\nu(\mathcal{A}) \leqslant s$. If $n \geqslant (2s+1)r - s$, then*

$$|\mathcal{A}| \leqslant \binom{n}{r} - \binom{n-s}{r}.$$

The aim of this paper is to extend Theorem 1.2 to some classes of simplicial complexes. Erdős-Ko-Rado type theorems for simplicial complexes have been studied by several authors; see e.g., [1], [2], [3], [12], [14]. Let us continue with some preliminaries from simplicial complexes.

A simplicial complex $\Delta$ on the ground set $V(\Delta) := [n]$ is a collection of subsets of $[n]$ that is closed under taking subsets; that is, if $F \in \Delta$ and $F' \subseteq F$, then also $F' \in \Delta$. Every element $F \in \Delta$ is called a *face* of $\Delta$. The *dimension* of a face $F$ is defined to be $\dim F := |F| - 1$. The *dimension* of $\Delta$, which is denoted by $\dim \Delta$, is defined to be $d - 1$, where $d = \max\{|F| : F \in \Delta\}$. A *facet* of $\Delta$ is a maximal face of $\Delta$ with respect to inclusion. Let $\mathcal{F}(\Delta)$ denote the set of facets of $\Delta$. It is clear that $\mathcal{F}(\Delta)$ determines $\Delta$. When $\mathcal{F}(\Delta) = \{F_1, \ldots, F_m\}$, we write $\Delta = \langle F_1, \ldots, F_m \rangle$ and say that $\Delta$ is *generated by* $F_1, \ldots, F_m$. A simplicial complex $\Delta$ is called *pure* if all facets of $\Delta$ have the same size. We say that $\Delta$ is a simplex if it consists of all subsets of $[n]$. Thus, a simplex has exactly one facet, namely $[n]$. The *link* of $\Delta$ with respect to a subset $F \subseteq [n]$, denoted by $\mathrm{lk}_\Delta F$, is the simplicial complex

$$\mathrm{lk}_\Delta F = \{G \subseteq [n] \setminus F : G \cup F \in \Delta\}.$$

Let $\Delta$ be a simplicial complex. The simplicial complex $\Delta^{(i)} := \{F \in \Delta : \dim F \leqslant i\}$ is the *$i$-skeleton* of $\Delta$. Also, the simplicial complex $\Delta^{[i]} := \langle F \in \Delta : \dim F = i \rangle$ is the *$i$-pure skeleton* of $\Delta$.

A face of $\Delta$ of size $r$ is an *$r$-face* of $\Delta$. We denote the number of $r$-faces of $\Delta$ by $f_r(\Delta)$.

**Note**. Many authors define an $r$-face to be a face with dimension $r$. We follow Swartz [13] and Woodroofe [14] in considering an $r$-face to be a face with size $r$ (rather than dimension $r$).

A simplicial complex $\Delta$ is said to be Cohen–Macaulay over a field $\mathbb{K}$ if for every $F \in \Delta$ and every $i$ less than $\dim(\mathrm{lk}_\Delta F)$, it holds that $\widetilde{H}_i(\mathrm{lk}_\Delta F; \mathbb{K}) = 0$, where $\widetilde{H}_i(\Delta; \mathbb{K})$ denotes the simplicial homology of $\Delta$ with coefficients in $\mathbb{K}$ (this definition of Cohen–Macaulayness coincides with the one which appears in the context of combinatorial commutative algebra via the Stanley–Reisner correspondence). It is well-known that every Cohen–Macaulay simplicial complex is pure (see for example [9, Lemma 8.1.5]). We say that a simplicial complex $\Delta$ is *sequentially Cohen–Macaulay over a field* $\mathbb{K}$ if every pure skeleton of $\Delta$ is Cohen–Macaulay over $\mathbb{K}$.

Woodroofe [14] defined the *depth* of $\Delta$ over $\mathbb{K}$ as

$$\mathrm{depth}_{\mathbb{K}}\Delta = \max\{\ell : \Delta^{(\ell)} \text{ is Cohen–Macaulay over } \mathbb{K}\}.$$

We note that $\mathrm{depth}_{\mathbb{K}}\Delta$ is at most the minimum facet dimension of $\Delta$, and equality holds if $\Delta$ is sequentially Cohen–Macaulay over $\mathbb{K}$.

Let $\Delta$ be a simplicial complex and $W$ be a subset of $V(\Delta)$. The *anti–star* of $\Delta$ with respect to $W$, denoted by $\mathrm{ast}_\Delta W$, is the simplicial complex

$$\mathrm{ast}_\Delta W = \{F \in \Delta : F \cap W = \emptyset\}.$$

When $W = \{v\}$ is a singleton, we sometimes write $\Delta \setminus v$ instead of $\mathrm{ast}_\Delta W$.

**Definition 1.3.** A simplicial complex $\Delta$ is said to be *EKR of type* $(r, s)$ if every family $\mathcal{A}$ of $r$-faces of $\Delta$ with $\nu(\mathcal{A}) \leqslant s$ satisfies the inequality

$$|\mathcal{A}| \leqslant f_r(\Delta) - \min f_r(\mathrm{ast}_\Delta W),$$

where the minimum is taken over all subsets $W$ of $V(\Delta)$ with $|W| = s$.

We restate Theorem 1.2 using the language of simplicial complexes:

**Theorem 1.4.** *Let $\Delta$ be a simplex on ground set $[n]$. If $n \geqslant (2s + 1)r - s$, then $\Delta$ is EKR of type $(r, s)$.*

In the subsequent sections, we extend Theorem 1.4 as follows. In Section 2, we focus on shifted simplicial complexes (see Definition 2.1). The main result of that section is Theorem 2.2, which asserts that every shifted simplicial complex $\Delta$ is EKR of type $(r, s)$, provided that the size of every facet of $\Delta$ is at least $(2s + 1)r - s$. Our main tool in the proof of Theorem 2.2 is (exterior) algebraic shifting. This method was used in [12] and [14] to prove other Erdős-Ko-Rado type theorems for simplicial complexes. In Section 3, we consider $i$-near-cone simplicial complexes (see Definition 3.1). We show in Theorem 3.3 that every $i$-near-cone simplicial complex is EKR of type $(r, i)$ if $\mathrm{depth}_{\mathbb{K}}\Delta \geqslant (2i + 1)r - i - 1$, for some field $\mathbb{K}$. In Section 4, we concentrate on the independence simplicial complexes associated to graphs. In Proposition 4.2, we characterize the graphs for which the independence complex $\Delta_G$ is $i$-near-cone and conclude in Corollary 4.4 that if $G$ is a graph having at least $(2i + 1)r - i$ connected components, including $i$ isolated vertices, then $\Delta_G$ is EKR of type $(r, i)$.

## 2 Shifted simplicial complexes

In this section, using shifting theory, we study the property of being EKR of type $(r, s)$ for shifted simplicial complexes. We first provide some definitions and basic facts from shifting theory.

**Definition 2.1.** A simplicial complex $\Delta$ on an ordered ground set $\{v_1, \ldots, v_n\}$ is *shifted* if whenever $\sigma$ is a face of $\Delta$ containing $v_i$, then $(\sigma \setminus \{v_i\}) \cup \{v_j\}$ is a face of $\Delta$ whenever $j < i$. An $r$-family $\mathcal{F}$ of subsets of $\{v_1, \ldots, v_n\}$ is shifted if it generates a shifted complex.

Consider the set of all simplicial complexes on a given ordered ground set $V$. A *shifting operation* associates to each such simplicial complex $\Delta$ a new simplicial complex $\operatorname{Shift}\Delta$ on the ground set $V$ such that

($S_1$) For every simplicial complex $\Delta$, $\operatorname{Shift}\Delta$ is a shifted complex.

($S_2$) If $\Delta$ is a shifted complex, then $\operatorname{Shift}\Delta = \Delta$.

($S_3$) $f_i(\operatorname{Shift}\Delta) = f_i(\Delta)$ for every $i$ with $0 \leqslant i \leqslant \dim\Delta + 1$.

($S_4$) If $\Gamma \subseteq \Delta$ are simplicial complexes, then $\operatorname{Shift}\Gamma \subseteq \operatorname{Shift}\Delta$.

If $\mathcal{A}$ is a family of $r$-subsets of $[n]$, then $\operatorname{Shift}\mathcal{A}$ is defined to be the set of $r$-faces of $\operatorname{Shift}(\Delta(\mathcal{A}))$, where $\Delta(\mathcal{A})$ is the simplicial complex generated by $\mathcal{A}$. In our proofs we need a shifting operation that satisfies the following extra property:

($S_5$) If $\mathcal{A}$ has the property that among every $s+1$ members of $\mathcal{A}$, there are two with nonempty intersection, then the same property holds for $\operatorname{Shift}\mathcal{A}$. In other words, $\nu(\operatorname{Shift}\mathcal{A}) \leqslant \nu(\mathcal{A})$.

Kalai proved (see [10, Theorem 6.3 and subsequent Remarks]) that a specific shifting operation called *exterior algebraic shifting* (with respect to a field $\mathbb{K}$) satisfies ($S_5$). We denote the exterior algebraic shift of $\Delta$, with respect to a field $\mathbb{K}$, by $\operatorname{Shift}_{\mathbb{K}}\Delta$. The precise definition of exterior algebraic shifting will not be important for us, but it can be found in [9] from a commutative algebraic perspective, or in [10] from a more elementary perspective.

We are now ready to prove that every shifted simplicial complex $\Delta$ is EKR of type $(r, s)$, provided that the size of facets of $\Delta$ are large enough. In the proof we do not rely on a specific shifting operator, but only require $(S_1, S_2, S_3, S_4, S_5)$ for the operator Shift.

**Theorem 2.2.** *Let $\Delta$ be a shifted complex having minimal facet size $k$. Then $\Delta$ is EKR of type $(r, s)$, for every natural numbers $r$ and $s$ with $k \geqslant (2s+1)r - s$.*

*Proof.* Suppose that the ordered set $\{v_1, \ldots, v_n\}$ is the ground set of $\Delta$. Let $\mathcal{A}$ be a family of $r$-faces of $\Delta$ with $\nu(\mathcal{A}) \leqslant s$. Note that by assumption $n \geqslant k > s$. Set $W = \{v_1, \ldots, v_s\}$. It is sufficient to prove that

$$|\mathcal{A}| \leqslant f_r(\Delta) - f_r(\operatorname{ast}_\Delta W).$$

Let $n$ be the size of the ground set of $\Delta$. We proceed by induction on $n$. Note that if $\Delta$ is a simplex, then Theorem 1.4 guarantees that the assertion is true. Also, there is nothing to prove if $r = 1$.

Thus assume that $\Delta$ is not a simplex and $r \geqslant 2$. It then follows from $(S_1)$, $(S_4)$ and $(S_5)$ that $\operatorname{Shift} \mathcal{A}$ is a shifted family of $r$-faces of $\operatorname{Shift} \Delta$ with $\nu(\operatorname{Shift} \mathcal{A}) \leqslant s$. On the other hand, $\Delta$ is a shifted complex, and thus we conclude from $(S_2)$ that $\operatorname{Shift} \Delta = \Delta$. This argument shows that $\operatorname{Shift} \mathcal{A}$ is in fact a shifted family of $r$-faces of $\Delta$, and $(S_3)$ shows that its size is $|\mathcal{A}|$. Let $\mathcal{A}_1$ be the set of all faces $\sigma \in \operatorname{Shift} \mathcal{A}$ with $v_n \in \sigma$ and set $\mathcal{A}_2 = \operatorname{Shift} \mathcal{A} \setminus \mathcal{A}_1$. We study the size of $\mathcal{A}_1$ and $\mathcal{A}_2$.

To study the size of $\mathcal{A}_1$, we set

$$\mathcal{A}'_1 = \{\sigma \setminus \{v_n\} : \sigma \in \mathcal{A}_1\}.$$

Hence $|\mathcal{A}'_1| = |\mathcal{A}_1|$. We claim that $\nu(\mathcal{A}'_1) \leqslant s$.

*Proof of the claim.* Suppose by contradiction that $\nu(\mathcal{A}'_1) \geqslant s + 1$. Hence, there exist $\sigma_1, \sigma_2, \ldots, \sigma_{s+1} \in \mathcal{A}_1$ such that $\sigma_i \cap \sigma_j = \{v_n\}$, for every pair of integers $i \neq j$. It follows that

$$|\sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_{s+1}| = (s + 1)r - s \leqslant k - sr \leqslant n - sr < n - s.$$

It follows that there exist $v_{\ell_1}, \ldots, v_{\ell_s}$ such that $v_{\ell_i} \notin \bigcup_{j=1}^{s+1} \sigma_j$ for every integer $i$ with $1 \leqslant i \leqslant s$. Set

$$\sigma'_j = (\sigma_j \setminus \{v_n\}) \cup \{v_{\ell_j}\},$$

for every $j$ with $1 \leqslant j \leqslant s$. Also, set $\sigma'_{s+1} = \sigma_{s+1}$. By the definition of shiftedness, we know that $\sigma'_j$ belongs to $\operatorname{Shift} \mathcal{A}$, for every $j$ with $1 \leqslant j \leqslant s + 1$. Further, $\sigma'_i \cap \sigma'_j = \emptyset$, for every $i \neq j$. This shows that $\sigma'_1, \ldots, \sigma'_{s+1}$ is a matching of $\operatorname{Shift} \mathcal{A}$. Hence, $\nu(\operatorname{Shift} \mathcal{A}) \geqslant s + 1$ which is a contradiction. This completes the proof of the claim.

It follows that $\mathcal{A}'_1$ is a family of $(r - 1)$-faces of $\operatorname{lk}_\Delta v_n$ with $\nu(\mathcal{A}'_1) \leqslant s$. Notice $\operatorname{lk}_\Delta v_n$ is a shifted complex on ground set $\{v_1, \ldots, v_{n-1}\}$. It is clear that the minimum facet size of $\operatorname{lk}_\Delta v_n$ is at least $k - 1$ and $k - 1 \geqslant (2s + 1)(r - 1) - s$. On the other hand, $r \geqslant 2$ and this shows that the size of the ground set of $\operatorname{lk}_\Delta v_n$ is at least $k - 1 \geqslant s + 1$. Thus, $W$ is a subset of the ground set of $\operatorname{lk}_\Delta v_n$. The induction hypothesis implies that

$$|\mathcal{A}_1| = |\mathcal{A}'_1| \leqslant f_{r-1}(\operatorname{lk}_\Delta v_n) - f_{r-1}(\operatorname{ast}_{\operatorname{lk}_\Delta v_n} W).$$

We now consider $\mathcal{A}_2$. It is clear that $\mathcal{A}_2$ is a a family of $r$-faces of $\Delta \setminus v_n$ with $\nu(\mathcal{A}_2) \leqslant s$. Since $\Delta$ is a shifted complex which is not a simplex, we conclude that the minimum facet size of $\Delta \setminus v_n$ is at least $k$. Hence the induction hypothesis implies that

$$|\mathcal{A}_2| \leqslant f_r(\Delta \setminus v_n) - f_r(\operatorname{ast}_{\Delta \setminus v_n} W).$$

Finally we have

$$\begin{aligned} |\mathcal{A}| = |\operatorname{Shift} \mathcal{A}| = |\mathcal{A}_1| + |\mathcal{A}_2| \leqslant f_{r-1}(\operatorname{lk}_\Delta v_n) - f_{r-1}(\operatorname{ast}_{\operatorname{lk}_\Delta v_n} W) \\ + f_r(\Delta \setminus v_n) - f_r(\operatorname{ast}_{\Delta \setminus v_n} W). \end{aligned}$$

The desired inequality now follows by observing that

$$f_{r-1}(\mathrm{lk}_\Delta v_n) + f_r(\Delta \setminus v_n) = f_r(\Delta)$$

and

$$f_{r-1}(\mathrm{ast}_{\mathrm{lk}_\Delta v_n} W) + f_r(\mathrm{ast}_{\Delta \setminus v_n} W) = f_r(\mathrm{ast}_\Delta W). \qquad \square$$

**Remark 2.3.** Let $\Delta$ be a shifted complex having minimal facet size $k \geqslant (2s+1)r - s$. Suppose that the ordered set $\{v_1, \ldots, v_n\}$ is the ground set of $\Delta$. Consider a family $\mathcal{A}$ of $r$-faces of $\Delta$ with $\nu(\mathcal{A}) \leqslant s$. It follows from the proof of Theorem 2.2 that

$$|\mathcal{A}| \leqslant f_r(\Delta) - f_r(\mathrm{ast}_\Delta W),$$

where $W = \{v_1, \ldots, v_s\}$. This observation will be used in the proof of Theorem 3.3.

## 3 Intersecting faces of $i$-near-cones

In this section we study the property of being EKR of type $(r, s)$ for $i$-near-cone simplicial complexes. The main result of this section is Theorem 3.3 which states that an $i$-near-cone is EKR of type $(r, i)$, provided that $\mathrm{depth}_{\mathbb{K}}\Delta$ is large enough. In the proof, we use exterior algebraic shifting and Theorem 2.2. Therefore in this section we fix a field $\mathbb{K}$ and by $\mathrm{Shift}\,\Delta$ we always mean the exterior algebraic shifting with respect to $\mathbb{K}$.

Let $\Delta$ be a simplicial complex and $v$ be a member of the ground set of $\Delta$. The complex $\Delta$ is a *near-cone* with respect to the *apex vertex* $v$ if $(\sigma \setminus \{w\}) \cup \{v\}$ is a face of $\Delta$ whenever $\sigma$ is a face of $\Delta$ and $w$ is an element of $\sigma$.

We next define the notion of $i$-near cone simplicial complexes. It was first introduced by Nevo [11]

**Definition 3.1.** Let $\Delta$ be a simplicial complex with some distinct elements $v_1, \ldots, v_i$ in the ground set of $\Delta$. The complex $\Delta$ is an *$i$-near-cone* with apex $v_1, \ldots, v_i$ if there exists a chain of nonempty simplicial complexes $\Delta(0) \supset \Delta(1) \supset \ldots \supset \Delta(i)$ with $\Delta(0) = \Delta$ such that whenever $1 \leqslant j \leqslant i$ the following conditions hold.

(i) $v_j \in \Delta(j-1)$,

(ii) $\Delta(j) = \mathrm{ast}_{\Delta(j-1)} v_j$,

(iii) $\Delta(j-1)$ is a near-cone with respect to $v_j$.

Let $\Delta$ be a simplicial complex with ground set $V(\Delta)$. For every subset $G \subseteq V(\Delta)$, the *restriction* of $\Delta$ to $G$ is defined to be the simplicial complex

$$\Delta_G = \{F \in \Delta : F \subseteq G\}.$$

The following proposition has a crucial role in the proof of Theorem 3.3.

**Proposition 3.2.** *Let $\Delta$ be an $i$-near cone. Assume that $v_1, \dots, v_i$ is the apex of $\Delta$ and set $F = \{v_1, \dots, v_i\}$. Suppose that the minimum size of facets of $\Delta$ and* $\mathrm{Shift}\,\Delta$ *is at least $k$ with $k \geqslant i$. View* $\mathrm{Shift}\,\Delta$ *as having ordered ground set $\{u_1, \dots, u_n\}$, and view* $\mathrm{Shift}(\mathrm{ast}_\Delta F)$ *as having ordered ground set $G = \{u_{i+1}, \dots, u_n\}$. For $r \leqslant k - 2i$,*

$$f_r((\mathrm{Shift}\,\Delta)_G) = f_r(\mathrm{Shift}(\mathrm{ast}_\Delta F)).$$

*Proof.* Consider a face $\sigma \in (\mathrm{Shift}\,\Delta)_G$, with $|\sigma| = r$. Since the minimum facet size of $\mathrm{Shift}\,\Delta$ is at least $k$ and $r \leqslant k - 2i < k - i$, we conclude that $\mathrm{Shift}\,\Delta$ has a face of size $r + i$ containing $\sigma$. It then follows from the definition of shiftedness that

$$\sigma \cup \{u_1, \dots, u_i\} \in \mathrm{Shift}\,\Delta.$$

Thus $\sigma \in \mathrm{lk}_{\mathrm{Shift}\,\Delta}\{u_1, \dots, u_i\}$. This shows that

$$f_r((\mathrm{Shift}\,\Delta)_G) \leqslant f_r(\mathrm{lk}_{\mathrm{Shift}\,\Delta}\{u_1, \dots, u_i\}).$$

The converse inequality is trivial, because

$$\mathrm{lk}_{\mathrm{Shift}\,\Delta}\{u_1, \dots, u_i\}) \subseteq (\mathrm{Shift}\,\Delta)_G.$$

Hence we conclude that

$$f_r((\mathrm{Shift}\,\Delta)_G) = f_r(\mathrm{lk}_{\mathrm{Shift}\,\Delta}\{u_1, \dots, u_i\}). \qquad (\dagger)$$

Now consider a face $\tau \in \mathrm{ast}_\Delta F$ with $|\tau| = r$. We use the notation from Definition 3.1 to prove that $\tau \cup F \in \Delta$. Since the minimal facet size of $\Delta$ is at least $k \geqslant r + 2i > r + i$, there exists $w \in V(\Delta) \setminus (\tau \cup F)$ such that $\tau \cup \{w\} \in \Delta$. It is clear from Definition 3.1 that $\tau \cup \{w\} \in \Delta(i) \subset \Delta(i-1)$. Since $\Delta(i-1)$ is a near-cone with respect to $v_i$, we conclude that

$$\tau \cup \{v_i\} = ((\tau \cup \{w\}) \setminus \{w\}) \cup \{v_j\} \in \Delta(i-1) \subseteq \Delta.$$

Let $j$ be the least integer such that $\tau \cup \{v_j, \dots, v_i\} \in \Delta$. We should prove that $j = 1$. Assume by contradiction that $j > 1$. Again, since the minimal facet size of $\Delta$ is at least $k \geqslant r + 2i > r + i$, there exists $x \in V(\Delta) \setminus (\tau \cup F)$ such that

$$\tau \cup \{v_j, \dots, v_i\} \cup \{x\} \in \Delta.$$

It is clear that

$$\tau \cup \{v_j, \dots, v_i\} \cup \{x\} \in \Delta(j-1) \subset \Delta(j-2).$$

Since $\Delta(j-2)$ is a near-cone with respect to $v_{j-1}$, we conclude that

$$\tau \cup \{v_{j-1}, v_j, \dots, v_i\} = ((\tau \cup \{v_j, \dots, v_i, x\}) \setminus \{x\}) \cup \{v_{j-1}\} \in \Delta(j-2) \subseteq \Delta,$$

which contradicts the choice of $j$. Therefore $\tau \cup F \in \Delta$, which yields that $\tau \in \mathrm{lk}_\Delta F$. This shows that

$$f_r(\mathrm{ast}_\Delta F) \leqslant f_r(\mathrm{lk}_\Delta F).$$

The converse inequality is trivial, because $\mathrm{lk}_\Delta F \subseteq \mathrm{ast}_\Delta F$. Hence, using $S_3$, we conclude that

$$f_r(\mathrm{Shift}(\mathrm{ast}_\Delta F)) = f_r(\mathrm{ast}_\Delta F) = f_r(\mathrm{lk}_\Delta F) = f_r(\mathrm{Shift}(\mathrm{lk}_\Delta F)). \qquad (\ddagger)$$

On the other hand, we know from [12, Proposition 3.6] that

$$f_r(\mathrm{lk}_{\mathrm{Shift}\,\Delta}\{u_1, \dots, u_i\}) = f_r(\mathrm{Shift}(\mathrm{lk}_\Delta F)).$$

The assertion now follows from the above equality together with equalities † and ‡. ☐

We are now ready to prove the main result of this section.

**Theorem 3.3.** *Let $r$ and $i$ be positive integers and $\Delta$ be an $i$-near-cone with*

$$\mathrm{depth}_{\mathbb{K}}\Delta \geqslant (2i+1)r - i - 1.$$

*Then $\Delta$ is EKR of type $(r, i)$.*

*Proof.* The case $r = 1$ is trivial. Hence, suppose that $r \geqslant 2$. Assume that $v_1, \dots, v_i$ is the apex of $\Delta$ and set $F = \{v_1, \dots, v_i\}$. Consider a family $\mathcal{A}$ of $r$-faces of $\Delta$ with $\nu(\mathcal{A}) \leqslant s$. It suffices to prove that $|\mathcal{A}| \leqslant f_r(\Delta) - f_r(\mathrm{ast}_\Delta F)$. In order to do this, we use algebraic shifting. Consider the simplicial complex $\mathrm{Shift}\,\Delta$ and assume that the ordered set $\{u_1, \dots, u_n\}$ is its ground set. It follows from $(S_5)$ that $\mathrm{Shift}\,\mathcal{A}$ a family of $r$-faces of $\mathrm{Shift}\,\Delta$ with $\nu(\mathrm{Shift}\,\mathcal{A}) \leqslant s$. It also follows from $(S_3)$ that its size is $|\mathrm{Shift}\,\mathcal{A}| = |\mathcal{A}|$. Let $k$ be the minimum size of facets of $\Delta$ and $\mathrm{Shift}\,\Delta$. By [5, Corollary 4.5] and the definition of depth, we conclude that

$$k \geqslant \mathrm{depth}_{\mathbb{K}}\Delta + 1 \geqslant (2i+1)r - i > r + 2i.$$

Set $W = \{u_1, \dots, u_i\}$ and $G = \{u_{i+1}, \dots, u_n\}$. It follows from $(S_3)$, Remark 2.3 and Proposition 3.2 that

$$\begin{aligned}|\mathcal{A}| = |\mathrm{Shift}\,\mathcal{A}| \leqslant f_r(\mathrm{Shift}\,\Delta) - f_r(\mathrm{ast}_{\mathrm{Shift}\,\Delta} W) &= f_r(\mathrm{Shift}\,\Delta) - f_r((\mathrm{Shift}\,\Delta)_G) \\ &= f_r(\Delta) - f_r(\mathrm{ast}_\Delta F).\end{aligned}$$

☐

The following corollary is an immediate consequence of Theorem 3.3 and proves that every sequentially Cohen-Macaulay $i$-near-cone $\Delta$ is EKR of type $(r, i)$, provided that the size of every facet of $\Delta$ is large enough. Note that if $\Delta$ is sequentially Cohen-Macaulay over $\mathbb{K}$ then $\mathrm{depth}_{\mathbb{K}}\Delta$ is the minimum facet dimension of $\Delta$.

**Corollary 3.4.** *Let $r$ and $i$ be positive integers and $\Delta$ be a sequentially Cohen-Macaulay $i$-near-cone having minimal facet size $k \geqslant (2i+1)r - i$. Then $\Delta$ is EKR of type $(r, i)$.*

## 4 Simplicial complexes associated to graphs

Let $G$ be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, the *neighbor set* of $v$ is $N_G(v) = \{u \mid \{u,v\} \in E(G)\}$ and We set $N_G[v] = N_G(v) \cup \{v\}$ and call it the *closed neighborhood* of $v$. The *degree* of $v$, denoted by $\deg_G(v)$ is the size of $N_G(v)$. The *independence simplicial complex* of $G$ is defined by

$$\Delta_G = \{A \subseteq V(G) \mid A \text{ is an independent set in } G\}.$$

We recall that $A \subseteq V(G)$ is an *independent set* in $G$ if none of its elements are adjacent. We say that a graph $G$ is sequentially Cohen-Macaulay (resp. near-cone, $i$-near-cone) if its independence simplicial complex $\Delta_G$ is sequentially Cohen-Macaulay (resp. near-cone, $i$-near-cone). In this section, we characterize the $i$-near-cone graphs. We first consider the near-cone case. The following lemma says that a graph $G$ with $n$ vertices is near cone if and only if it has a vertex $v$ such that every neighborhood of $v$ has degree $n-1$

**Lemma 4.1.** *A graph $G$ is near-cone with apex $v \in V(G)$ if and only if for every $u \in N_G(v)$, we have $N_G[u] = V(G)$.*

*Proof.* Notice that $\Delta_G$ is near-cone with apex $v$ if and only if $(A \setminus \{w\}) \cup \{v\}$ is an independent set of $G$, for every independent set $A \in \Delta_G$ and every vertex $w \in A$. This equivalent to say that $v$ is not adjacent to any vertex of $A \setminus \{w\}$, for every independent set $A \in \Delta_G$ and every vertex $w \in A$. This means that $v$ is not adjacent to any vertex of $A$, for every independent set $A \in \Delta_G$ with $|A| \geqslant 2$. Therefore, $\Delta_G$ is near-cone with apex vertex $v$ if and only if for every vertex $u \in N_G(v)$, the largest independent set of $G$, containing $u$ is the singleton $\{u\}$, i.e., $N_G[u] = V(G)$. □

The following proposition is an immediate consequence of Lemma 4.1 and characterizes $i$-near-cone graphs. Recall that for every graph $G$ and every vertex $v \in V(G)$, the graph $G \setminus v$ is obtained from $G$ by deleting the vertex $v$ and every edge adjacent to $v$.

**Proposition 4.2.** *A graph $G$ is an $i$-near-cone if and only if there are distinct vertices $v_1, \ldots, v_i \in V(G)$ and a chain of subgraphs $G(0) \supset G(1) \supset \cdots \supset G(i)$ with $G(0) = G$ such that whenever $1 \leqslant j \leqslant i$ the following conditions hold.*

*(i) $v_j$ is a vertex of $G(j-1)$,*

*(ii) $G(j) = G(j-1) \setminus v_j$,*

*(iii) For every $u \in N_{G(j-1)}(v_j)$, we have $N_{G(j-1)}[u] = V(G(j-1))$.*

*In particular, $G$ is an $i$-near-cone if it has $i$ isolated vertices.*

For a graph $G$, we denote the minimal facet size of $\Delta_G$ by $\operatorname{minind}(G)$. It is clear that $\operatorname{minind}(G)$ is equal to the minimum size of maximal independent sets of $G$. As an immediate consequence of Corollary 3.4 and Proposition 4.2, we conclude the following result.

**Corollary 4.3.** *Let $r$ and $i$ be positive integers and $G$ be a sequentially Cohen-Macaulay graph having $i$ isolated vertices. If* $\mathrm{minind}(G) \geqslant (2i+1)r - i$, *then $\Delta_G$ is EKR of type $(r, i)$.*

A list of sequentially Cohen-Macaulay graphs can be found in [14, Page 1224–1225]. In particular every chordal graph (i.e., the graph which has no induced cycle of length at least 4) is sequentially Cohen-Macaulay. Also, if every connected component of $G$ is sequentially Cohen-Macaulay, then $G$ is sequentially Cohen-Macaulay too.

Let $\Delta_1$ and $\Delta_2$ be two simplicial complexes with disjoint vertex sets. The *join* of $\Delta_1$ and $\Delta_2$ is defined to be

$$\Delta_1 * \Delta_2 = \{F \cup G : F \in \Delta_1 \text{ and } G \in \Delta_2\}.$$

It is know by [14, Lemma 2.12] that for every field $\mathbb{K}$, we have $\mathrm{depth}_{\mathbb{K}}\Delta_1 * \Delta_2 = \mathrm{depth}_{\mathbb{K}}\Delta_1 + \mathrm{depth}_{\mathbb{K}}\Delta_2 + 1$. On the other hand one can easily see that if $G$ is a graph with connected components $G_1, \ldots, G_t$, then $\Delta_G = \Delta_{G_1} * \cdots * \Delta_{G_t}$. This yields that for every field $\mathbb{K}$ and every graph $G$ with $t$ connected components, the quantity $\mathrm{depth}_{\mathbb{K}}\Delta_G$ is at least $t-1$.

**Corollary 4.4.** *Let $r$ and $i$ be positive integers and $G$ be a graph having $i$ isolated vertices. If the number of connected components of $G$ is at least $(2i+1)r - i$, then $\Delta_G$ is EKR of type $(r, i)$.*

*Proof.* The assertion follows immediately from Theorem 3.3, Proposition 4.2 and the above argument. □

Let $G$ be a graph. The *complementary graph* $\overline{G}$ is the graph with $V(\overline{G}) = V(G)$ and $E(\overline{G})$ consists of those 2-element subsets $\{u, v\}$ of $V(G)$ for which $\{u, v\} \notin E(G)$. Woodroofe [14, Lemma 4.4] proves that if $G$ is graph with at least two vertices which has connected complement, then for every field $\mathbb{K}$, we have $\mathrm{depth}_{\mathbb{K}}\Delta_G \geqslant 1$. As a consequence, we conclude the following result.

**Corollary 4.5.** *Let $r, i, m$ and $t$ be positive integers and $G$ be a graph with connected components $G_1, \ldots, G_t$. Assume further that*

*(i)* $t \geqslant (2i+1)r - i - m$,

*(ii)* $G_1, \ldots, G_i$ *are isolated vertices and*

*(iii)* *For every integer $j$ with $i+1 \leqslant j \leqslant i+m$, the graph $G_j$ has at least two vertices and the complementary graph $\overline{G_j}$ is connected.*

*Then $\Delta_G$ is EKR of type $(r, i)$.*

*Proof.* The assumptions (i), (iii), together with [14, Lemmas 2.12 and 4.14] imply that for every field $\mathbb{K}$, we have $\mathrm{depth}_{\mathbb{K}}\Delta_G \geqslant (2i+1)r - i - 1$. The assertion now follows from Theorem 3.3 and Proposition 4.2. □

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