Anti-van der Waerden numbers of 3-term arithmetic progressions

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Abstract

The *anti-van der Waerden number*, denoted by $\text{aw}([n],k)$, is the smallest r such that every exact r-coloring of $[n]$ contains a rainbow k-term arithmetic progression. Butler et al. showed that $\lceil \log_3 n \rceil + 2 \leqslant \text{aw}([n],3) \leqslant \lceil \log_2 n \rceil + 1$, and conjectured that there exists a constant C such that $\text{aw}([n],3) \leqslant \lceil \log_3 n \rceil + C$. In this paper, we show this conjecture is true by determining $\text{aw}([n],3)$ for all n. We prove that for $7 \cdot 3^{m-2} + 1 \leqslant n \leqslant 21 \cdot 3^{m-2}$,

$$\text{aw}([n],3) = \begin{cases} m+2, & \text{if } n = 3^m \\ m+3, & \text{otherwise.} \end{cases}$$

Keywords: arithmetic progression; rainbow coloring; Behrend construction; unitary coloring

1 Introduction

Let n be a positive integer and let $G \in \{[n], \mathbb{Z}_n\}$, where $[n] = \{1, \dots, n\}$. A *k-term arithmetic progression* (k-AP) of G is a sequence in G of the form

$$a, a+d, a+2d, \dots, a+(k-1)d,$$

where $d \geqslant 1$. For the purposes of this paper, an arithmetic progression is referred to as a set of the form $\{a, a+d, a+2d, \dots, a+(k-1)d\}$. An *$r$-coloring* of G is a function $c : G \to [r]$, and such a coloring is called *exact* if c is surjective. Given $c : G \to [r]$, an arithmetic progression is called *rainbow* (under c) if $c(a+id) \neq c(a+jd)$ for all $0 \leqslant i < j \leqslant k-1$.

The *anti-van der Waerden number*, denoted by $\mathrm{aw}(G,k)$, is the smallest r such that every exact r-coloring of G contains a rainbow k-AP. If G contains no k-AP, then $\mathrm{aw}(G,k) = |G|+1$; this is consistent with the property that there is a coloring of G with $\mathrm{aw}(G,k)-1$ colors that has no rainbow k-AP.

An r-coloring of G is *unitary* if there is an element of G that is uniquely colored. The smallest r such that every exact unitary r-coloring of G contains a rainbow k-AP is denoted by $\mathrm{aw}_u(G,k)$. Similar to the anti-van der Waerden number, $\mathrm{aw}_u(G,k) = |G|+1$ if G has no k-AP.

Problems involving counting and the existence of rainbow arithmetic progressions have been well-studied. The main results of Axenovich and Fon-Der-Flaass [1] and Axenovich and Martin [2] deal with the existence of 3-APs in colorings that have uniformly sized color classes. Fox, Jungić, Mahdian, Nešetril, and Radoičić also studied anti-Ramsey results of arithmetic progressions in [6]. In particular, they showed that every 3-coloring of $[n]$ for which each color class has density more than $1/6$, contains a rainbow 3-AP. Fox et al. also determined all values of n for which $\mathrm{aw}(\mathbb{Z}_n,3) = 3$.

The specific problem of determining anti-van der Waerden numbers for $[n]$ and $\mathbb{Z}_n$ was studied by Butler et al. in [4]. It is proved in [4] that for $k \geqslant 4$, $\mathrm{aw}([n],k) = n^{1-o(1)}$ and $\mathrm{aw}(\mathbb{Z}_n,k) = n^{1-o(1)}$. These results are obtained using results of Behrend [3] and Gowers [5] on the size of a subset of $[n]$ with no k-AP. Butler et al. also expand upon the results of [6] by determining $\mathrm{aw}(\mathbb{Z}_n,3)$ for all values of n. These results were generalized to all finite abelian groups in [7]. Butler et al. also provides bounds for $\mathrm{aw}([n],3)$, as well as many exact values (see Table 1).

In this paper, we determine the exact value of $\mathrm{aw}([n],3)$, which answers questions posed in [4] and confirms the following conjecture:

Conjecture 1. [4] There exists a constant C such that $\mathrm{aw}([n],3) \leqslant \lceil \log_3 n \rceil + C$, for all $n \geqslant 3$.

Our main result, Theorem 2, also determines $\mathrm{aw}_u([n],3)$ which shows the existence of extremal colorings of $[n]$ that are unitary.

Theorem 2. *For all integers $n \geqslant 2$,*

$$\mathrm{aw}_u([n],3) = \mathrm{aw}([n],3) = \begin{cases} m+2, & \text{if } n = 3^m \\ m+3, & \text{if } n \neq 3^m \text{ and } 7\cdot 3^{m-2}+1 \leqslant n \leqslant 21\cdot 3^{m-2}. \end{cases}$$

In Section 2, we provide lemmas that are useful in proving Theorem 2 and Section 3 contains the proof of Theorem 2.

2 Main Tools

In [4, Theorem 1.6] it is shown that $3 \leqslant \mathrm{aw}(\mathbb{Z}_p,3) \leqslant 4$ for every prime number p and that if $\mathrm{aw}(\mathbb{Z}_p,3) = 4$ then $p \geqslant 17$. Furthermore, it is shown that the value of $\mathrm{aw}(\mathbb{Z}_n,3)$ is determined by the values of $\mathrm{aw}(\mathbb{Z}_p,3)$ for the prime factors p of n. We have included this theorem below with some notation change.

$n \setminus k$	3	4	5	6	7	8	9	10	11	12	13	14
3	3											
4	4											
5	4	5										
6	4	6										
7	4	6	7									
8	5	6	8									
9	4	7	8	9								
10	5	8	9	10								
11	5	8	9	10	11							
12	5	8	10	11	12							
13	5	8	11	11	12	13						
14	5	8	11	12	13	14						
15	5	9	11	13	14	14	15					
16	5	9	12	13	15	15	16					
17	5	9	13	13	15	16	16	17				
18	5	10	14	14	16	17	17	18				
19	5	10	14	15	17	17	18	18	19			
20	5	10	14	16	17	18	19	19	20			
21	5	11	14	16	17	19	20	20	20	21		
22	6	12	14	17	18	20	21	21	21	22		
23	6	12	14	17	19	20	21	22	22	22	23	
24	6	12	15	18	20	20	22	23	23	23	24	
25	6	12	15	19	21	21	23	23	24	24	24	25

Table 1: Values of $\mathrm{aw}([n], k)$ for $3 \leqslant k \leqslant \frac{n+3}{2}$.

Theorem 3 ([4]). *Let n be a positive integer with prime decomposition*

$$n = 2^{e_0} p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$$

for $e_i \geqslant 0$, $i = 0, \dots, s$, where primes are ordered so that $\mathrm{aw}(\mathbb{Z}_{p_i}, 3) = 3$ for $1 \leqslant i \leqslant \ell$ and $\mathrm{aw}(\mathbb{Z}_{p_i}, 3) = 4$ for $\ell + 1 \leqslant i \leqslant s$. Then

$$\mathrm{aw}(\mathbb{Z}_n, 3) = \begin{cases} 2 + \sum\limits_{j=1}^{\ell} e_j + \sum\limits_{j=\ell+1}^{s} 2e_j, & \text{if } n \text{ is odd} \\ 3 + \sum\limits_{j=1}^{\ell} e_j + \sum\limits_{j=\ell+1}^{s} 2e_j, & \text{if } n \text{ is even.} \end{cases}$$

We use Theorem 3 to prove the following lemma.

Lemma 4. *Let $n \geqslant 3$, then $\mathrm{aw}(\mathbb{Z}_n, 3) \leqslant \lceil \log_3 n \rceil + 2$ with equality if and only if $n = 3^j$ or $2 \cdot 3^j$ for $j \geqslant 1$.*

Proof. Suppose $n = 2^{e_0} p_1^{e_1} p_2^{e_2} \dots p_s^{e_s}$ with $e_i \geqslant 0$ for $i = 0, \dots, s$, where primes $p_1, p_2, \dots, p_s$ are ordered so that $\mathrm{aw}(\mathbb{Z}_{p_i}, 3) = 3$ for $1 \leqslant i \leqslant \ell$ and $\mathrm{aw}(\mathbb{Z}_{p_i}, 3) = 4$ for $\ell + 1 \leqslant i \leqslant s$. We consider two cases depending on parity of n.

Case 1. Suppose n is odd, that is $e_0 = 0$. Then $\mathrm{aw}(\mathbb{Z}_n, 3) = 2 + \sum\limits_{j=1}^{\ell} e_j + \sum\limits_{j=\ell+1}^{s} 2e_j$ by Theorem 3. Since $\mathrm{aw}(\mathbb{Z}_p, 3) = 3$ for odd primes $p \leqslant 13$, we have $p_i \geqslant 17$ for $i \geqslant \ell + 1$, and clearly $p_i \geqslant 3$ for $i \leqslant \ell$, therefore

$$3^{\mathrm{aw}(\mathbb{Z}_n,3)} = 3^{2+\sum_{j=1}^{\ell} e_j + \sum_{j=\ell+1}^{s} 2e_j} = 9 \cdot 3^{e_1} \cdots 3^{e_\ell} \cdot 9^{e_{\ell+1}} \cdots 9^{e_s} \leqslant 9 \cdot p_1^{e_1} \cdots p_s^{e_s} = 9n.$$

Note that the equality holds if and only if n is a power of 3, that is $e_j = 0$ for $2 \leqslant j \leqslant s$. Therefore, $\mathrm{aw}(\mathbb{Z}_n, 3) \leqslant \lceil \log_3 n \rceil + 2$ for odd n, with equality if and only if $n = p_1^{e_1}$.

Case 2. Suppose n is even, that is $e_0 \geqslant 1$. Then $\mathrm{aw}(\mathbb{Z}_n, 3) = 3 + \sum_{j=1}^{\ell} e_j + \sum_{j=\ell+1}^{s} 2e_j$ by Theorem 3. If $n = 2^{e_0} \cdot 3^j$ for $j \geqslant 1$, then by direct computation $\mathrm{aw}(\mathbb{Z}_n, 3) = 3 + j \leqslant 2 + \lceil \log_3 n \rceil$, with equality if and only if $e_0 = 1$. So suppose there is i such that $p_i \neq 3$, and let $h = \frac{n}{2^{e_0} p_i^{e_i}}$.

If $i \leqslant \ell$ then $p_i \geqslant 5$, and so $3 \cdot 3^{e_i} < 2^{e_0} p_i^{e_i}$ for all $e_0 \geqslant 1$ and $e_i \geqslant 1$. Therefore, since h is odd, by the previous case

$$3^{\mathrm{aw}(\mathbb{Z}_n,3)} = 3 \cdot 3^{e_i} \cdot 3^{\mathrm{aw}(\mathbb{Z}_h,3)} \leqslant 3 \cdot 3^{e_i} \cdot 9h < 2^{e_0} p_i^{e_i} \cdot 9h = 9n.$$

If $i \geqslant \ell + 1$ then $p_i \geqslant 17$, and so $3 \cdot 9^{e_i} < 2^{e_0} p_i^{e_i}$ for all $e_0 \geqslant 1$ and $e_i \geqslant 1$. Then by the previous case

$$3^{\mathrm{aw}(\mathbb{Z}_n,3)} = 3 \cdot 9^{e_i} \cdot 3^{\mathrm{aw}(\mathbb{Z}_h,3)} \leqslant 3 \cdot 9^{e_i} \cdot 9h < 2^{e_0} p_i^{e_i} \cdot 9h = 9n. \qquad \square$$

A set of consecutive integers I in $[n]$ is called an *interval* and $\ell(I)$ is the number of integers in I. Given a coloring c of some finite nonempty subset S of $[n]$, a *color class* of a color i under c in S is denoted $c_i(S) := \{x \in S : c(x) = i\}$. A coloring c of $[n]$ is *special* if $n = 7q + 1$ for some positive integer q, $c(1)$ and $c(n)$ are both uniquely colored, and there are two colors α and β such that $c_\alpha([n]) = \{q + 1, 2q + 1, 4q + 1\}$ and $c_\beta([n]) = \{3q + 1, 5q + 1, 6q + 1\}$.

Lemma 5. *Let N be an integer and c be an exact r-coloring of $[N]$ with no rainbow 3-AP, where 1 and N are colored uniquely. Then either the coloring c is special or $|\{c(x) : x \equiv i \pmod 3$ and $x \in [N]\}| \geqslant r - 1$ for $i = 1$ or $i = N$.*

Proof. Observe that N is even, otherwise $\{1, (N+1)/2, N\}$ is a rainbow 3-AP. We partition the interval $[N]$ into four subintervals $I_1 = \{1, \dots, \lceil N/4 \rceil\}$, $I_2 = \{\lceil N/4 \rceil + 1, \dots, N/2\}$, $I_3 = \{N/2+1, \dots, \lfloor 3N/4 \rfloor\}$, and $I_4 = \{\lfloor 3N/4 \rfloor + 1, \dots, N\}$. Notice that every color other than $c(1)$ and $c(N)$ must be used in the subinterval I_2. To see this, assume i is the missing color in I_2 distinct from $c(1)$ and $c(N)$. Let x be the largest integer in $c_i(I_1)$. Since N is even, we have $2x - 1 \leqslant 2\lceil N/4 \rceil - 1 \leqslant N/2$, and so $2x - 1 \in I_1 \cup I_2$. If $2x - 1 \in I_1$, then $c(2x - 1) \neq i$ since x is the largest integer in I_1 with color i; hence, $\{1, x, 2x - 1\}$ is a rainbow 3-AP. If $2x - 1 \in I_2$, then $c(2x - 1) \neq i$ since color i is missing in I_2; hence, the 3-AP $\{1, x, 2x - 1\}$ is a rainbow. If there is no such integer x in I_1, then the integers colored with i must be in the second half of the interval $[N]$, so we choose the smallest such integer y in $c_i(I_3 \cup I_4)$. Then $\{2y - N, y, N\}$ is a rainbow 3-AP since $c(2y - N) \neq i$, because $2y - N \in I_1 \cup I_2$. Similarly, every color other than $c(1)$ and $c(N)$ must be used in the subinterval I_3.

Throughout the proof we mostly drop (mod 3) and just say congruent even though we mean congruent modulo 3. We consider the following three cases.

Case 1: $N \equiv 0 \pmod 3$. Assume $|\{c(x) : x \equiv i \pmod 3 \text{ and } x \in [N]\}| < r - 1$ for both $i = 1$ and $i = N$. So there are two colors, say *red* and *blue*, such that no integer in $[N]$ colored with *red* is congruent to 1, and no integer in $[N]$ colored with *blue* is congruent to 0. We further partition the interval I_2 into subintervals $I_{2(i)}$ and $I_{2(ii)}$ so that $\ell(I_{2(i)}) \leqslant \ell(I_{2(ii)}) \leqslant \ell(I_{2(i)})+1$, and partition the interval I_3 into subintervals $I_{3(i)}$ and $I_{3(ii)}$ so that $\ell(I_{3(ii)}) \leqslant \ell(I_{3(i)}) \leqslant \ell(I_{3(ii)}) + 1$. Then we have the following observations:

(i) $x \equiv 0$ for all $x \in c_{red}(I_3 \cup I_4)$ and $y \equiv 1$ for all $y \in c_{blue}(I_1 \cup I_2)$.
If there is an integer r in $I_3 \cup I_4$ colored with *red* and congruent to 2, then $2r - N \equiv 1$, and so $c(2r - N)$ is not *red* by our assumption. Therefore the 3-AP $\{2r - N, r, N\}$ is rainbow. Similarly, if there is an integer b in $I_1 \cup I_2$ colored with *blue* and congruent to 2, then $2b - 1 \equiv 0$, and so $c(2b - 1)$ is not *blue*, forming a rainbow 3-AP $\{1, b, 2b - 1\}$.

(ii) $x \equiv 2$ for all $x \in c_{red}(I_2)$ and $y \equiv 2$ for all $y \in c_{blue}(I_3)$.
If there is an integer r in $c_{red}(I_2)$ congruent to 0, then $2r - 1 \equiv 2$ and $2r - 1 \in I_3 \cup I_4$ since $2r - 1 \geqslant N/2 + 1$. Therefore, $2r - 1$ is not colored with *red* by the previous observation, and so the 3-AP $\{1, r, 2r - 1\}$ is a rainbow. Similarly, if there is an integer b in $c_{blue}(I_3)$ congruent to 1, then using N we obtain the rainbow 3-AP $\{2b - N, b, N\}$, because $2b - N \equiv 2$ and $2b - N \leqslant N/2$.

(iii) $c_{red}(I_{3(ii)}) = c_{blue}(I_{2(i)}) = \emptyset$.
If there is an integer r in $I_{3(ii)}$ colored with *red*, then $2r - N \equiv 0$, by observation *(i)*. Furthermore, $2r - N \leqslant N/2$ and $2r - N \geqslant 2(N/2 + \ell(I_{3(i)}) + 1) - N \geqslant (2\ell(I_{3(i)}) + 1) + 1 \geqslant \lceil N/4 \rceil + 1$. So $2r - N \in I_2$ and hence it is not colored with *red* by observation (*ii*). Therefore, $\{2r - N, r, N\}$ is a rainbow 3-AP. Similarly, if there is an integer b in $I_{2(i)}$ colored with *blue*, then $2b - 1 \equiv 1$ and $N/2 + 1 \leqslant 2b - 1 \leqslant \lfloor 3N/4 \rfloor$. So $2b - 1 \in I_3$ and hence it is not colored with *blue* by observation (*ii*). Therefore, $\{1, b, 2b - 1\}$ is a rainbow 3-AP.

(iv) $c_{red}(I_{2(ii)}) = c_{blue}(I_{3(i)}) = \emptyset$.
Suppose there is an integer r in $I_{2(ii)}$ colored with *red*. Since the coloring of I_2 contains both *red* and *blue* and there is no integer in $I_{2(i)}$ colored with *blue*, by (*iii*), there must be an integer b in $I_{2(ii)}$ colored with *blue*. By (*i*) and (*ii*), $b \equiv 1$ and $r \equiv 2$. Without loss of generality, suppose $b > r$. Then $2r - b \equiv 0$ and $2r - b \in I_2$ since $\ell(I_{2(ii)}) \leqslant \ell(I_{2(i)}) + 1$. So $2r - b$ is not colored *red* or *blue* and hence the 3-AP $\{2r - b, r, b\}$ is rainbow. Therefore, there is no integer in $I_{2(ii)}$ that is colored with *red*. Similarly, there is no integer in $I_{3(i)}$ that is colored with *blue*.

Recall that every color other than $c(1)$ and $c(N)$ is used in both intervals I_2 and I_3. Therefore, sets $c_{red}(I_{2(i)})$, $c_{blue}(I_{2(ii)})$, $c_{red}(I_{3(i)})$, and $c_{blue}(I_{3(ii)})$ are nonempty. Using the above observations we next show that in fact these integers colored with *blue* and *red* in each subinterval are unique. Let $B = \{b_1, \ldots, b_2\}$ be the shortest interval in $I_{2(ii)}$ which contains all integers colored with *blue* and let $R = \{r_1, \dots, r_2\}$ be the shortest interval in $I_{3(i)}$ which contains all integers colored with *red*. Choose the largest integer x in $c_{red}(I_{2(i)})$ and consider two 3-APs $\{x, b_1, 2b_1 - x\}$ and $\{x, b_2, 2b_2 - x\}$. Since x is congruent to 2 and both b_1 and b_2 are congruent to 1, we have that both $2b_1 - x$ and $2b_2 - x$ are congruent to 0 and are contained in I_3, otherwise the 3-APs are rainbow. Since all integers colored with *blue* in I_3 are congruent to 2 by *(ii)*, we have that $2b_1 - x$ and $2b_2 - x$ are both colored with *red* and so contained in R. Therefore, $2\ell(B) - 1 \leqslant \ell(R)$. Now using the smallest integer in $c_{blue}(I_{3(ii)})$, we similarly have that $2\ell(R) - 1 \leqslant \ell(B)$. Since $\ell(B) \geqslant 1$ and $\ell(R) \geqslant 1$, we have that $\ell(R) = \ell(B) = 1$, i.e. there are unique integers b in $c_{blue}(I_{2(ii)})$ and r in $c_{red}(I_{3(i)})$.

Now for any integer $\tilde{r}$ from $c_{red}(I_{2(i)})$ the integer $2\tilde{r} - 1$ must be colored with *red*, otherwise the 3-AP $\{1, \tilde{r}, 2\tilde{r} - 1\}$ is rainbow. Since $2\tilde{r} - 1 \in I_3$, it must be equal to the unique *red* colored integer r of I_3. Therefore, there is exactly one such $\tilde{r}$ in $I_{2(i)}$, i.e. $c_{red}(I_{2(i)}) = \{\tilde{r}\}$. Similarly, using N there is a unique integer $\tilde{b}$ in $I_{3(ii)}$ colored with *blue*. Since $\{1, \tilde{r}, r\}$, $\{\tilde{r}, b, r\}$, $\{b, r, \tilde{b}\}$, and $\{b, \tilde{b}, N\}$ are all 3-APs, $N = 7(\ell(\{b, \dots, r\}) - 1) + 1 = 7(r - b) + 1$.

Observe that if $\tilde{r}$ is even, the integer $(\tilde{r} + N)/2$ in 3-AP $\{\tilde{r}, (\tilde{r} + N)/2, N\}$ must be *red* and congruent to 1 since $\tilde{r} \equiv 2$ by *(ii)*, contradicting our assumption. So $\tilde{r}$ is odd, and hence the integer $r' = (\tilde{r} + 1)/2$ in I_1 must be colored with *red*. Notice that there cannot be another integer x larger than r' in $c_{red}(I_1)$, otherwise $2x - 1$ will be another integer colored with *red* in I_2 distinct from $\tilde{r}$. Now, since $\ell(\{r', \dots, \tilde{r}\}) = \ell(\{b, \dots, r\})$ we have that $\{r', r, N\}$ is a 3-AP, and so r' must be even. Suppose there are integers smaller than r' in $c_{red}(I_1)$, and let z be the largest of them. Then $2z - 1$ is also in $c_{red}(I_1)$ and must be equal to or larger than r' in I_1. However, that is impossible because r' is even and there is no integer in $c_{red}(I_1)$ larger than r'. So r' is a unique integer in I_1 colored with *red*. Similarly, there is a unique integer b' in I_4 colored with *blue*. Therefore the 8-AP can be formed using integers $1, r', \tilde{r}, b, r, \tilde{b}, b', N$ since $\ell(\{1, \dots, r'\}) = \ell(\{r', \dots, \tilde{r}\}) = \ell(\{\tilde{r}, \dots, b\}) = \ell(\{b, \dots, r\}) = \ell(\{r, \dots, \tilde{b}\}) = \ell(\{\tilde{b}, \dots, b'\}) = \ell(\{b', \dots, N\})$.

In order for this coloring to be special, it remains to show that $c_{blue}(I_1) = c_{red}(I_4) = \emptyset$. If $c_{blue}(I_1) \neq \emptyset$, then choose the largest integer y in it and consider the 3-AP $\{1, y, 2y - 1\}$. Since $2y - 1$ must be in $c_{blue}(I_2)$ and the only integer in this set is b, we have $2y - 1 = b$. However, we know that b is even because $b = 2\tilde{b} - N$, a contradiction. Similarly, if $c_{red}(I_4) \neq \emptyset$ choose the smallest integer x in it and consider the 3-AP $\{2x - N, x, N\}$. Since $2x - N$ must be in $c_{red}(I_3)$ and the only integer in this set is r, we have $2x - N = r$. However, we know that r is odd because $r = 2\tilde{r} - 1$, a contradiction. This implies that $c_{red}([N]) = \{r', \tilde{r}, r\}$ and $c_{blue}([N]) = \{b, \tilde{b}, b'\}$, so the coloring is special.

Case 2: $N \equiv 2 \pmod 3$. This case is analogous to Case 1.

Case 3: $N \equiv 1 \pmod 3$. Assume $|\{c(x) : x \equiv i \pmod 3 \text{ and } x \in [N]\}| < r - 1$ i.e. there are two colors, say *red* and *blue*, such that no integer in $[N]$ colored with *red* or *blue* is congruent to 1. Recall that every color other than $c(1)$ and $c(N)$ appears in I_2 and I_3. First, notice that all integers colored with *red* or *blue* in I_2 must be congruent modulo 3. Otherwise, choosing a *red* colored integer and a *blue* colored integer, we obtain a 3-AP whose third term is colored with *red* or *blue* and is congruent to 1 contradicting our assumption. Similarly, this is also the case for I_3. So suppose all integers in $c_{red}(I_2) \cup c_{blue}(I_2)$ and $c_{red}(I_3) \cup c_{blue}(I_3)$ are congruent modulo 3 to integers $p \not\equiv 1$ and $q \not\equiv 1$, respectively. Pick the largest integers from $c_{red}(I_2)$ and $c_{blue}(I_2)$ and form a 3-AP whose third term is in I_3. Then the third term is colored with *red* or *blue* and is congruent to p. Therefore, $p \equiv q \not\equiv 1$.

We further partition the interval I_2 into subintervals $I_{2(i)}$ and $I_{2(ii)}$, so that $\ell(I_{2(i)}) \leqslant \ell(I_{2(ii)}) \leqslant \ell(I_{2(i)}) + 1$. If there exists $x \in c_{red}(I_{2(i)}) \cup c_{blue}(I_{2(i)})$, the integer $2x - 1$ must be colored with $c(x)$ and contained in I_3, so $2x - 1 \equiv p$ while $x \equiv p \not\equiv 1$, a contradiction. So $c_{red}(I_{2(i)}) \cup c_{blue}(I_{2(i)}) = \emptyset$. However, then the smallest integers of $c_{red}(I_{2(ii)})$ and $c_{blue}(I_{2(ii)})$ form a 3-AP whose first term is contained in $I_{2(i)}$ and is colored with *red* or *blue*, a contradiction. This completes the proof of the lemma. $\square$

3 Proof of the main result

Given a positive integer n, define the function f as follows:

$$f(n) = \begin{cases} m+2, & \text{if } n = 3^m \\ m+3, & \text{if } n \neq 3^m \text{ and } 7 \cdot 3^{m-2} + 1 \leqslant n \leqslant 21 \cdot 3^{m-2}. \end{cases}$$

In this section, we prove Theorem 2 by showing that $\mathrm{aw}([n], 3) = f(n)$ for all n. Throughout the proof we mostly drop (mod 3), although all equivalences will happen modulo 3.

First, we show that $f(n) \leqslant \mathrm{aw}_u([n], 3)$ by inductively constructing a unitary coloring of $[n]$ with $f(n) - 1$ colors and no rainbow 3-AP. The result is true for $n = 1, 2, 3$, by definition. Suppose $n > 3$ and that the result holds for all positive integers less than n. Let $n = 3h - s$, where $s \in \{0, 1, 2\}$ and $2 \leqslant h < n$.

Let $r = \mathrm{aw}_u([h], 3)$. So there is an exact unitary $(r-1)$-coloring c of $[h]$ with no rainbow 3-AP. Let *red* be a color not used in c. Define the coloring c_1 of $[n]$ such that if $x \equiv 1 \pmod 3$, then $c_1(x) = c((x+2)/3)$, otherwise color x with *red*. When $s \neq 0$, define the coloring c_2 of $[n]$ as follows:

- if $x \not\equiv 0 \pmod 3$, then $c_2(x) = red$,
- if $x \equiv 0 \pmod 3$, then
  - $c_2(x) = c(x/3 + 1)$, if $c(h)$ is the only unique color in c,
  - $c_2(x) = c(x/3)$, otherwise.

Notice that c_2 is a unitary $\mathrm{aw}_u([h-1], 3)$-coloring when $s \neq 0$ and c_1 is a unitary r-coloring of $[n]$. Now consider a 3-AP $\{a, b, 2b - a\}$ in $[n]$. If $a \equiv b \not\equiv 1$, then a and b are colored with *red*, and so the 3-AP is not a rainbow. If $a \equiv b \equiv 1$, then $2b - a \equiv 1$, so this set corresponds to a 3-AP in $[h]$ with coloring c, and hence the 3-AP is not rainbow. If $a \not\equiv b$, then $2b - a$ is not congruent to a or b, so two of the terms of the 3-AP are colored with *red*, and hence the 3-AP is not rainbow under c_1. Similarly, this 3-AP is not rainbow under c_2. Therefore, c_1 and c_2 are unitary colorings of $[n]$ with no rainbow 3-AP.

Also note that $\mathrm{aw}_u([n],3) \geqslant \mathrm{aw}_u([h],3)+1$ under c_1 and $\mathrm{aw}_u([n],3) \geqslant \mathrm{aw}_u([h-1],3)+1$ under c_2. We proceed with three cases determined by $\frac{n}{3}$.

Case 1. First suppose $7 \cdot 3^{m-2}+1 \leqslant n \leqslant 3^m-3$ or $3^m \leqslant n \leqslant 21 \cdot 3^{m-2}$. By the induction hypothesis and using the coloring c_1,

$$\mathrm{aw}_u([n],3) \geqslant \mathrm{aw}_u([h],3)+1 \geqslant f(h)+1 = f(n).$$

Case 2. Suppose $n = 3^m - t$ where $t \in \{1,2\}$. Notice that $h = 3^{m-1}$, so by induction and using coloring c_2,

$$\mathrm{aw}_u([n],3) \geqslant \mathrm{aw}_u([h-1],3)+1 \geqslant f(h-1)+1 = f(3^{m-1}-1)+1 = (m+2)+1 = f(n).$$

The upper bound, $\mathrm{aw}([n],3) \leqslant f(n)$, is also proved by induction on n. For small n, the result follows from Table 1. Assume the statement is true for all values less than n, and let $7 \cdot 3^{m-2}+1 \leqslant n \leqslant 21 \cdot 3^{m-2}$ for some m. Let $\mathrm{aw}([n],3) = r+1$, so there is an exact r-coloring $\hat{c}$ of $[n]$ with no rainbow 3-AP. We need to show that $r \leqslant f(n)-1$. Let $[n_1, n_2, \ldots, n_N]$ be the shortest interval in $[n]$ containing all r colors under $\hat{c}$. Define c to be an r-coloring of $[N]$ so that $c(j) = \hat{c}(n_j)$ for $j \in \{1, \ldots, N\}$. By minimality of N the colors of 1 and N are unique. If $[N]$ has at least $r-1$ colors congruent to 1 or N, then $[n]$ has at least $r-1$ colors congruent to n_1 or n_N, respectively, so $r \leqslant \mathrm{aw}(\lfloor n/3 \rfloor, 3)$ and by induction $r \leqslant f(\lfloor n/3 \rfloor) \leqslant f(n)-1$. So suppose that is not the case, then by Lemma 5 we have that the coloring c is special.

Let $N = 7q+1$ for some $q \geqslant 1$, and let the 8-AP in this special coloring be $\{1, r_1, r_2, b_1, r_3, b_2, b_3, N\}$, where r_1, r_2, r_3 are the only integers colored *red*, b_1, b_2, b_3 are the only integers colored *blue* and $q = r_1 - 1$. If $n \geqslant 9q$, then the 8-AP can be extended to a 9-AP in n by adding the 9th element to either the beginning or the ending. Without loss of generality, suppose $\{1, r_1, r_2, b_1, r_3, b_2, b_3, N, 2N-b_3\}$ correspond to a 9-AP in $[n]$. Since the coloring has no rainbow 3-AP, the color of $2N-b_3$ is *blue* or $c(N)$, so we have a 4-coloring of this 9-AP. However, $\mathrm{aw}([9],3) = 4$ and hence there is a rainbow 3-AP in this 9-AP which is in turn a rainbow 3-AP in $[n]$. Therefore, $n \leqslant 9q-1$.

By uniqueness of the *red* colored integer r_1 in interval $\{1, \ldots, r_2-1\}$, the colors of integers in interval $\{r_1+1, \ldots, r_2-1\}$ is the same as the reversed colors of integers in $\{2, \ldots, r_1-1\}$, i.e. $c(r_1+i) = c(r_1-i)$ for $i = 1, \ldots, q-1$. Similarly, coloring of integers in interval $\{r_2+1, \ldots, b_1-1\}$ is the reversed of the coloring of integers in interval $\{r_1+1, \ldots, r_2-1\}$, and so on. This gives a rainbow 3-AP-free $(r-2)$-coloring of $\mathbb{Z}_{2q}$. Therefore, $r-2 \leqslant \mathrm{aw}(\mathbb{Z}_{2q},3)-1$.

If $q = 3^i$ for some i, then n can not be a power of 3 because $7 \cdot 3^i + 1 \leqslant n \leqslant 9 \cdot 3^i - 1$. Suppose $n = 3^m$, then $2q$ is not twice a power of 3 and clearly $2q$ is not a power of 3. Therefore, by Lemma 4 we have

$$\begin{aligned} r &\leqslant \mathrm{aw}(\mathbb{Z}_{2q},3)+1 \leqslant \lceil \log_3(2q) \rceil + 2 \leqslant \lceil \log_3(2n/7) \rceil + 2 \\ &= \lceil \log_3(2 \cdot 3^m/7) \rceil + 2 = m+1 \leqslant f(n)-1. \end{aligned}$$

Suppose now that $n \neq 3^m$. If $q = 3^i$ for some i then $i \leqslant m-2$. Otherwise, if $i \geqslant m-1$ then $q \geqslant 3^{m-1} \geqslant \frac{n}{7}$ which contradicts the fact that $q < \frac{n}{7}$. Therefore, $2q \leqslant 2 \cdot 3^{m-2} = 18 \cdot 3^{m-4}$ and so by induction and Lemma 4,

$$r \leqslant \mathrm{aw}(\mathbb{Z}_{2q}, 3) + 1 = \mathrm{aw}([2q], 3) + 1 \leqslant m + 2 \leqslant f(n) - 1.$$

If q is not a power of 3, then again using Lemma 4, $r \leqslant \mathrm{aw}(\mathbb{Z}_{2q}, 3) + 1 \leqslant \mathrm{aw}([2q], 3)$. Notice that $6 \cdot 3^{m-3} + \frac{2}{7} \leqslant \frac{2}{7}n \leqslant 18 \cdot 3^{m-3}$, and so $\mathrm{aw}([2q], 3) \leqslant m + 2$ by induction. Therefore, $r \leqslant m + 2 \leqslant f(n) - 1$. This completes the proof of the main theorem. □

4 Concluding Remarks

We have determined the exact value of the anti-van der Waerden number $\mathrm{aw}([n], 3)$ for all n, which confirms a conjecture in [4]. Bounds on $\mathrm{aw}([n,], k)$ for $k \geqslant 4$ are given in [4], however the exact values are not known in general. These bounds can be improved and it would be interesting to know exact values $\mathrm{aw}([n], k)$ for $k \geqslant 4$.

The values of $\mathrm{aw}(G, k)$ has been recently studied for finite abelian groups [7]. It would be interesting to investigate this number for other groups.

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