TO REVIEWER A: REVISIONS OF NEW COMBINATORIAL FORMULAS FOR CLUSTER MONOMIALS OF TYPE A QUIVERS.

We made the following revisions (in blue) suggested by the referee.

Major recommended changes:

1) At the end of the first paragraph of the introduction, the authors writev"We give a brief summary of the pros and cons of four such models." As currently written, I believe it would be more accurate to write "We give a brief summary of the history of four such models."

However, I think to motivate this papers publication, it is more important to list actual pros and cons here in addition to the history. In particular, since the authors are taking the effort to introduce such different combinatorial objects and formulas for them, I think it is important to convince the reader that there are in fact advantages and disadvantages to these different models and how they complement each other. The reader should be convinced that these four models are different from each other in essential ways as opposed to re-expressions of each other. Or a clearer map of how these objects divide into different structural types.

By analogy, if this was a paper about Catalan numbers, I would want the authors to make it clear that the objects of non-crossing partitions, Dyck paths, and 2xn Standard Young Tableaux are simply related to one another but that non-nesting partitions, or triangulations, etc. have quite a different orbit structure under fundamental symmetries.

— Done. According to the suggestion of the referee, we extended these paragraphs to include more pros and cons, and how they are related to each other.

2) While the authors focus on cluster monomial formulas for three out of the four combinatorial interpretations, Section 7 focused only on cluster variable formulas. I realize this is partially due to the fact that the authors glued two papers together but reading this as one manuscript, the difference was jarring.

It would much improve the paper if the authors were able to extend their bijection between GCS and broken lines to the general cluster monomial case. Since I realize that might be mathematically daunting, if that is not easily possible, I would recommend the authors at least address the obstacles to extending this formula in this way and possibly give an example or two showing how a naive extension of the bijection fails.

— Done. We added in Remark 7.4 an explanation for the obstacle to extend this formula. We do not know even a naive extension of the bijection in the non-cluster-variable case.

3) The authors dedicate Section 3 to discussing the object of pipelines and how these correspond to cluster monomials. However, what the authors call pipelines, which as far as I can tell is a name exclusive to this paper, are actually *laminations* which have been studied extensively in a number of previous papers.

For example, the authors should look at work of Fock-Goncharov, including page 11 (Reconstruction) of arXiv:0510312.

Accordingly, I think Section 3 should be rewritten appropriately with laminations rather than pipelines and citing this previous literature. Since the authors do a mostly (see comment 4 below) good job of

explaining laminations, the authors could still include Section 3 in some form but rewrite it as a self-contained exposition rather than as a new combinatorial gadget.

— We discussed on this issue, compared with the work of Fock-Goncharov, and still feel unsure about the referee's suggestion. Our pipelines are not the same as laminations, even though they are very similar. The main difference is that: the two ends of our pipelines are marked points, and that two pipelines may meet at marked points; but curves of a lamination do not end at marked points, and two curves do not meet at endpoints.

So instead, we keep the terminology pipeline in the current draft, and wait for the referee to give us more detail about his/her suggestion. Meanwhile, we add a detailed footnote of comparision between pipelines and a lamination at the beginning of Section 3.1.

4) At the end of Section 3.1, the authors write "It is easy to see that the d-vector $(a_1, ..., a_n)$ of any cluster monomial satisfies the following..."

However, I think seeing Property A is somewhat subtle. In fact the authors even bring up more explicitly "it is not hard to verify that $\sigma_{kij} + \sigma_{ijk} <= a_i$" later in the pre-amble of Section 4 but I think not only should this inequality be explicitly shown or at least another word on this, such as "by case-by-case analysis and the definition of σ_{ijk}", I think this should be included in Section 3 as opposed to deferred to Section 4.

As a related point, the authors are careful to write "Moreover, for any edge i− >j of Q that is not ... (the vertex is a frozen vertex)." in Section 4, but in fact the construction of Proposition 3.2 only makes sense if the frozen vertices are added so that σ_{ijk} can always be defined.

For example, in Figure 3, without adding the frozen vertex (indicated as side 9 as a typo), the arrow 3 $-> 4$ does not immediately lead to $\sigma(3, 2, 0) = 2, \sigma(2, 0, 3) = 0, \sigma(0, 3, 2) = 0$ which explains the lamination pattern in the triangle 3-4-9 in contrast to the lamination pattern in the triangle with sides 1 and 2.

Lastly, the description of $\mathcal{W}$ appeared a little ambiguous to me. Do I understand correctly that if Q has no 3-cycles whatsoever, than $\mathcal{W} = Z^n$? Are arbitrary negative values or triples of positive (a_i, a_j, a_k) such that $a_i + a_j + a_k = odd$ but the triangle inequalities do not hold allowed?

— Done. We added a detailed proof of Property A, include the inequality in Section 3, Remark 3.2. Thanks for the reviewer for pointing out the issue that Proposition 3.2 (which is Proposition 3.3 in the current version) only makes sense if the frozen vertices are added! We corrected the proof accordingly.

We reworded the definition of $\mathcal{W}$ at the beginning of §3.2 so there is no more ambiguity.

5) In the last line of Section 4.1, on page 11, the authors write "This formula specializes to the formula given in [11] for a linear quiver."

Did the authors instead mean to cite [1] here instead of [11]?

— Yes! We corrected the citation.

More importantly, comparing Sections 4-6 to the preprint [1], it is very similar. I dont think this is a mathematical issue since [1] was never published (as far as this reviewer knows) and the authors have indeed extended the result from the linear A_n quiver to general A_n quivers.

However, given that the authorship of [1] is not the same as the authorship of the current write-up, I believe the authors should dedicate more of the beginning of Section 4 and/or Section 1 crediting the results of [1] and comparing the new results to those for the linear A_n case.

— We dedicated more at both beginning of §4 and in §1 crediting the result of [1]; and we compared them at the beginning of §4.

Minor edits:

1) In the last sentence of the abstract, the word "We" in "We give a bijection" should be lower-case.

— Done.

2) Missing word "a" on page 2 in "In Theorem 4.1, using a sequence of 0-1 sequences called *a* GCS "

— Done.

3) On page 4, make it clearer if the authors want to consider frozen variables to be cluster variables or contained in clusters or to be separate from them.

— Done. We now made it clear that frozen variables are not cluster variables.

4) Throughout Section 3 (starting with Step 1 on page 8), the authors write "marking point" where "marked point" would be a better use.

— Done.

5) Between (3.2) and (3.3), the sentence "Let S be the multiset of all such sequences" was a little unclear. I might make it clearer that each of these sequences (in the multiset) corresponds to one such Λ. (By the way, as currently written I believe Λ is a single "pipe" as opposed to a "pipeline".)

— Done. We revised the definition of S. We intend to define Λ to be a pipeline, which is a long curve that lies over several triangles; in comparison, a pipe lies in a single triangle. For example, in the last figure of Figure 3, there are 5 pipelines, four of which contain 5 pipes each, and one of which contains 4 pipes.

6) Missing word "a" at the bottom of page 8 in Example 3.3: "For *a* clearer illustration, "

— Done.

7) In Figure 3, the side of the triangle 3-4 also contains a 9. Explain what this 9 in terms of the quiver (as recommended above) or edit this typo.

— Done. It is a typo.

8) "Pipe" rather than "Pipeline" just below Figure 3 in "There are 5 pipelines, passing through ", but again, I recommend rewriting in terms of laminations anyway.

— It is arguable.

9) In Lemma 3.4, the notation $[\mathbf{a}]_+$ appears, but I don't believe this has yet been defined for $\mathbf{a}$ a vector (as opposed to a number).

— You are right! We added the definition in Lemma 3.4.

10) In Remark 4.7, the authors write "of any type A quiver". Can the authors make it clearer if they mean any orientation of a type A acyclic, i.e Dynkin, quiver or if the authors really mean any mutation-equivalent type A quiver (possibly including some 3-cycles).

— Done. We mean "any mutation-equivalent type A quiver (possibly including some 3-cycles)". We made that clear in Remark 4.7.

11) In the first paragraph of Section 5, "via perfect matching" $->$ "via perfect matchings".

— Done.

12) On the bottom of page 15, the authors write "We shall construction a bijective map" instead of "We shall construct a bijective map".

— Done.

13) In Remark 7.4, the authors write "only consider those appear in the theta" and "if we consider those appear in the theta".

In both of these cases, I recommend using "appearing" instead of "appear".

— Done.

14) In Example 8.1, the authors write "3 x 3 = 27".

— Done.