New combinatorial formulas for cluster monomials of type A quivers

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Abstract

Lots of research focuses on the combinatorics behind various bases of cluster algebras. This paper studies the natural basis of a type A cluster algebra, which consists of all cluster monomials. We introduce a new kind of combinatorial formula for the cluster monomials in terms of the so-called globally compatible collections. We give bijective proofs of these formulas by comparing with the well-known combinatorial models of the T-paths and of the perfect matchings in a snake diagram. For cluster variables of a type A cluster algebra, we give a bijection that relates our new formula with the theta functions constructed by Gross, Hacking, Keel and Kontsevich.

1 Introduction

Cluster algebras were first introduced by S. Fomin and A. Zelevinsky in [8] to design an algebraic framework for understanding total positivity and canonical bases for quantum groups. A cluster algebra is a subring of a rational function field generated by a distinguished set of Laurent polynomials called cluster variables. The long-standing Positivity Conjecture, now proved in [15] and [10], asserts that the coefficients in the Laurent expansion of any cluster variable with respect to any fixed cluster are positive integers. From the combinatorial point of view, the Positivity Conjecture suggests that these coefficients should count some combinatorial objects. Lots of research focuses on building such combinatorial models. We give a brief summary of the pros and cons of four such models.

- **T-paths:** In [19], Schiffler (independently Carroll–Price and Fomin–Zelevinsky in their unpublished work) obtained a formula for the cluster variables of a cluster algebra of finite type A (see §2 for the definition) in terms of T-paths. This formula has been modified and generalized to cluster algebras coming from surfaces [16, 17, 20, 21, 11]. The formula is computation-friendly, but it does not seem to generalize to cluster algebras not coming from surfaces.

- **Perfect matchings of a snake diagram:** A description that is similar to the T-path model but has a more graph-theoretic flavor [16]. Interesting combinatorics, for example the snake graph calculus [4, 5], arises in the study of this model. This formula is simply bijective to the T-path formula, but uses more classical graph-theoretical notion "perfect matching" and is easier to compute; like the T-path formula, it is also restricted to cluster algebras coming from surfaces.

- **Compatible pairs in a Dyck path:** In [14], the cluster variables of rank 2 quivers are described in terms of Dyck paths. A more general construction of the so-called compatible pairs is used in the study of greedy bases in [13], and another generalization called GCC is used in [1, 12]. This formula is computation-friendly and easy to implement, and it applies to rank 2 cluster algebras that do not necessarily come from surfaces. Meanwhile, the main drawback is that we could not yet find a generalization that gives the cluster variables for higher rank non-type-A cluster algebras. The combinatorics is quite different than the T-paths and perfect matchings, but we shall give a bijection in this paper showing that they indeed coincide in the type A case (Theorem 4.4 and Theorem 8.3).

- **Broken lines and Theta functions:** Discovered in [10], they are the most general combinatorial models so far. They are so powerful that can be used to proved several well-known conjectures including the positivity conjecture. On the other hand, they are mainly of theoretical importance but do not yet give a satisfying combinatorial model (at least not in the sense of the previous three models): for example, the finiteness of the number of broken lines is not immediate from the definition, and it is difficult to implement even for rank 2 cluster algebras.

Our ultimate goal is to find a combinatorial model that is both general and effective in computation. Even though this goal appears out of reach for now, we feel that the model of maximal Dyck paths and compatible pairs has the potential to be generalized. This motivates the main goal of this paper:

For a type A quiver, give a new formula for the cluster monomials using a combinatorial model similar to compatible pairs, and find the bijections to other known models.

We reach this goal by proving three equivalent formulas.

– In Theorem 3.1, we give a formula using a sequence of 0-1 sequences called a GCS (globally compatible sequence), where each vertex of the quiver is assigned a 0-1 sequence satisfying a certain compatibility condition.

– In Theorem 3.5, we give a formula using globally compatible collections (GCCs) in Dyck paths. This formula has a similar flavor to the combinatorial formula for greedy bases in [13].

– In §2, we first use a combinatorial gadget called pipelines to decompose the **d**-vector of a cluster monomial into the ones of cluster variables, then give a formula for cluster variables using GCCs in Theorem 3.6.

We would like to point that that the above results are extending the results on the equioriented type A quivers given in [1].

Moreover, for cluster variables of a type A quiver, we construct a bijection between GCSs (which is equivalent to GCCs) and broken lines in Theorem 6.10 (the even rank case) and 6.13 (the general case), which relates our new formula with the theta functions constructed in [10]. The simplicity of this bijection came as a surprise for us: namely, under our setting, the i-th number in a GCS (which is a 0-1 sequence) is 0 if and only if the corresponding broken line bends at the i-th coordinate hyperplane $\mathbf{e}_i^\perp$. This suggests that there could be further connections between our new combinatorial formulas and theta functions, and thus could provide a new approach to understanding broken lines (which are difficult to describe explicitly in general); however, as explained in Remark 6.4, right now we are only able to construct a bijection for cluster variables of a type A quiver because the broken lines are special in this case.

The paper is organized as follows. In §2 we recall the definition of cluster algebra and some facts about type A quivers. In §3 we define the **d**-vector of a cluster monomial and introduce its decomposition using pipelines. §4 consists of the statements of the main results of the paper. In §5 we prove the GCC formula for cluster variables (Theorem 3.6) by establishing a bijection from GCCs to perfect matchings. Then in §6 we give the proof of the other main results of §4. In §7 we give the bijection between GCSs and broken lines. Then we give some examples in §8. In the appendix, we give another proof of Theorem 3.6 using T-paths.

In this section, we recall some definitions and fix notations about quivers and skew-

symmetric cluster algebras (§2.1) and some special type A quivers (§2.2).

1.1 Quivers and skew-symmetric cluster algebras

Recall that a finite oriented graph is a quadruple $Q = (Q_0, Q_1, h, t)$ formed by a finite set of vertices Q_0, a finite set of arrows Q_1 and two maps h and t from Q_1 to Q_0 which send an arrow α respectively to its head $h(\alpha)$ and its tail $t(\alpha)$. An arrow α whose head and tail coincide is a *loop*; a *2-cycle* is a pair of distinct arrows β and γ such that $h(\beta) = t(\gamma)$ and $t(\beta) = h(\gamma)$. Similarly, it is clear how to define n-cycles for $n \geqslant 3$. A vertex is a *source* (respectively a *sink*) if it is not the head (resp. the tail) of any arrow.

In this paper, a *quiver* is a finite oriented graph without loops or 2-cycles.

Given a quiver Q and a vertex $v \in Q_0$, the mutation $\mu_v(Q)$ is the new quiver Q' obtained as follows:

1. For every path of the form $u \to v \to w$, add a new arrow from u to w.

2. Reverse all arrows incident to v.

3. Remove all 2-cycles.

Let $Q = (Q_0, Q_1, h, t)$ be a quiver. Let $Q_0 = \{v_1, \ldots, v_n\} = \{1, \ldots, n\}$ (for simplicity, we denote v_i by i in this paper if no confusion arises; and later we also use notation $I = I_{\mathrm{uf}} = Q_0$ to denote the same set). Let $F = \mathbb{Q}(x_1, \ldots, x_n)$ be the field of rational functions in $x_1, x_2, \ldots, x_n$ with rational coefficients. A *seed* is a pair $(\mathbf{u}, Q)$ where $u = \{u_1, u_2, \ldots, u_n\}$ is a set of elements of F which freely generate the field F. For any vertex $i \in Q_0$, we denote

$$\prod_{j \to i} u_j = \prod_{\alpha \in Q_1, h(\alpha) = i} u_{t(\alpha)}, \qquad \prod_{i \to j} u_j = \prod_{\alpha \in Q_1, t(\alpha) = i} u_{h(\alpha)}.$$

The mutation $\mu_i(\mathbf{u}, Q)$ is the seed $(\mathbf{u}', Q')$ where $Q' = \mu_i(Q)$ and $\mathbf{u}'$ is obtained from $\mathbf{u}$ by replacing u_i by

$$u_i' = \frac{\displaystyle\prod_{j \to i} u_j + \prod_{i \to j} u_j}{u_i}.$$

Let $(\{x_1 \ldots, x_n\}, Q)$ be the initial seed. A *cluster* is a set $\mathbf{u}'$ which appears in a seed $(\mathbf{u}', Q')$ obtained from the initial seed by iterated mutations. An element in a cluster is called a *cluster variable*. A *cluster monomial* is a product of cluster variables in the same cluster. The (coefficient-free) *cluster algebra* $\mathcal{A}(Q)$ associated with Q is the subring of F generated by all cluster variables.

Next we recall the definition of the cluster algebra $\mathcal{A}_{\mathrm{prin}}$ with principal coefficient corresponding to the coefficient-free cluster algebra $\mathcal{A} = \mathcal{A}(Q)$. Let $I_{\mathrm{uf}} = Q_0 = \{1, \ldots, n\}$, $I = \{1, \ldots, 2n\}$. Define a quiver $\tilde{Q}$ with the vertex set $\tilde{Q}_0 := I$ and the edge set

$$\tilde{Q}_1 := Q_1 \cup \{(n+i, i) | i = 1, \ldots, n\}.$$

In other words, $\tilde{Q}$ is obtained from Q by adding an arrow from $n + i$ to i for each $i = 1, \ldots, n$. We call $i \in I_{\mathrm{uf}}$ unfrozen vertices and $i \in I \setminus I_{\mathrm{uf}}$ frozen vertices. Starting with the initial seed $(\{x_1, \ldots, x_{2n}\}, \tilde{Q})$, we mutate it iteratively similar as above, with the restriction that we only use the mutations μ_i for $1 \leqslant i \leqslant n$ (that is, only mutate at the unfrozen vertices). For each new seed $(\{x'_1, \ldots, x'_{2n}\}, \tilde{Q}')$, the first n rational functions $x'_1, \ldots, x'_n$ form a cluster. (Note that the frozen variables do not change after mutation, that is, $x'_i = x_i$ for $n + 1 \leqslant i \leqslant 2n$; and we do not consider them to be cluster variables.) The union of all clusters gives the set of cluster variables. The cluster algebra $\mathcal{A}_{\mathrm{prin}}$ is the subring of $\mathbb{Q}(x_1, \ldots, x_{2n})$ generated over $\mathbb{QP} := \mathbb{Q}[x_{n+1}^{\pm 1}, \ldots, x_{2n}^{\pm 1}]$ by all cluster variables.

Let $Q' = (Q'_0, Q'_1, h', t')$ be another quiver. We say that Q is a *subquiver* of Q' if

$$Q_0 \subseteq Q'_0 \text{ and } Q_1 \subseteq Q'_1$$

and $h(e) = h'(e)$ and $t(e) = t'(e)$ for any arrow $e \in Q_1$. We say that Q is a *full subquiver* of Q' if Q can be obtained from Q' by removing vertices $Q'_0 \setminus Q_0$ and their incident arrows.

1.2 Special classes of type A quivers

Here we define type A, linear, completely extended linear, and extended linear quivers. The relation among the four classes can be described as follows:

$$\{\text{type } A\} \supset \{\text{extended linear}\} \supset \{\text{completely extended linear}\}$$
$$\circlearrowleft \{\text{linear}\}$$

1.2.1 Type A quivers

By definition, type A quivers are those that are mutation equivalent to quivers of the form $\bullet \to \bullet \to \cdots \to \bullet$. In [3], A. Buan and D. Vatne showed that a type A quiver is a connected quiver such that

- all nontrivial simple cycles in the underlying graph have length 3, and the corresponding directed subgraphs are oriented (3-cycles);

- the vertex degrees of the underlying graph are at most 4; moreover, a degree-4 vertex belongs to two 3-cycles, a degree-3 vertex belongs to one 3-cycle.

1.2.2 Linear quivers

For two integers a and b, we denote $[a, b] = \{a, a + 1, \ldots, b\}$ if $a \leqslant b$, and $[a, b] = \emptyset$ if $a > b$.

A *linear quiver* is a quiver with n vertices $\{v_1, v_2, \ldots, v_n\}$ and $n - 1$ arrows in which any two consecutive vertices v_i and v_{i+1} ($i \in [1, n-1]$) are connected by a single arrow in either direction and there are no others arrows.

In order to have a convenient description for a linear quiver Q, we construct a sequence $\{\delta_i\}_{1 \leqslant i \leqslant n-1}$ such that $\delta_i = 0$ if there is an arrow going from the vertex v_i to the vertex v_{i+1}, and $\delta_i = 1$ otherwise. For example, for the quiver $1 \to 2 \to 3 \to 4 \leftarrow 5$, we have $(\delta_1, \ldots, \delta_4) = (0, 0, 0, 1)$.

1.2.3 Completely extended linear quiver

We define a *completely extended linear quiver* Q' as obtained from a linear quiver Q by attaching a 3-cycle to every edge, a 3-cycle to v_1, and a 3-cycle to v_n. (So Q' has $2n+3$ vertices.) By abuse of terminology, we also call the pair (Q, Q') a completely extended linear quiver, whenever we need to specify the linear quiver Q.

Let (Q, Q') be a completely extended linear quiver with $Q_0 = \{v_1, \ldots, v_n\}$ and $Q'_0 = \{v_1, \ldots, v_{2n+3}\}$. If $v_i \in Q'_0 \setminus Q_0$ is adjacent to both v_j and v_{j+1} then we also denote $v_{j,j+1} = v_i$. For the 3-cycle attached to v_1, the head (resp. tail) of the outgoing (resp. incoming) arrow is denoted $v_{1,0}$ (resp. $v_{1,1}$). We define $v_{n,0}$ and $v_{n,1}$ similarly.

Example 1.1. The quiver (Q, Q') in Figure 1 is a completely extended linear quiver, where Q is the linear part $1 \to 2 \to 3 \leftarrow 4$. In there, $v_{1,0} := 5$, $v_{1,1} := 6$, $v_{1,2} := 7$, $v_{2,3} := 8$, $v_{3,4} := 9$, $v_{4,0} := 10$ and $v_{4,1} := 11$.
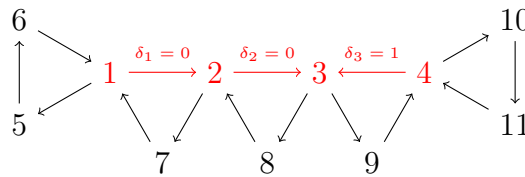

Figure 1: A completely extended linear quiver

For convenience, if $v_{j,k} = v_i$, then we denote the variable $x_{j,k} = x_i$.

1.2.4 Extended linear quivers

An *extended linear quiver* (Q, P) is obtained from a completely extended linear quiver (Q, Q') by removing some vertices (or none) in $Q'_0 \setminus Q_0$ and the arrows incident with them. Equivalently, we can characterize P as a quiver obtained from Q by adding some (or none) of the following:

- a 3-cycle or an edge hung on v_1, or

- a 3-cycle or an edge hung on v_n, or

- 3-cycles attached to some edges of Q.

There is an obvious way to obtain a completely extended linear quiver (Q, Q') from an extended linear quiver (Q, P) (up to relabeling vertices in $Q'_0 \setminus P_0$). An example is shown in Figure 2.
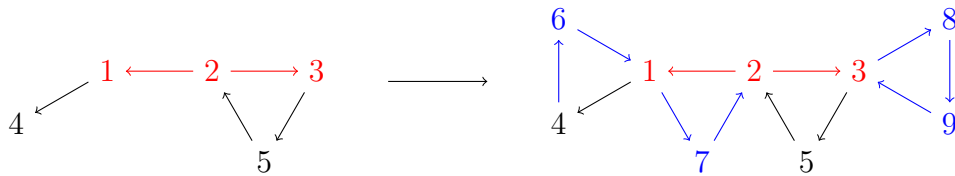
Figure 2: An extended linear quiver before and after being completed

2 Parametrization of Cluster Monomials by d-vectors

2.1 Cluster monomials and d-vectors

It is well known that any cluster algebra associated to a type A quiver with n vertices can be constructed from a triangulation on a $(n+3)$-gon P. A *diagonal* on the $(n+3)$-gon P is a line segment connecting two non-adjacent vertices. Two diagonals are said to be crossing if they intersect in the interior of P. A triangulation is a maximal set of non-crossing diagonals together with the boundary edges of P. A connected curve on the polygon is called a *pseudo-diagonal* if it is isotopic to a diagonal (and its endpoints are the same as those of the diagonal) and if its interior is in the interior of the polygon. Two pseudo-diagonals are said to be *crossing* if they intersect in the interior of the polygon. Note that if two pseudo-diagonals are not crossing, then the corresponding diagonals either coincide or do not cross in the interior of the polygon.

Given a triangulation of P, assume the non-crossing diagonals are $T_1, \ldots, T_n$ and the boundary edges are $T_{n+1}, \ldots, T_{2n+3}$. Then it induces a type A quiver Q with n vertices labeled by $T_1, \ldots, T_n$, such that there is an arrow from T_i to T_j if and only if there is a triangle with T_i and T_j being its two sides, and T_j is obtained from T_i by rotating $(< 180°)$ counterclockwisely about their common endpoint. (For example, for the triangulation in Figure 14, the corresponding quiver is $T_1 \to T_2 \to T_3 \leftarrow T_4$.) The triangulation also induces a larger quiver Q' with $2n+3$ vertices labeled by $T_1, \ldots, T_{2n+3}$ following the same rule as above. We call $T_{n+1}, \ldots, T_{2n+3}$ the frozen vertices of Q'.

Take P and Q as above. It is also known that the cluster variables of $\mathcal{A}(Q)$ are in natural bijection with all the diagonals of P. Using this bijection, a cluster monomial of $\mathcal{A}(Q)$ can be identified with a finite set of pairwise non-crossing (and non-identical) pseudo-diagonals, or equivalently with

$$\{(D_1, d_1), \ldots, (D_m, d_m)\},$$

where m is a non-negative integer, $D_1, \ldots, D_m$ are pairwise non-crossing diagonals, and $d_1, \ldots, d_m$ are positive integers. It is easy to see that given $\{(D_1, d_1), \ldots, (D_m, d_m)\}$, we can always find a finite set of pairwise non-crossing (and non-identical) pseudo-diagonals, such that d_i of these pseudo-diagonals are isotopic to D_i for $1 \leqslant i \leqslant m$.

The following definition uses a natural intersection number, which was already considered in [7, 16, 20, 21, 17].

Definition 2.1. For two diagonals D, E on the $(n + 3)$-gon, we define the intersection number $i(D, E)$ of D and E as follows:

$$i(D, E) := \begin{cases} 1, & \text{if } D \text{ and } E \text{ cross each other;} \\ -1, & \text{if } D \text{ and } E \text{ are the same;} \\ 0, & \text{otherwise.} \end{cases}$$

Then the **d**-vector of the cluster monomial $\{(D_1, d_1), \ldots, (D_m, d_m)\}$ is defined by

$$(a_1, \ldots, a_n) = (\sum_{j=1}^{m} d_j i(D_j, T_1), \ldots, \sum_{j=1}^{m} d_j i(D_j, T_n)).$$

It is easy to see that the **d**-vector $(a_1, \ldots, a_n)$ of any cluster monomial satisfies the following:

Property A. For any 3-cycle $i \to j \to k \to i$ in Q such that a_i, a_j, a_k are positive and satisfy the triangle inequalities (i.e., the sum of any two numbers is strictly greater than the third), the sum $a_i + a_j + a_k$ is even.

Proof. Assume $i \to j \to k \to i$ is a 3 cycle in Q such that a_i, a_j, a_k are positive and satisfy the triangle inequalities. Correspondingly, the three diagonals T_i, T_j, T_k form a triangle. Since $a_i \geqslant 0$, T_i is not in $\{D_1, \ldots, D_m\}$. Similar conclusion holds for T_j and T_k. There are two cases to consider:
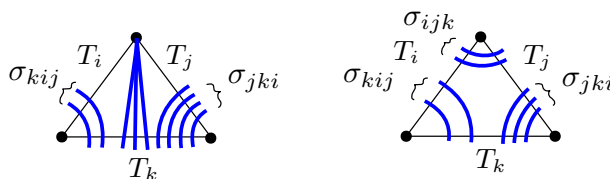

Figure 3: Two cases.

Case 1: there is at least one diagonal in $\{D_1, \ldots, D_m\}$ that crosses an edge and shares a vertex with the triangle formed by T_i, T_j, T_k. Without loss of generality, assume D_ℓ crosses T_k and shares a vertex with T_i and T_j (see Figure 3 Left, where we draw pseudo-diagonals to illustrate the multiplicity). In this case all the a_i diagonals (with multiplicity) crossing T_i must cross T_k (otherwise they will cross D_ℓ). Similarly, all the a_j diagonals (with multiplicity) crossing T_j must cross T_k. Thus the total number of diagonals crossing T_k is $a_k = a_i + a_j + a_\ell$, contradicting the triangle inequality assumption.

Case 2: each diagonal in $\{D_1, \ldots, D_m\}$ either cross two of T_i, T_j, T_k, or none (see Figure 3 Right). In this case, assume σ_{ijk} is the number of diagonals (with multiplicity) crossing T_i and T_j, etc. (So $\sigma_{ijk} = \sigma_{jik}$.) Then $a_i = \sigma_{ijk} + \sigma_{kij}$, $a_j = \sigma_{ijk} + \sigma_{jki}$, $a_k = \sigma_{jki} + \sigma_{kij}$, therefore $a_i + a_j + a_k = 2(\sigma_{ijk} + \sigma_{jki} + \sigma_{kij})$ is even. $\square$

2.2 Construction of pipelines

In this subsection, the notion of pipelines will be introduced (which we do not claim any originality[1]) to help us understanding d-vectors of cluster monomials.

Let $\mathcal{W}$ be the set of all integer vectors $(a_1, \ldots, a_n)$ satisfying Property A, that is, $a_i + a_j + a_k$ is even when both of the following conditions are satisfied:

(i) $i \to j \to k \to i$ is a 3-cycle, and

(ii) a_i, a_j, a_k are positive and satisfy the triangle inequalities.

(In particular, if Q has no 3-cycles, then $\mathcal{W} = \mathbb{Z}^n$.)

In this subsection we prove that the cluster monomials of $\mathcal{A}(Q)$ are in bijection with $\mathcal{W}$. We will define a map from $\mathcal{W}$ to the cluster monomials, which would then induce the immediate bijection. Let $[x]_+ = \max(x, 0)$ for any real number x. Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathcal{W}$. We define a function $\sigma : \mathbb{R}_{\geqslant 0}^3 \to \mathbb{R}_{\geqslant 0}$ as follows:

$$\sigma(x, y, z) = \frac{[x + y - z]_+ - [x - y - z]_+ - [y - x - z]_+}{2} = \begin{cases} x, & \text{if } y > x + z; \\ y, & \text{if } x > y + z; \\ 0, & \text{if } z > x + y; \\ \dfrac{x + y - z}{2}, & \text{otherwise.} \end{cases} \tag{1}$$

For convenience, we denote

$$\sigma_{ijk}^{\mathbf{a}} = \sigma(a_i, a_j, a_k).$$

(If no confusion shall arise, we denote $\sigma_{ijk} = \sigma_{ijk}^{\mathbf{a}}$. Note that it coincides with σ_{ijk} appeared in the proof of Property A, Case 2.)

Remark 2.2. By case-by-case analysis for the cases shown in Figure 3, it is obvious that the equality $\sigma_{kij} + \sigma_{jki} \leqslant [a_k]_+$ (and similar inequalities obtained by permuting i, j, k) holds for any 3-cycle $i \to j \to k \to i$.

Proposition 2.3. *Let Q be a type A quiver. Then*

*1. $\mathbf{a} \in \mathbb{Z}^n$ is the **d**-vector of some cluster monomial of $\mathcal{A}(Q)$ if and only if $\mathbf{a} \in \mathcal{W}$.*

*2. Two distinct cluster monomials have different **d**-vectors.*

Proof. Let M be the set of monomials of $\mathcal{A}(Q)$. An element in M can be identified with a set $\{(D_1, d_1), \ldots, (D_m, d_m)\}$. Let $f : M \to \mathcal{W}$ be the map that sends a cluster monomial to its **d**-vector. It follows from Property A that the image of f is indeed in $\mathcal{W}$.

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[1]A set of pipelines (to be defined in Proposition 2.3) are, in many aspects, similar to a lamination as in [6], and a pipeline is similar to a curve in a lamination (we thank one of the referees to pointing it out to us). For example, our construction of the set of pipelines given in Proposition 2.3 is essentially the same as the "Reconstruction" of the lamination given in [6, page 11]. The main difference is the following: the two ends of our pipelines are marked points, and that two pipelines may meet at marked points; in contrast, two curves of a lamination do not end at marked points, and two curves do not meet at endpoints. Moreover, we do not discuss homotopy equivalence of pipelines in this paper.

Next we construct a map $g : \mathcal{W} \to M$. Given $\mathbf{a} \in \mathcal{W}$, we construct *the set of pipelines associated to* $\mathbf{a}$ in three steps. We consider the triangulation of the $(n+3)$-gon corresponding to Q, and T_i, T_j, T_k appeared below are diagonals or boundary edges; in other words, $1 \leqslant i, j, k \leqslant 2n + 3$.

Step 1: If $a_i > 0$ then draw a_i marked points on T_i to separate it into $a_i + 1$ segments. If $a_i < 0$ then draw $-a_i$ pipes so that these pipes are pairwise non-crossing pseudo-diagonals isotopic to T_i and that they do not cross any other T_j $(j \neq i)$.

Step 2: If T_i and T_j are two sides of a triangle with the third side T_k, then for $1 \leqslant r \leqslant \sigma_{ijk}$, we join the two r-th marked points on T_i and T_j (ordering in the increasing distance from the common endpoint of T_i and T_j) by a pipe inside the triangle. Draw these pipes so that they are disjoint from each other and from the pipes constructed in Step 1.

Step 3: Suppose that T_i, T_j, T_k form a triangle. Then for each marked point on T_k that is not connected by a pipe to any marked point on T_i or T_j (see Figure 3 Right), we draw a pipe from the marked point to the common endpoint of T_i and T_j. Draw these pipes inside the triangle in such a way that they are non-crossing with each other and with the pipes constructed in Step 1,2.

A pipeline is a union of pipes connected consecutively through the marked points (but not through the vertices of the $(n+3)$-gon). Then the pipelines are pairwise non-crossing. Since the endpoints of each pipeline are non-adjacent vertices of the polygon, every pipeline is a pseudo-diagonal. Hence the set of pipelines corresponds to a cluster monomial, which we define as the image $g(\mathbf{a})$.

It follows immediately from the above construction that g is the inverse of f. So f is bijective. The surjectivity of f implies (1) and the injectivity of f implies (2). $\qquad \square$

The above propostion allows us to denote the (unique) cluster monomial with $\mathbf{d}$-vector $(a_1, \ldots, a_n)$ by $x[a_1, \ldots, a_n]$ or $x[\mathbf{a}]$.

Each pipeline Λ corresponds to a 0-1 sequence $\mathbf{b} = \mathbf{b}_\Lambda = (b_1, \ldots, b_n)$ such that

$$ b_i = \begin{cases} 0, & \text{if the pipeline } \Lambda \text{ is disjoint from } T_i; \\ 1, & \text{otherwise.} \end{cases} \tag{2} $$

Note that Λ corresponds to a linear full subquiver of Q. Two pipelines sharing the same endpoints correspond to the same 0-1 sequence. Let S be the multiset of 0-1 sequences corresponding to all the pipelines in the set of pipelines associated to $\mathbf{a}$. Then

$$ x[\mathbf{a}] = \prod_{\mathbf{b} \in S} x[\mathbf{b}]. \tag{3} $$

Example 2.4. The first two pictures in Figure 4 are a type A quiver with 7 vertices and its corresponding triangulation on the 10-gon. For a clearer illustration, the 10-gon is drawn as a concave polygon. The bottom illustrates the construction of pipelines for $\mathbf{a} = (3, 3, 3, 2, 4, 3, 1)$.

The set of pipelines associated to $\mathbf{a}$ consists of 5 pipelines, namely those passing through edges sets $\{1, 2, 3, 4\}$, $\{1, 2, 5, 6\}$, $\{1, 2, 5, 7\}$, $\{3, 4, 5, 6\}$, $\{3, 5, 6\}$, respectively. So the cluster monomial $x[\mathbf{a}]$ is decomposed as $x[3, 3, 3, 2, 4, 3, 1] = x[1, 1, 1, 1, 0, 0, 0] \cdot$
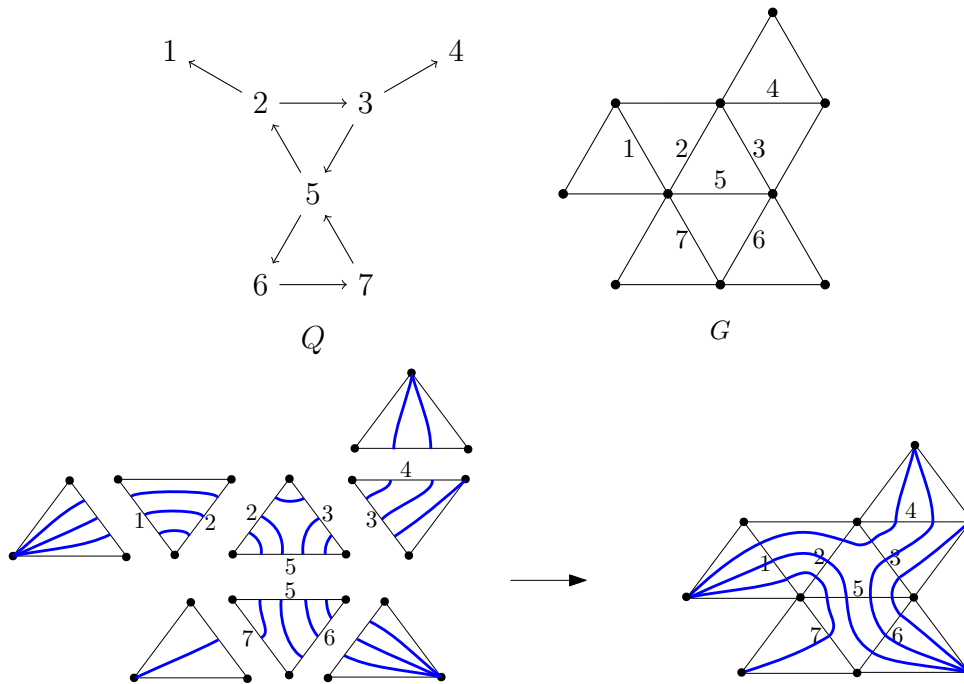
Figure 4: The construction of pipelines

$x[1, 1, 0, 0, 1, 1, 0] \cdot x[1, 1, 0, 0, 1, 0, 1] \cdot x[0, 0, 1, 1, 1, 1, 0] \cdot x[0, 0, 1, 0, 1, 1, 0]$, and the multiset S (which is indeed a set in this example) consists of the five 0-1 sequences appeared above.

Lemma 2.5. *For* $\mathbf{a} \in \mathbb{Z}^n$, *assume that* $[\mathbf{a}]_+ := [[a_1]_+, \ldots, [a_n]_+]$ *satisfies Property A and* $x[[\mathbf{a}]_+] = \prod x[\mathbf{b}]$ *is a factorization into cluster variables in the same cluster. Then*

$$x[\mathbf{a}] = x[[\mathbf{a}]_+] \prod_i x_i^{[-a_i]_+} = \prod x[\mathbf{b}] \prod_i x_i^{[-a_i]_+},$$

and that $x[\mathbf{b}]$ *and* x_i $(a_i < 0)$ *are in the same cluster.*

Proof. If $a_i < 0$, then there is no $\mathbf{b}$ satisfying $b_i > 0$. Thus no pseudo-diagonals corresponding to pipelines for $[\mathbf{a}]_+$ will cross the diagonal T_i. Therefore, after adding T_i we still get a set of non-crossing pseudo-diagonals, which means that $x[\mathbf{b}]$ and x_i $(a_i < 0)$ are in the same cluster. It then follows that $x[\mathbf{a}] = x[[\mathbf{a}]_+] \prod_i x_i^{[-a_i]_+}$. $\qquad\square$

Remark 2.6. Assume that Q is a full subquiver of Q'. We compare cluster variables in various cluster algebras.

For simplicity, we denote the vertex sets $Q_0 = \{1, \ldots, n\}$ and $Q'_0 = \{1, \ldots, n'\}$. By definition, the cluster algebra with coefficients, denoted $\mathcal{A}(Q, Q')$, is generated by only those cluster variables in $\mathcal{A}(Q')$ obtained from iteratively mutating the initial cluster variables $x_1, \ldots, x_n$ only at vertices in $\{1, \ldots, n\}$. (Vertices in $\{n+1, \ldots, n'\}$ are called frozen vertices. The coefficients are in $\mathbb{Z}[x_{n+1}^{\pm 1}, \ldots, x_{n'}^{\pm 1}]$.) Thus, there is a natural bijection sending a cluster variable in $\mathcal{A}(Q, Q')$ of the form $x[d_1, \ldots, d_n]$ to the cluster variable $x[d_1, \ldots, d_n, 0, \ldots, 0] \in \mathcal{A}(Q')$ of the same expression.

There is also a natural bijection sending a cluster variable $x[d_1, \ldots, d_n] \in \mathcal{A}(Q, Q')$ to the cluster variable $x[d_1, \ldots, d_n] \in \mathcal{A}(Q)$, given by substituting x_i by 1 for $i \in [n+1, n']$. More generally, if Q is a full subquiver of P, and P is a full subquiver of Q', then there is a natural bijection sending a cluster variable $x[d_1, \ldots, d_n] \in \mathcal{A}(Q, Q')$ to $x[d_1, \ldots, d_n] \in \mathcal{A}(Q, P)$ given by substituting x_i by 1 for $i \in Q'_0 \setminus P_0$.

If Q is a full subquiver of Q'', and Q' is the vertex-induced subquiver of Q'' whose vertex set consists of vertices in Q_0 and those adjacent to Q_0, then the cluster variables in $\mathcal{A}(Q, Q')$ have the same expressions as those in $\mathcal{A}(Q, Q'')$.

3 Globally Compatible Collections

In this section, we give three formulas for the cluster monomial $x[\mathbf{a}]$ for $\mathbf{a} \in \mathcal{W}$. All results here will be proved in §6. These formulas extend the results on the equioriented type A quivers given in [1]. For general type A quivers, a new situation we need to handle is the appearance of 3-cycles.

First we reduce to a special case. We can replace $\mathbf{a}$ by $[\mathbf{a}]_+$, thus can assume $a_i \geqslant 0$, because of Lemma 2.5. Moreover, for any edge $i \to j$ of Q that is not in a 3-cycle, we can add a vertex k and two arrows $j \to k$ and $k \to i$ (the vertex k is a frozen vertex). Indeed, assume the modified quiver is Q'. Then by Remark 2.6, once we have a formula for cluster monomials for Q', we can set $x_i = 1$ for all $i \in Q'_0 \setminus Q_0$ and obtain a formula for cluster monomials for Q. In the rest of the paper, we assume that $\mathbf{a} = [\mathbf{a}]_+$ and

Q is of type A with more than one vertex, and every edge of Q is in a 3-cycle.   (4)

3.1 A formula using 0-1 sequences

Fix a deg-2 vertex i_0 of Q (which exists because of (4)). For $i \in Q_0$, denote by $d(i)$ be the distance (i.e., the length of the shortest directed path) from i_0 to i. Let $\mathbf{s}_i = (s_{i,1}, s_{i,2}, \ldots, s_{i,a_i}) \in \{0,1\}^{a_i}$ be a 0-1 sequence, and define

$$|\mathbf{s}_i| = \sum_{r=1}^{a_i} s_{i,r}, \qquad |\bar{\mathbf{s}}_i| = \sum_{r=1}^{a_i} (1 - s_{i,r}) = a_i - |\mathbf{s}_i|.$$

We say that a sequence of 0-1 sequences $\mathbf{s} := (\mathbf{s}_1, \ldots, \mathbf{s}_n)$ is a *globally compatible sequence (abbreviated GCS)* if the following holds for any 3-cycle $i \to j \to k \to i$:

- If $d(i) < d(j) < d(k)$, then $(s_{i,t}, s_{j,t}) \neq (1, 0)$ for $1 \leqslant t \leqslant \sigma_{ijk}$;

- If $d(j) < d(k) < d(i)$, then $(s_{i,a_i+1-t}, s_{j,a_j+1-t}) \neq (1, 0)$ for $1 \leqslant t \leqslant \sigma_{ijk}$;

- If $d(k) < d(i) < d(j)$, then $(s_{i,a_i+1-t}, s_{j,t}) \neq (1, 0)$ for $1 \leqslant t \leqslant \sigma_{ijk}$.

Theorem 3.1. *For any $\mathbf{d}$-vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geqslant 0}^n$ (i.e., $\mathbf{a} \in \mathcal{W} \cap \mathbb{Z}_{\geqslant 0}^n$), we have the following formula for the corresponding cluster monomial:*

$$x[\mathbf{a}] = \left(\prod_{l=1}^n x_l^{-a_l} \right) \sum_{\mathbf{s}} \left(\prod_i x_i^{e_i} \right), \quad where \ e_i = \sum_{i \to j} |\bar{\mathbf{s}}_j| + \sum_{k \to i} |\mathbf{s}_k| - \sum_{i \to j \to k \to i} \sigma_{jki}, \quad (5)$$

and **s** *runs through all GCSs.*

This formula specializes to the formula given in [1] for a linear quiver.

3.2 A formula using Dyck paths

We recall the following definition from [12].

Let (a_1, a_2) be a pair of nonnegative integers. Let $c = \min(a_1, a_2)$. The *maximal Dyck path* of type $a_1 \times a_2$, denoted by $\mathcal{D} = \mathcal{D}^{a_1 \times a_2}$, is a lattice path from $(0, 0)$ to (a_1, a_2) that is as close as possible to the diagonal joining $(0, 0)$ and (a_1, a_2), but never goes above it. A *corner* is a subpath consisting of a horizontal edge followed by a vertical edge.

Definition 3.2. Let $\mathcal{D}_1$ (resp. $\mathcal{D}_2$) be the set of horizontal (resp. vertical) edges of a maximal Dyck path $\mathcal{D} = \mathcal{D}^{a_1 \times a_2}$. We label $\mathcal{D}$ with the *corner-first index* in the following sense:

(a) edges in $\mathcal{D}_1$ are indexed as $u_1, \dots, u_{a_1}$ such that u_i is the horizontal edge of the i-th corner for $i \in [1, c]$ and u_{c+i} is the i-th of the remaining horizontal ones for $i \in [1, a_1 - c]$,

(b) edges in $\mathcal{D}_2$ are indexed as $v_1, \dots, v_{a_2}$ such that v_i is the vertical edge of the i-th corner for $i \in [1, c]$ and v_{c+i} is the i-th of the remaining vertical ones for $i \in [1, a_2 - c]$.

(Here we count corners from bottom left to top right, count vertical edges from bottom to top, and count horizontal edges from left to right.)
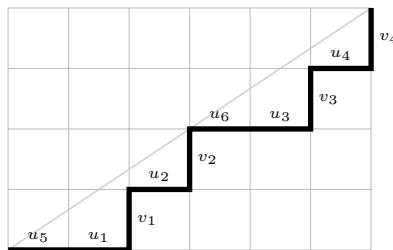

Figure 5: A maximal Dyck path

Definition 3.3. Let $S_1 \subseteq \mathcal{D}_1$, $S_2 \subseteq \mathcal{D}_2$, $s \in \mathbb{Z}_{\geqslant 0}$. We say that S_1 and S_2 are *s-compatible* if for every $1 \leqslant r \leqslant s$, either $u_r \notin S_1$ or $v_r \notin S_2$. In other words, neither of the first s corners are in the subpath $S_1 \cup S_2$.

Note that the following definition of global compatibility is different from [12].

Definition 3.4. Let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}_{\geqslant 0}^n$ be a **d**-vector. For each pair $i \to j$ in Q, let $\mathcal{D}^{(i,j)}$ be a maximal Dyck path $\mathcal{D}^{a_i \times a_j}$. We label $\mathcal{D}^{(i,j)}$ with the corner-first index (described

in Definition 3.2), whose horizontal edges are denoted $u_1^{(i,j)}, \ldots, u_{a_i}^{(i,j)}$ and vertical edges are denoted by $v_1^{(i,j)}, \ldots, v_{a_j}^{(i,j)}$. We say that the collection

$$\left\{ S_\ell^{(i,j)} \subseteq \mathcal{D}_\ell^{(i,j)} \mid i \to j \text{ is an arrow, } \ell \in \{1,2\} \right\}$$

is a *globally compatible collection (abbreviated GCC)* if and only if for any $k \to i \to j$ in Q:

- if $j \to k$ is also an arrow in Q, then $S_1^{(i,j)}$ and $S_2^{(i,j)}$ are σ_{ijk}-compatible, and

$$v_r^{(k,i)} \in S_2^{(k,i)} \iff u_{a_i+1-r}^{(i,j)} \notin S_1^{(i,j)}, \text{ for all } r \in [1, a_i];$$

- otherwise,

$$v_r^{(k,i)} \in S_2^{(k,i)} \iff u_r^{(i,j)} \notin S_1^{(i,j)}, \text{ for all } r \in [1, a_i].$$

Theorem 3.5. *Assume $n > 1$. For any $\mathbf{d}$-vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geqslant 0}^n$, we have the following formula for the corresponding cluster monomial:*

$$x[\mathbf{a}] = \left(\prod_{l=1}^n x_l^{-a_l} \right) \sum \left(\prod_{i \to j} x_i^{\left| S_2^{(i,j)} \right|} x_j^{\left| S_1^{(i,j)} \right|} \right) \cdot \prod_{i \to j \to k \to i} x_i^{-\sigma_{jki}}, \tag{6}$$

where the sum runs over all GCCs. (Note that because of the rotational symmetry, each 3-cycle contributes three terms to the last product.)

3.3 A more explicit formula for cluster variables

The third method of computing the cluster monomial with given $\mathbf{d}$-vector is to first decompose the cluster monomial into a product of cluster variables using pipelines and then use a formula for cluster variables using GCCs.

For given $\mathbf{d}$-vector, we construct pipelines as in the subsection 2.2. Then each pipeline corresponds to a cluster variable. We give the GCC formula of the cluster variable $x[\mathbf{a}]$ in a completely extended linear quiver (Q, Q'), where $\mathbf{a} = (a_1, \ldots, a_{n'})$ such that $a_i = 1$ if $i \in Q$, and $a_i = 0$ otherwise. For any arrow $(i + \delta_i) \to (i + 1 - \delta_i)$ of Q we attach a Dyck path $\mathcal{D}^{(i)} = \mathcal{D}^{1 \times 1}$, which consists of one horizontal edge and one vertical edge. (Recall that δ_i is defined in §1.2.2.) Then Definition 3.4 specializes to the following:

Let $S_{i,r} \subseteq \mathcal{D}_r^{(i)}$ for $i \in [1, n-1]$, $r \in [1, 2]$. We say that the collection $\{S_{i,r}\}$ is a *GCC* if

$$(|S_{i,1}|, |S_{i,2}|) \neq (1,1) \text{ for } i \in [1, n-1], \text{ and the following holds for } i \in [2, n-1]: \tag{7}$$

(a) if $(\delta_{i-1}, \delta_i) = (0, 0)$, then $|S_{i-1,2}| \neq |S_{i,1}|$;

(b) if $(\delta_{i-1}, \delta_i) = (1, 1)$, then $|S_{i-1,1}| \neq |S_{i,2}|$;

(c) if $(\delta_{i-1}, \delta_i) = (0, 1)$, then $|S_{i-1,2}| = |S_{i,2}|$;

(d) if $(\delta_{i-1}, \delta_i) = (1, 0)$, then $|S_{i-1,1}| = |S_{i,1}|$.

Theorem 3.6. *Let (Q, Q') be a completely extended linear quiver, and $n = |Q_0| > 1$, $n' = |Q_0'|$. Define $\mathbf{a}_Q = (a_1, \ldots, a_{n'})$ such that $a_i = 1$ if $i \in Q$, and $a_i = 0$ otherwise. Then the cluster variable with $\mathbf{d}$-vector $\mathbf{a}_Q$ is*

$$x[\mathbf{a}_Q] = \left(\prod_{i=1}^{n} x_i^{-1} \right) \sum \left(\prod_{i=0}^{n} y_i \right), \tag{8}$$

where the sum runs over all GCCs $\{S_{i,r}\}$,

$$y_i := x_{i+\delta_i}^{|S_{i,2}|} x_{i+1-\delta_i}^{|S_{i,1}|} x_{i,i+1}^{1-|S_{i,1}|-|S_{i,2}|} = \begin{cases} x_i, & \text{if } (|S_{i,1}|, |S_{i,2}|) = (\delta_i, 1-\delta_i) \\ x_{i+1}, & \text{if } (|S_{i,1}|, |S_{i,2}|) = (1-\delta_i, \delta_i) \\ x_{i,i+1}, & \text{if } (|S_{i,1}|, |S_{i,2}|) = (0, 0) \end{cases}$$

for $i \in [1, n-1]$, and

$$y_0 := \begin{cases} x_{1,0}, & \text{if } |S_{1,1+\delta_1}| = 1 - \delta_1, \\ x_{1,1}, & \text{otherwise,} \end{cases} \qquad y_n := \begin{cases} x_{n,0}, & \text{if } |S_{n-1,2-\delta_{n-1}}| = \delta_{n-1}, \\ x_{n,1}, & \text{otherwise.} \end{cases}$$

Remark 3.7. Note that the above theorem induces a formula for the cluster variables of any type A quiver, that is, any mutation-equivalent type A quiver (possibly including some 3-cycles). Indeed, let $\tilde{Q}$ be any type A quiver and $x[\mathbf{a}]$ be a non-initial cluster variable (the formulas for initial cluster variables are trivial) with $\mathbf{d}$-vector $\mathbf{a} = (a_1, \ldots, a_{n'})$, with $n' = |\tilde{Q}_0|$. Then the subset of vertices $\{i \mid a_i = 1\}$ is equal to the set of vertices Q_0 of a linear full subquiver Q. By relabeling vertices if necessary, we assume $Q_0 = \{1, \ldots, n\}$ (thus $a_1 = \cdots = a_n = 1$ and $a_{n+1} = \cdots = a_{n'} = 0$ and for convenience we denote $\mathbf{a} = 1^n 0^{n'-n}$. By removing vertices in $\tilde{Q}_0$ but not in or adjacent to Q_0, we get an extended linear quiver (Q, P); from this extended linear quiver we can obtain a completely extended linear quiver Q'. Define $n' = |Q_0'|$, $m = |P_0|$. Thanks to Remark 2.6, a formula for cluster variable $x[1^n 0^{n'-n}] \in \mathcal{A}(Q, Q')$ induces a formula for $x[1^n 0^{m-n}] \in \mathcal{A}(Q, P)$ by substituting $x_i = 1$ for $i \in Q_0' \setminus P_0$, which is also a formula for $x[\mathbf{a}] = x[1^n 0^{n'-n}] \in \mathcal{A}(\tilde{Q})$. (See Example 7.2.)

Remark 3.8. We also give a formula for cluster variables of any type A quiver in terms of GCS as follows. Let $\tilde{Q}$ be a type A quiver and Q be a linear full subquiver of $\tilde{Q}$, as in Remark 3.7. Define

$$\mathbf{a}_Q = (a_i), \quad \text{where } a_i = \begin{cases} 1, & \text{if } i \in Q_0; \\ 0, & \text{if } i \notin Q_0. \end{cases} \tag{9}$$

Relabelling vertices if necessary, we assume that Q is $1 \longleftrightarrow 2 \longleftrightarrow \cdots \longleftrightarrow n$ where each arrow can go either direction. We will give a formula of the cluster variable $x[\mathbf{a}_Q]$.

Let $\mathbf{s} = (s_i) \in \{0,1\}^{n'}$, $0 \leqslant s_i \leqslant a_i$ for every $1 \leqslant i \leqslant n'$. We say that $\mathbf{s}$ is a GCS if $(s_i, s_j) \neq (1,0)$ for every arrow $i \to j$ in Q_1. Because $s_i = 0$ for $i > n$, by abuse of notation we also view $\mathbf{s}$ as an element in $\{0,1\}^n$.

Let $\deg_Q^-(i)$ be the number of arrows in $\tilde{Q}_1$ whose tails are in Q_0 and heads are the vertex i, let $\deg_Q^+(i)$ be the number of arrows in $\tilde{Q}_1$ whose heads are in Q_0 and tails are the vertex i. Let K be the set of vertices $k \in \tilde{Q}_0 \setminus Q_0$ such that there exists a 3-cycle $k \to i \to j \to k$ with $s_i = s_j = 1$; or equivalently,

$$K = \{k \in \tilde{Q}_0 \setminus Q_0 \mid \deg_Q^-(k) = \deg_Q^+(k) = 1\}.$$

Denote $\bar{s}_i = a_i - s_i$. We have the following formula, where arrows $i \to j$ are in $\tilde{Q}$:

$$x[\mathbf{a}_Q] = \left(\prod_{r \in Q_0} x_r^{-1} \right) \frac{\sum_{\mathbf{s}} \left(\prod_{i \to j} x_i^{\bar{s}_j} x_j^{s_i} \right)}{\prod_{k \in K} x_k} = \sum_{\mathbf{s}} z_{\mathbf{s}}, \quad \text{where } z_{\mathbf{s}} = \frac{\prod_{i \to j} x_i^{\bar{s}_j} x_j^{s_i}}{\prod_{r \in Q_0 \cup K} x_r} \qquad (10)$$

and $\mathbf{s}$ runs through all GCSs.

4 A bijection between perfect matchings and GCCs

In this section, we first recall the construction of snake diagram and the formula of cluster variables using perfect matching as in [16], then give a bijective proof of Theorem 3.6 via perfect matchings.

Associated to a completely extended linear quiver (Q, Q'), we recursively construct the snake diagram by gluing n-tiles together as follows: we first put the 2nd-tile to the right side of the 1st-tile; suppose the ith-tile is placed, we add the $(i+1)^{\mathrm{th}}$-tile to the right side or on top of the ith-tile such that the $(i-1)^{\mathrm{th}}$, $(i)^{\mathrm{th}}$ and $(i+1)^{\mathrm{th}}$-tiles are in the same row or column if and only if $\delta_{i-1} \neq \delta_i$.

Next, we label the edges as follows.

- The common edge of the ith-tile and the $(i+1)^{\mathrm{th}}$-tile is labeled $T_{i,i+1}$.

- Denote by $Pl^{(i)}$ the parallelogram bounded by the main diagonals of the ith-tile and the $(i+1)^{\mathrm{th}}$-tile and two boundary edges. Any edge forming an angle of $135°$ with the main diagonal of the ith-tile will be labeled $T_i^{(j)}$ (where j indicates the parallelogram to which the edge belongs).

- For convenience, we let $Pl^{(0)}$ be the right triangle with legs $T_{1,0}$ and $T_{1,1}$, and let $Pl^{(n)}$ the right triangle with legs $T_{n,0}$ and $T_{n,1}$.

- The edges of the first and the last tiles are labeled as in Figure 7.

A perfect matching of the snake diagram is a set of edges such that each vertex is incident to exactly one edge in the set.

For the fixed completely extended linear quiver (Q, Q'), let $\mathcal{M}$ be the set of all perfect matchings in the associated snake diagram, and $\mathcal{G}$ be the set of all GCCs. We shall
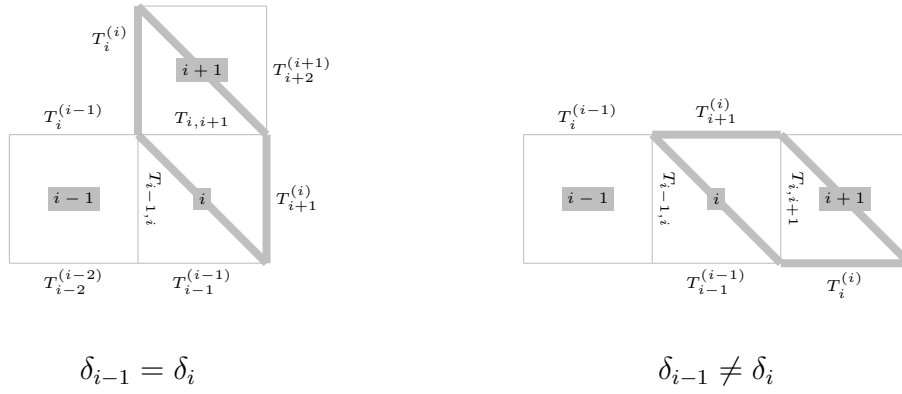
$$\delta_{i-1} = \delta_i \qquad\qquad \delta_{i-1} \neq \delta_i$$

Figure 6: Labels of edges in $(i-1)^{\text{th}}$, i^{th} and $(i+1)^{\text{th}}$-tiles in two cases

First tile   Last tile
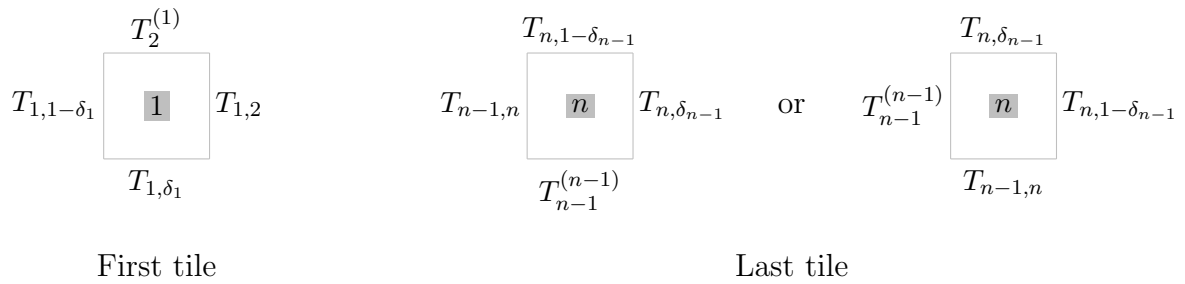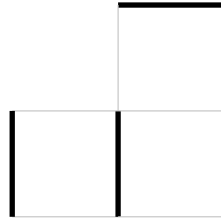
Figure 7: Labels of edges in the first and last tiles

Figure 8: The associated snake diagram of the quiver $1 \to 2 \to 3$ with a perfect matching.

construct a bijective map $\psi_{\mathcal{M},\mathcal{G}} : \mathcal{M} \to \mathcal{G}$ and its inverse $\psi_{\mathcal{G},\mathcal{M}}$. First we prove a simple lemma.

Lemma 4.1. *For any perfect matching γ and $i \in [0, n]$, there is exactly one edge of γ lies in $Pl^{(i)}$.*

Proof. The statement is obviously true for $i = 0$ and $i = n$. Suppose the statement is false for some $i \in [1, n-1]$. Since γ is a perfect matching, we are in one of the following two cases.

Case 1: both $T_i^{(i)}, T_{i+1}^{(i)}$ are in γ. If we remove $Pl^{(i)}$ (4 vertices and 3 edges), then the rest graph has two components which have odd number of vertices and have perfect matchings. This is a contradiction.

Case 2: none of the three edges $T_i^{(i)}, T_{i+1}^{(i)}, T_{i,i+1}$ lies in γ. Then we remove the three edges (but do not remove the vertices) and apply the same argument as in Case 1. $\qquad\square$

Remark 4.2. Thanks to Lemma 4.1, we can write a perfect matching γ as $\{\gamma_1, \ldots, \gamma_n\}$ where $\gamma_i \in Pl^{(i)}$.

Definition 4.3. - (i) We define a map $\psi_{\mathcal{M},\mathcal{G}} : \mathcal{M} \to \mathcal{G}$ by sending $\gamma \in \mathcal{M}$ to $\{S_{i,r}\} \in \mathcal{G}$ such that for $i \in [1, n-1]$,

$$(|S_{i,1}|, |S_{i,2}|) = \begin{cases} (\delta_i, 1 - \delta_i), & \text{if } T_i^{(i)} \in \gamma, \\ (1 - \delta_i, \delta_i), & \text{if } T_{i+1}^{(i)} \in \gamma, \\ (0, 0), & \text{if } T_{i,i+1} \in \gamma. \end{cases}$$

(By Lemma 4.1, exactly one of the three cases occurs.)

(ii) We define a map $\psi_{\mathcal{G},\mathcal{M}} : \mathcal{G} \to \mathcal{M}$ by sending $\{S_{i,r}\} \in \mathcal{G}$ to the set of edges $\gamma = \{\gamma_0, \gamma_1, \ldots, \gamma_n\}$ such that

$$\gamma_i = \begin{cases} T_i^{(i)}, & \text{if } (|S_{i,1}|, |S_{i,2}|) = (\delta_i, 1 - \delta_i) \\ T_{i+1}^{(i)}, & \text{if } (|S_{i,1}|, |S_{i,2}|) = (1 - \delta_i, \delta_i) \\ T_{i,i+1}, & \text{if } (|S_{i,1}|, |S_{i,2}|) = (0, 0) \end{cases}$$

for $i \in [1, n-1]$, and
$$\gamma_0 = \begin{cases} T_{1,0} & \text{if } |S_{1,1+\delta_1}| = 1 - \delta_1, \\ T_{1,1} & \text{otherwise,} \end{cases} \qquad \gamma_n = \begin{cases} T_{n,0} & \text{if } |S_{n-1,2-\delta_{n-1}}| = \delta_{n-1}, \\ T_{n,1} & \text{otherwise.} \end{cases}$$

We assign a weight $w(u)$ for each edge u of the snake diagram as follows for all i, j:

$$w(T_j^{(i)}) = x_j, \quad w(T_{j,j+1}) = x_{j,j+1} \tag{11}$$

For a perfect matching $\gamma = (\gamma_0, \ldots, \gamma_n)$, define its weight $w(\gamma) = \prod_{i=0}^{n} w(\gamma_i)$. In [16] it is proved that the cluster variable with $\mathbf{d}$-vector $\mathbf{a}$ is

$$x[\mathbf{a}] = \left(\prod_{i=1}^{n} x_i^{-1}\right) \sum_{\gamma} w(\gamma). \tag{12}$$

Theorem 4.4. *The maps $\psi_{\mathcal{M},\mathcal{G}}$ and $\psi_{\mathcal{G},\mathcal{M}}$ are well-defined and are inverses of each other. Moreover, $w(\gamma_i) = y_i$, thus $\psi_{\mathcal{M},\mathcal{G}}$ induces a bijective proof of Theorem 3.6 using (12).*

Proof. (i) We show that $\psi_{\mathcal{M},\mathcal{G}}$ is well-defined, that is, $\psi_{\mathcal{M},\mathcal{G}}(\gamma) = \{S_{i,r}\}$ satisfies the condition (7). It's clear from the construction that for every $i \in [1, n-1]$, $(|S_{i,1}|, |S_{i,2}|) \neq (1, 1)$. Next, we prove (a) and (c) of (7), since (b) and (d) can be proved similarly.

For (a), we suppose $(\delta_{i-1}, \delta_i) = (0, 0)$ and need to show that $T_{i-1}^{(i-1)} \in \gamma \Leftrightarrow T_{i+1}^{(i)} \notin \gamma$. This is true because the two edges $T_{i-1}^{(i-1)}$ and $T_{i+1}^{(i)}$ are incident to the same deg-2 vertex, thus exactly one of them is in γ. (See the left diagram in Figure 6.)

For (c), we suppose $(\delta_{i-1}, \delta_i) = (0, 1)$ and need to show that $T_{i-1}^{(i-1)} \in \gamma \Leftrightarrow T_{i+1}^{(i)} \in \gamma$. These two edges are opposite edges of a tile which is the middle of three tiles in a row or a column. Deleting these two edges will separate the snake diagram into two graphs with even number of vertices each. Thus the two edges must be both in γ or not in γ. (See the right diagram in Figure 6.)

(ii) We show that $\psi_{\mathcal{G},\mathcal{M}}$ is well-defined, that is, $\psi_{\mathcal{G},\mathcal{M}}(\{S_{i,r}\}) = \gamma$ is a perfect matching. Since the snake diagram has $2n + 2$ vertices and γ has $n + 1$ edges, it suffices to show that all edges in γ are disjoint. We assume the contrary that γ_c shares a vertex with γ_d for some $0 \leqslant c < d \leqslant n$. Since $\gamma_c \in Pl^{(c)}$ and $\gamma_d \in Pl^{(d)}$, $Pl^{(c)}$ and $Pl^{(d)}$ much be consecutive, thus $d = c + 1$.

We first assume $1 \leqslant c \leqslant n - 2$. We shall discuss two cases $(\delta_c, \delta_{c+1}) = (0, 0)$ and $(0, 1)$, and omit the other two cases $(1, 0)$ and $(1, 1)$ since the proof is similar.

Case $(\delta_c, \delta_{c+1}) = (0, 0)$: since $\{S_{i,r}\}$ is a GCC, we must have

$$(|S_{c,1}|, |S_{c,2}|, |S_{c+1,1}|, |S_{c+1,2}|) = (0, 1, 0, 1), (0, 1, 0, 0), (1, 0, 1, 0), \text{ or } (0, 0, 1, 0).$$

Correspondingly,

$$(\gamma_c, \gamma_{c+1}) = (T_c^{(c)}, T_{c+1}^{(c+1)}), (T_c^{(c)}, T_{c+1,c+2}), (T_{c+1}^{(c)}, T_{c+2}^{(c+1)}), \text{ or } (T_{c,c+1}, T_{c+2}^{(c+1)}).$$

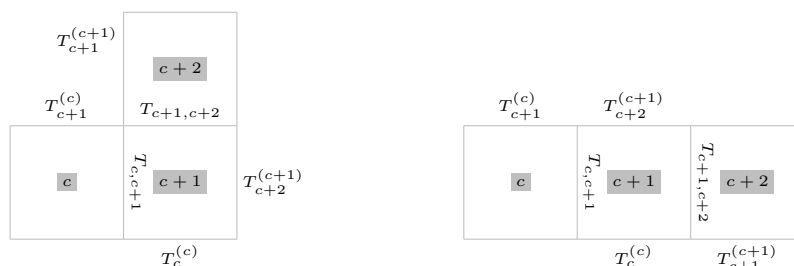It is obvious from Figure 9 that γ_c and γ_{c+1} are disjoint, a contradiction as expected.

Figure 9: Left: $(\delta_c, \delta_{c+1}) = (0, 0)$ and Right: $(\delta_c, \delta_{c+1}) = (0, 1)$

Case $(\delta_c, \delta_{c+1}) = (0, 1)$: similar as the above case,

$$(\gamma_c, \gamma_{c+1}) = (T_c^{(c)}, T_{c+2}^{(c+1)}), (T_{c+1}^{(c)}, T_{c+1}^{(c+1)}), (T_{c+1}^{(c)}, T_{c+1,c+2}), (T_{c,c+1}, T_{c+1}^{(c+1)}) \text{ or } (T_{c,c+1}, T_{c+1,c+2}).$$

We get the expected contradiction by observing Figure 9.

The cases of $c = 0$ and $c = n - 1$ are proved by a similar argument.

(iii) The fact that $\psi_{\mathcal{M},\mathcal{G}}$ and $\psi_{\mathcal{G},\mathcal{M}}$ are inverse of each other, and $w(\gamma_i) = y_i$, follows easily from their definitions. □

Example 4.5. If $\gamma = \{T_1, T_3, T_6, T_8, T_{10}\}$ then $\psi_{\mathcal{M},\mathcal{G}}(\gamma) = ((0, 1), (0, 0), (1, 0))$ as shown in Figure 10.
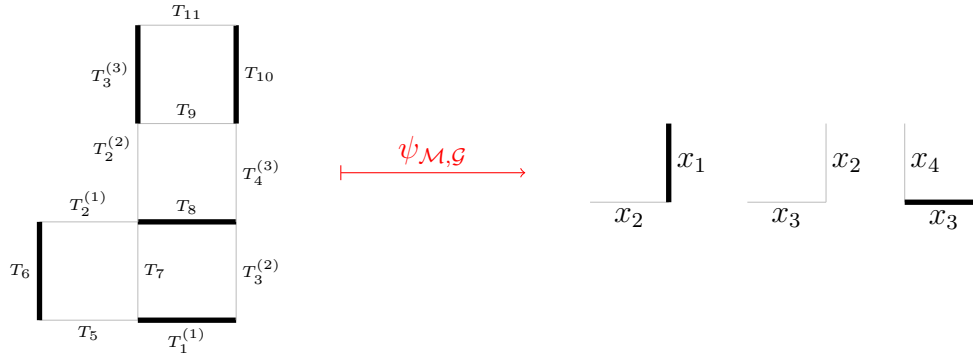
Figure 10: An example of the map $\psi_{\mathcal{M},\mathcal{G}}$

5   Proof of Main Theorems 3.1 and 3.5

We have explained in §3.3 that, in order to compute cluster variables, it suffices to have the formula for a completely extended linear quiver, namely Theorem 3.6. This theorem follows from Theorem 4.4. In this section, we show how to derive Theorem 3.1 and Theorem 3.5 from Theorem 3.6.

5.1   Proof of Theorem 3.5

Let $x'[\mathbf{a}]$ be the right hand side of the formula in Theorem 3.5. We shall show that (i) the GCCs for the $\mathbf{d}$-vector $\mathbf{a}$ are in one-to-one correspondence with the collections of GCCs for the $\mathbf{d}$-vectors $\mathbf{b}$'s described in (2); (ii) $x'[\mathbf{a}] = \prod_{\mathbf{b}} x'[\mathbf{b}]$; and (iii) $x'[\mathbf{b}] = x[\mathbf{b}]$ using Theorem 3.6. It then follows that $x'[\mathbf{a}] = x[\mathbf{a}]$.

(i) Let $\{S_\ell^{(i,j)}\}$ be any GCC for the $\mathbf{d}$-vector $\mathbf{a}$. For each pipeline Λ, let $\mathbf{b} = \mathbf{b}_\Lambda$, we construct a GCC $\{S_\ell^{(i,j),\Lambda}\}$ for the $\mathbf{d}$-vector $\mathbf{b}$ by requiring the following for each arrow $i \to j$:

– if Λ intersects the edge i at the r-th marked point, then $|S_1^{(i,j),\Lambda}| = 1$ if and only if $u_r^{(i,j)} \in S_1^{(i,j)}$,

– if Λ intersects the edge j at the r-th marked point, then $|S_2^{(i,j),\Lambda}| = 1$ if and only if $v_r^{(i,j)} \in S_2^{(i,j)}$,

(in both case the marked points are ordered in the increasing distance from the common endpoint of i and j).

To verify that $\{S_\ell^{(i,j),\Lambda}\}$ is a GCC for the $\mathbf{d}$-vector $\mathbf{b}$, we need to check the conditions in Definition 3.4. The only nontrivial condition to check is that for a 3-cycle $k \to i \to j \to k$, $S_1^{(i,j),\Lambda}$ and $S_2^{(i,j),\Lambda}$ are $\sigma_{ijk}^{\mathbf{b}}$-compatible. That reduces to showing that $(|S_1^{(i,j),\Lambda}|, |S_1^{(i,j),\Lambda}|) \neq (1,1)$ in the case $(b_i, b_j, b_k) = (1,1,0)$. In this case, Λ intersects edges i and j at the r-th marked points for some $r \leqslant \sigma_{ijk}^{\mathbf{a}}$. Then either $u_r^{(i,j)} \notin S_1^{(i,j)}$ or $v_r^{(i,j)} \notin S_2^{(i,j)}$. In the former case, $|S_1^{(i,j),\Lambda}| = 0$; in the latter case, $|S_2^{(i,j),\Lambda}| = 0$. Therefore $(|S_1^{(i,j),\Lambda}|, |S_2^{(i,j),\Lambda}|) \neq (1,1)$.

It is easy to see that a unique GCC $\{S_\ell^{(i,j)}\}$ is determined if we take any collection of GCCs $\{S_\ell^{(i,j),\Lambda}\}$ for all pipelines Λ. So we have the desired one-to-one correspondence.

(ii) We show that $\prod_{\mathbf{b}} x'[\mathbf{b}] = x'[\mathbf{a}]$. Since $\sum \mathbf{b} = \mathbf{a}$, it suffices to show that, for each GCC $\{S_\ell^{(i,j)}\}$, letting $\{S_\ell^{(i,j),\Lambda}\}$ be defined as in (i), the following holds (recall that $\mathbf{b} = \mathbf{b}_\Lambda$ depends on Λ):

$$\prod_\Lambda \left(\prod_{i \to j} x_i^{\left|S_2^{(i,j),\Lambda}\right|} x_j^{\left|S_1^{(i,j),\Lambda}\right|} \cdot \prod_{i \to j \to k \to i} x_i^{-\sigma_{jki}^{\mathbf{b}}} \right) = \prod_{i \to j} x_i^{\left|S_2^{(i,j)}\right|} x_j^{\left|S_1^{(i,j)}\right|} \cdot \prod_{i \to j \to k \to i} x_i^{-\sigma_{jki}^{\mathbf{a}}}.$$

Since the left hand side is equal to

$$\prod_{i \to j} x_i^{\sum_\Lambda \left|S_2^{(i,j),\Lambda}\right|} x_j^{\sum_\Lambda \left|S_1^{(i,j),\Lambda}\right|} \cdot \prod_{i \to j \to k \to i} x_i^{-\sum_\Lambda \sigma_{jki}^{\mathbf{b}}},$$

it suffices to show that $\sum_\Lambda |S_2^{(i,j),\Lambda}| = |S_2^{(i,j)}|$, $\sum_\Lambda |S_1^{(i,j),\Lambda}| = |S_1^{(i,j)}|$, and $\sum_\Lambda \sigma_{jki}^{\mathbf{b}} = \sigma_{jki}^{\mathbf{a}}$. The first two are clear. To show the last equality: first note that if Λ is disjoint from the edge j, then $b_j = 0$ and thus $\sigma_{jki}^{\mathbf{b}} = 0$. So we only need to consider those Λ's that intersect j. Let Λ_r ($1 \leqslant r \leqslant a_j$) be the pipeline that intersects j at the r-th marked point (ordered in the increasing distance to the common endpoint of j and k). If $r \leqslant \sigma_{jki}^{\mathbf{a}}$, then $(b_j, b_k, b_i) = (1, 1, 0)$, thus $\sigma_{ijk}^{\mathbf{b}} = 1$; otherwise, either $b_j = 0$ or $b_k = 0$, thus $\sigma_{ijk}^{\mathbf{b}} = 0$. Therefore $\sum_\Lambda \sigma_{jki}^{\mathbf{b}} = \sigma_{jki}^{\mathbf{a}}$.

(iii) We show that $x'[\mathbf{b}] = x[\mathbf{b}]$. By Remark 2.6, it suffices to show that, in the setting of Theorem 3.6, the right hand side of (8) is equal to $x'[\mathbf{a}]$. It breaks down to show that, for $i' \in [0, n]$, the following equality holds for the i'-th 3-cycle $i \to j \to k \to i$ in Q' (for $i' \in [1, n-1]$, the i'-th 3-cycle is the one that contains vertices v_i, v_{i+1} and $v_{i,i+1}$; the 0-th 3-cycle is $v_1 \to v_{1,0} \to v_{1,1} \to v_1$; the n-th 3-cycle is $v_n \to v_{n,0} \to v_{n,1} \to v_n$):

$$\left(x_i^{\left|S_2^{(i,j)}\right|} x_j^{\left|S_1^{(i,j)}\right|} \right) \left(x_j^{\left|S_2^{(j,k)}\right|} x_k^{\left|S_1^{(j,k)}\right|} \right) \left(x_k^{\left|S_2^{(k,i)}\right|} x_i^{\left|S_1^{(k,i)}\right|} \right) \cdot x_i^{-\sigma_{jki}} x_j^{-\sigma_{kij}} x_k^{-\sigma_{ijk}} = y_{i'} \qquad (13)$$

We shall only prove the case when $i' \in [1, n-1]$ and $\delta_{i'} = 0$, because other cases can be proved in a similar way. In this case, the i'-th 3-cycle is $v_i \to v_{i+1} \to v_{i,i+1} \to v_i$ (where $i = i'$), and the left hand side of (13) is equal to

$$\left(x_i^{\left|S_{i,2}\right|} x_j^{\left|S_{i,1}\right|} \right) \left(x_j^0 x_k^{1 - \left|S_{i,2}\right|} \right) \left(x_k^{1 - \left|S_{i,1}\right|} x_i^0 \right) \cdot x_k^{-1} = x_i^{\left|S_{i,2}\right|} x_j^{\left|S_{i,1}\right|} x_k^{1 - \left|S_{i,1}\right| - \left|S_{i,2}\right|} = y_{i'}.$$

So (13) holds.

5.2 Proof that Theorem 3.5 implies Theorem 3.1

We give a bijection between GCSs and GCCs. Let $\mathbf{s}$ be a GCS. Consider a 3-cycle $i \to j \to k \to i$, labeled in the way that $d(k) < d(i) < d(j)$. Then we define

$$S_1^{(k,i)} = \{u_r \in \mathcal{D}_1^{(k,i)} \big| s_{k,r} = 1\}, \qquad S_2^{(k,i)} = \{v_r \in \mathcal{D}_2^{(k,i)} \big| s_{i,r} = 0\}$$
$$S_1^{(i,j)} = \{u_r \in \mathcal{D}_1^{(i,j)} \big| s_{i,a_i+1-r} = 1\}, \qquad S_2^{(i,j)} = \{v_r \in \mathcal{D}_2^{(i,j)} \big| s_{j,r} = 0\}$$
$$S_1^{(j,k)} = \{u_r \in \mathcal{D}_1^{(j,k)} \big| s_{j,a_j+1-r} = 1\}, \qquad S_2^{(j,k)} = \{v_r \in \mathcal{D}_2^{(j,k)} \big| s_{k,a_k+1-r} = 0\}$$

It is then easy to check that the conditions of GCSs and GCCs, as well as the two theorems, are equivalent under this bijection.

6 A bijection between GCSs and broken lines

This section is devoted to the construction of a bijection between GCSs and broken lines in the even rank case (Theorem 6.10) and the general case (Theorem 6.13). In §6.1, we give a description of the g-vector of a cluster variable, which will determine the initial direction of the broken lines. In §6.2, we recall the necessary facts on scattering diagrams, broken lines, and theta functions. The rest of the section is to state and prove Theorem 6.10 and 6.13.

In this section, we let $\tilde{Q}$ be a type A quiver and Q be a linear full subquiver of $\tilde{Q}$. Relabelling vertices if necessary, we assume that Q is

$$Q: \quad 1 \longleftrightarrow 2 \longleftrightarrow \cdots \longleftrightarrow n \quad \text{(each arrow can go either direction).} \tag{14}$$

6.1 g-vectors

It is shown in [10, Theorem 7.5 (4)] that, if m is the g-vector of a cluster variable, then the cluster variable is exactly ϑ_m. So we first study g-vectors.

Lemma 6.1. *Let* $\mathbf{a}_Q$ *be defined as in* (9). *Then the g-vector* $m_0 = (g_r)$ *of the cluster variable* $x[\mathbf{a}_Q]$ *is given by*

$$g_r = \begin{cases} \deg_Q^-(r) - 1, & \text{if } r \in Q_0; \\ 1, & \text{if } r \notin Q_0 \text{ and } (\deg_Q^+(r), \deg_Q^-(r)) = (0,1); \\ 0, & \text{if } r \notin Q_0 \text{ and } (\deg_Q^+(r), \deg_Q^-(r)) \neq (0,1). \end{cases}$$

Proof. We use the description of g-vectors in [17, §13.1]:

$$g_\gamma = \deg \frac{x(P_-)}{\text{cross}(T^\circ, \gamma)}$$

It is easy to verify that P_- corresponds to the GCS $\mathbf{s} = (s_i)$ where $s_i = 1$ if and only if $i \in Q$ (this is the unique GCS satisfying $\bar{s}_i = 0$). Thus

$$x(P_-) = \frac{\left(\prod_{i \to j} x_i^{\bar{s}_j} x_j^{s_i} \right)}{\prod_{k \in K} x_k} = \frac{\prod_{i \to j} x_j^{s_i}}{\prod_{k \in K} x_k} \tag{15}$$

Meanwhile, $\text{cross}(T^\circ, \gamma) = \prod_{l=1}^n x_l^{-a_l}$. So we can compute the g-vector case by case:

If $r \in Q$: the power of x_r in (15) is equal to the number of arrows $i \to r$ with $s_r = 1$, which is $\deg_Q^-(r)$. Since $a_r = 1$, $g_r = \deg_Q^-(r)$.

If $r \notin Q$ and $(\deg_Q^+(r), \deg_Q^-(r)) = (0,1)$: then $r \notin K$, thus the power of x_r in (15) is $\deg_Q^-(r) = 1$.

If $r \notin Q$ and $(\deg_Q^+(r), \deg_Q^-(r)) \neq (0,1)$: then $(\deg_Q^+(r), \deg_Q^-(r)) = (1,1), (1,0)$ or $(0,0)$. If it is $(1,1)$, then $r \in K$, thus the power of x_r in (15) is $\deg_Q^-(r) - 1 = 0$; if it is $(1,0)$ or $(0,0)$, then $\deg_Q^- = 0$ and $r \notin K$, thus x_r does not appear in (15). In all cases, we get $g_r = 0$. $\qquad\square$

Remark 6.2. The above is equivalent to the following description of the g-vector of $x[\mathbf{a}_Q]$:

$$g_r = \begin{cases} -1, & \text{if } r \in Q \text{ is not the head of any arrow in } Q; \\ 1, & \text{if "}r \in Q \text{ is the head of two arrows in } Q\text{", or "}r \notin Q \text{ is adjacent to only} \\ & \text{one vertex in } Q, \text{ either } 1 \to r \text{ or } n \to r\text{";} \\ 0, & \text{otherwise.} \end{cases}$$

6.2 Scattering diagrams and broken lines

We only recall the necessary facts needed in our paper, specialized in our setting. For more reference, see [10].

Recall that in §1.1 we defined $I = I_{\mathrm{uf}} = \{1, \ldots, n\}$ for a coefficient free cluster algebra $\mathcal{A}$ of rank n, and $I = \{1, \ldots, 2n\}$ and $I_{\mathrm{uf}} = \{1, \ldots, n\}$ for a principal coefficient cluster algebra $\mathcal{A}_{\mathrm{prin}}$ of rank n.

Let $M \cong \mathbb{Z}^{n'}$, $N = \mathrm{Hom}(M, \mathbb{Z})$ with a basis $(e_i)_{i \in I}$, $M_{\mathbb{R}} := M \otimes \mathbb{R}$ with dual basis $(f_i)_{i \in I}$. N is equipped with a skew-symmetric form $\{\cdot, \cdot\}$. Let N_{uf} be a sublattice of N with basis $(e_i)_{i \in I_{\mathrm{uf}}}$. Define

$$N^+ := \Big\{ \sum_{i \in I_{\mathrm{uf}}} a_i e_i \,|\, a_i \geqslant 0, \sum a_i > 0 \Big\}.$$

We assume the Fundamental Assumption: the map $p_1^* : N_{\mathrm{uf}} \to M$ given by $n \mapsto \{n, \cdot\}$ is injective.

Choose a strictly convex top-dimensional cone $\sigma \subseteq M_{\mathbb{R}}$, with associated monoid $P := \sigma \cap M$, such that $p_1^*(e_i) \in P \setminus \{0\}$ for all $i \in I_{\mathrm{uf}}$. Let $\widehat{\mathbb{Z}[P]}$ be the completion of $\mathbb{Z}[P]$ at the maximal monomial ideal $\mathfrak{m}$ generated by $\{x^m | m \in P \setminus \{0\}\}$.

A *wall* is a pair $(\mathfrak{d}, f_{\mathfrak{d}})$ where $\sigma \in M_{\mathbb{R}}$ is a $(\mathrm{rank}N - 1)$-dimensional convex rational polyhedral cone contained in $n_0^\perp$ for some primitive vector $n_0 \in N^+$, and $f_{\mathfrak{d}} = 1 + \sum_{k>0} c_k z^{kp_1^*(n_0)}$.

A *scattering diagram* $\mathfrak{D}$ is a collection of walls such that there are only finite many walls $(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D}$ satisfying $f_{\mathfrak{d}} \not\equiv 1 \mod \mathfrak{m}^k$ for each $k > 0$.

Given a scattering diagram $\mathfrak{D}$, define its *support* to be

$$\mathrm{Supp}(\mathfrak{D}) := \bigcup \mathfrak{d},$$

and define its *singular locus* to be

$$\mathrm{Sing}(\mathfrak{D}) := \bigcup \partial \mathfrak{d} \cup \bigcup_{\dim \mathfrak{d}_1 \cap \mathfrak{d}_2 = n-2} \mathfrak{d}_1 \cap \mathfrak{d}_2.$$

For a smooth path $\gamma : [0, 1] \to M_{\mathbb{R}} \setminus \mathrm{Sing}(\mathfrak{D})$ whose endpoints are not in $\mathrm{Supp}(\mathfrak{D})$ and that it is transversal to each wall that it crosses, we define an automorphism $\theta_{\gamma, \mathfrak{D}}$ of $\widehat{\mathbb{Z}[P]}$ as follows: for each $k > 0$, we can find numbers $0 < t_1 \leqslant t_2 \leqslant \cdots \leqslant t_s < 1$ such that $\gamma(t_i) \in \mathfrak{d}_i$, with $(\mathfrak{d}_i, f_{\mathfrak{d}_i}) \in \mathfrak{D}$, $f_{\mathfrak{d}_i} \not\equiv 1 \mod \mathfrak{m}^k$, and $\mathfrak{d}_i \neq \mathfrak{d}_j$ if $t_i = t_j$, and s taken as large as possible. For each i, define an automorphism θ_i to be

$$\theta_i(x^m) := x^m f_{\mathfrak{d}_i}^{m \cdot n_i}$$

where n_i is the primitive vector annihilating $\mathfrak{d}_i$ and satisfying $\gamma'(t_i) \cdot n_i < 0$. Define

$$\theta_{\gamma,\mathfrak{D}}^{(k)} = \theta_s \theta_{s-1} \cdots \theta_1, \quad \text{and} \quad \theta_{\gamma,\mathfrak{D}} = \lim_{k\to\infty} \theta_{\gamma,\mathfrak{D}}^{(k)}.$$

A *consistent scattering diagram* $\mathfrak{D}$ is a scattering diagram such that $\theta_{\gamma,\mathfrak{D}}$ only depends on the endpoints of γ.

Two scattering diagrams $\mathfrak{D}$, $\mathfrak{D}'$ are *equivalent* if $\theta_{\gamma,\mathfrak{D}} = \theta_{\gamma,\mathfrak{D}'}$ for all paths γ for which both sides are defined. Let $p^* : N \to M$ be given by $n \mapsto \{n, \cdot\}$. A wall $\mathfrak{d}$ is *incoming* if $p^*(n_0) \in \mathfrak{d}$, where $n_0 \perp \mathfrak{d}$ and $n_0 \in N^+$; otherwise, the wall is *outgoing*. Define the initial scattering diagram

$$\mathfrak{D}_{\text{in}} := \{(e_i^\perp, 1 + z^{v_i}) | i \in I_{\text{uf}}\}, \quad \text{where } v_i := p_1^*(e_i). \tag{16}$$

Note that all walls in $\mathfrak{D}_{\text{in}}$ are incoming.

Definition 6.3. Let $\mathfrak{D} \supset \mathfrak{D}_{\text{in}}$ be a consistent scattering diagram such that $\mathfrak{D} \setminus \mathfrak{D}_{\text{in}}$ consists only of outgoing walls.

(It is proved in [10, Theorem 1.7] that $\mathfrak{D}$ exists and is unique up to equivalence.)

Let $\mathfrak{D}$ be as defined in Definition 6.3, $m_0 \in M \setminus \{0\}$ and $\mathbf{Q} \in M_\mathbb{R} \setminus \text{Supp}(\mathfrak{D})$. A *broken line* for m_0 with endpoint $\mathbf{Q}$ is a piecewise linear continuous proper path $\gamma : (-\infty, 0] \to \mathbb{R}^n \setminus \text{Sing}(\mathfrak{D})$ with a finite number of domains of linearity. This path comes along with the data of, for each domain of linearity $L \subseteq (-\infty, 0]$ of γ, a monomial $c_L z^{m_L} \in \mathbb{Z}[M]$. This data satisfies the following properties:

1. $\gamma(0) = \mathbf{Q}$.

2. If L is the first (and therefore unbounded) domain of linearity of γ, then $c_L z^{m_L} = z^{m_0}$.

3. For t in a domain of linearity L, $\gamma'(t) = -m_L$.

4. Let $t \in (-\infty, 0)$ be a point at which γ is not linear, passing from domain of linearity L to L'. Let $\mathfrak{D}_t = \{(\mathfrak{d}, f_\mathfrak{d}) \in \mathfrak{D} | \gamma(t) \in \mathfrak{d}\}$. Then $c_{L'} z^{m_{L'}}$ is a term in the formal power series
$$c_L z^{m_L} \prod_{(\mathfrak{d}, f_\mathfrak{d}) \in \mathfrak{D}_t} f_\mathfrak{d}^{|\langle n_0, m_L \rangle|}.$$

We denote the monomial attached to the final domain of linearity of γ by $\text{Mono}(\gamma)$.

For given $m_0 \in M \setminus \{0\}$ and $\mathbf{Q} \in M_\mathbb{R} \setminus \text{Supp}(\mathfrak{D})$, the *theta function* is defined to be

$$\vartheta_{\mathbf{Q},m_0} = \sum_\gamma \text{Mono}(\gamma),$$

where γ runs over all broken lines for m_0 with endpoint $\mathbf{Q}$.

Remark 6.4. The broken lines that we shall construct in this paper will not bend on walls that are not in the initial scattering diagram $\mathfrak{D}_{\text{in}}$ defined in (16). A priori, there may exist broken lines that do bend on walls in $\mathfrak{D} \setminus \mathfrak{D}_{\text{in}}$ (i.e., outgoing walls); but we shall argue that there are no such broken lines in our setting (where we only consider those appearing in the theta function corresponding to cluster variables; see §6.4.2). In general, broken lines can also bend on outgoing walls if we consider those appearing in the theta function corresponding to cluster monomials. Because of this obstacle, we could not yet extend the cluster variable formulas Theorem 6.10 and 6.13 (which explicitly decribe the broken lines using GCSs) to cluster monomials.

6.3 A bijection between GCSs and broken lines (in the even rank case)

In this subsection, we give a bijection between GCSs and broken lines for the cluster variable $x[\mathbf{a}_Q]$ when n' (the rank of the cluster algebra $\mathcal{A}$) is even. This is exactly the case when the exchange matrix B is of full rank, which guarantees that the Fundamental Assumption in [10] is satisfied. (Indeed, since [2, Lemma 3.2] asserts that the rank is invariant under mutation, it is suffices to consider the linear quiver $1 \to 2 \to \cdots \to n'$, in which case the determinant of B is 1 if n' is even, 0 if n' is odd.)

Definition 6.5. For a given GCS $\mathbf{s}$, we say that $r \in [1, n]$ is an *adjustable* position if
    (a) the r-th coordinate of $\mathbf{s}$ is 0, and
    (b) the sequence obtained from $\mathbf{s}$ by replacing the r-th coordinate by 1 is still a GCS.

Remark 6.6. An adjustable position always exists as long as some coordinate of $\mathbf{s} = (s_j)$ is 0. This follows from the following equivalent definition: let $Q^{\mathbf{s}}$ be the full subquiver of Q with vertex set $\{j \in Q_0 \,|\, s_j = 0\}$. Then r is an adjustable position if and only if r is a sink of $Q^{\mathbf{s}}$. Since Q is acyclic, $Q^{\mathbf{s}}$ is also acyclic, thus $Q^{\mathbf{s}}$ has at least one sink.

Definition 6.7. For a given GCS $\mathbf{s}$, let $\ell = n - |\mathbf{s}|$. Define $\mathbf{s}^{(\ell)} = \mathbf{s}$, and define w_i $(1 \leqslant i \leqslant \ell)$ and $\mathbf{s}_i$ $(0 \leqslant i \leqslant \ell - 1)$ backward recursively as follows. Assume $\mathbf{s}^{(i)}$ is defined for some i satisfying $1 \leqslant i \leqslant \ell$. Define $w_i \in [1, n]$ to be the smallest adjustable position, and the corresponding new GCS by $\mathbf{s}^{(i-1)}$. (It is clear that $w_1, \ldots, w_\ell$ are mutually distinct, and the set $\{w_1, \ldots, w_\ell\} = \{j \in Q_0 \,|\, s_j = 0\}$.)

Definition 6.8. (i) Define a function $g_Q : \{0, 1\}^n \to \{0, \pm 1\}^{n'}$ sending $\mathbf{s}$ to $(g_r)_{1 \leqslant r \leqslant n'}$ where

$$g_r = \begin{cases} \deg_{Q,\mathbf{s}}^{1\to}(r) + \deg_{Q,\mathbf{s}}^{0\leftarrow}(r) - 1, & \text{if } r \in Q_0, \text{ or } r \text{ and two vertices in } Q_0 \text{ form a 3-cycle;} \\ \deg_{Q,\mathbf{s}}^{1\to}(r) + \deg_{Q,\mathbf{s}}^{0\leftarrow}(r), & \text{otherwise.} \end{cases}$$

(17)

Here $\deg_{Q,\mathbf{s}}^{1\to}(i)$ is the number of arrows $j \to i$ in Q_1' such that $j \in Q_0$ and $s_j = 1$, and $\deg_{Q,\mathbf{s}}^{0\leftarrow}(i)$ is the number of arrows $j \leftarrow i$ in Q_1' such that $j \in Q_0$ and $s_j = 0$. (Remark 6.9(2) explains why $g_r \in \{0, \pm 1\}$.)
    (ii) Define $m_i = (m_{i,r}) := g_Q(\mathbf{s}^{(i)}) \in \{0, \pm 1\}^{n'}$ for $0 \leqslant i \leqslant \ell$.

Remark 6.9. (1) Note that $\mathbf{s}^{(0)} = [1, \ldots, 1] \in \{0, 1\}^n$. We claim that $m_0 = (g_r)$ coincides with the definition given in Lemma 6.1. Indeed, note that $\deg_{Q,\mathbf{s}}^{0\leftarrow}(r) = 0$. If $r \in Q_0$, nothing needs to be proved. We assume $r \notin Q_0$ in the rest of the paragraph. If $(\deg_Q^+(r), \deg_Q^-(r)) = (0, 1)$, then we are in the second case of (17), $g_r = \deg_{Q,\mathbf{s}}^{1\rightarrow}(r) = \deg_Q^-(r) = 1$; if $(\deg_Q^+(r), \deg_Q^-(r)) = (0, 0)$ or $(1, 0)$, then we are in the second case of (17), thus $g_r = \deg_{Q,\mathbf{s}}^{1\rightarrow}(r) = \deg_Q^-(r) = 0$; if $(\deg_Q^+(r), \deg_Q^-(r)) = (1, 1)$, then we are in the first case of (17), thus $g_r = \deg_{Q,\mathbf{s}}^{1\rightarrow}(r) - 1 = \deg_Q^-(r) - 1 = 0$.

(2) It is easy to check that every coordinate of $m_i = (g_r)$ satisfies $-1 \leqslant g_r \leqslant 1$. Indeed, $g_r \geqslant -1$ is obvious; to check $g_r \leqslant 1$, note that every vertex r is adjacent to at most two vertices in Q_0, so we only need to show that in the second case of (17), it is impossible to have $\deg_{Q,\mathbf{s}}^{1\rightarrow}(r) + \deg_{Q,\mathbf{s}}^{0\leftarrow}(r) \geqslant 2$. But this equality implies that r is adjacent to at least two vertices in Q_0; it follows from the description of type A quivers that r and two vertices in Q_0 form a 3-cycle, which contradicts the condition of the second case.

The following main theorem gives a bijective construction of $\vartheta_{\mathbf{Q}, m_0} = x[\mathbf{a}_Q]$.

Theorem 6.10. *Assume the rank n' of the cluster algebra is even, and $\mathbf{Q} = (q_1, q_2, \ldots, q_{n'})$ such that*

$$0 < q_i \ll q_1 \ll q_2 \ll \cdots \ll q_n, \text{ for each } i = n+1, \ldots n'. \tag{18}$$

(Here $x \ll y$ means $0 < x/y \leqslant \varepsilon$, where $\varepsilon > 0$ is a fixed real number satisfying $(1 + \varepsilon)^n < 2$.) Let m_0 be defined as in Lemma 6.1. Then there is a bijection φ between the set of GCS and the set of broken lines for m_0 with endpoint $\mathbf{Q}$, such that each GCS $\mathbf{s}$ is sent to a broken line $\gamma = \varphi(\mathbf{s})$ satisfying $\mathrm{Mono}(\gamma) = z_{\mathbf{s}}$ (defined in (10)): using notation in Definition 6.7, the broken line γ can be explicitly described as follows:
   (i) *it has $\ell + 1$ domains of linearity $L_0, L_1, \ldots, L_\ell$ (where L_0 is unbounded);*
   (ii) *it bends from the domain of linearity L_{i-1} to L_i at a point on the wall $\mathfrak{d}_{w_i}$;*
   (iii) *$\gamma'(t) = -m_i$ for $t \in L_i$;*
   (iv) *the monomial attached to L_i is z^{m_i}.*

6.4 Proof of Theorem 6.10

6.4.1 We show that (i)–(iv) determine a valid broken line.

The setting of [10] for a type A cluster algebra is specialized as follows: the lattice $N_{\mathrm{uf}} = N = \mathbb{Z}^{n'}$ is equipped with a basis $e_1, \ldots, e_{n'}$ and with a skew-symmetric bilinear form $\{\cdot, \cdot\} : N \times N \to \mathbb{Q}$ satisfying $\{e_i, e_j\} = \epsilon_{ij} = -b_{ij}$. (Since type A is skew-symmetric, all the multipliers $d_1 = \cdots = d_{n'} = 1$ as noted in [10, p114]). The dual lattice M has a basis $f_1, \ldots, f_{n'}$ dual to $e_1, \ldots, e_{n'}$. The vector v_j defined in [10, p29] is

$$v_i := \{e_i, \cdot\} = \sum_{j=1}^{n'} b_{ji} f_j.$$

Step 1: check $z^{m_{i-1}} f_{\mathfrak{d}_{w_i}}^{\langle n_0, m_{i-1} \rangle}$ contains a term z^{m_i}, for $1 \leqslant i \leqslant \ell$. (Here n_0 is the primitive vector annihilating the tangent space to $\mathfrak{d}_{w_i}$ and that $\langle n_0, m_{i-1} \rangle$ is positive.) Since $n_0 = $

$\pm e_{w_i}$, and by Remark 6.9(2), all coordinates of m_{i-1} take value in $\{-1, 0, 1\}$, we must have $\langle n_0, m_{i-1} \rangle = 1$. Meanwhile, by the initial scattering diagram described [10, p31], $f_{\mathfrak{d}_{w_i}} = 1 + z^{v_{w_i}}$. So

$$z^{m_{i-1}} f_{\mathfrak{d}_{w_i}}^{\langle n_0, m_{i-1} \rangle} = z^{m_{i-1}} + z^{m_{i-1} + v_{w_i}},$$

and we are left to show that $v_{w_i} = m_i - m_{i-1}$, or equivalently $b_{rw_i} = m_{i,r} - m_{i-1,r}$, or equivalently

$$b_{rw_i} = X + Y \tag{19}$$

where $X = \deg_{Q,\mathbf{s}^{(i)}}^{1\to}(r) - \deg_{Q,\mathbf{s}^{(i-1)}}^{1\to}(r)$, $Y = \deg_{Q,\mathbf{s}^{(i-1)}}^{0\leftarrow}(r) - \deg_{Q,\mathbf{s}^{(r-1)}}^{0\leftarrow}(r)$. Since $\mathbf{s}^{(i)}$ and $\mathbf{s}^{(i-1)}$ only differ in the w_i-th coordinate (where the former has coordinate 0 and the latter has coordinate 1), we have

$$X = \begin{cases} -1, & \text{if there is an arrow } w_i \to r; \\ 0, & \text{otherwise.} \end{cases} \quad \text{and } Y = \begin{cases} 1, & \text{if there is an arrow } r \to w_i; \\ 0, & \text{otherwise.} \end{cases}$$

Then (19) can be shown case by case: (1) if $b_{rw_i} = 0$, then $r = w_i$ or r is not adjacent to w_i, in either situation we have $X = Y = 0$, thus (19) holds; (2) if $b_{rw_i} = 1$: then $X = 0$, $Y = 1$, and (19) holds; (3) if $b_{rw_i} = -1$, then $X = -1$, $Y = 0$ and (19) still holds.

<u>Step 2</u>: denoting by $\mathbf{Q}_i \in \mathfrak{d}_{w_i}$ the point where γ bends from the domain of linearity L_{i-1} to L_i, check that $\mathbf{Q}_i - \mathbf{Q}_{i+1} \in \mathbb{R}^+ m_i$ for $i = 1, \ldots, \ell$ (assume $\mathbf{Q}_{\ell+1} = \mathbf{Q}$).

Note that $\mathbf{Q}_i = (q_j^{(i)})$ are determined by the following conditions:

$$\mathbf{Q}_{\ell+1} = \mathbf{Q}, \quad q_{w_i}^{(i)} = 0, \quad \mathbf{Q}_i - \mathbf{Q}_{i+1} \in \mathbb{R} \, m_i.$$

For convenience, we introduce the following definition: for $x, y, r \in \mathbb{R}$, $y > 0$ and $1 \leqslant r < 2$, define

$$x \approx_r y \iff 2 - r \leqslant \frac{x}{y} \leqslant r \quad \left(\iff \left| \frac{x}{y} - 1 \right| \leqslant r - 1 \right).$$

Note that $x \approx_r y$ implies $x > 0$.

Lemma 6.11. (i) *The w_i-th coordinate of m_i is -1.*
  (ii) $\mathbf{Q}_i - \mathbf{Q}_{i+1} = q_{w_i}^{(i+1)} m_i$.
  (iii) $q_{w_i}^{(i')} \approx_{(1+\varepsilon)^{\ell+1-i'}} q_{w_i}$ *for $1 \leqslant i < i' \leqslant \ell + 1$. As a consequence, for all $1 \leqslant i \leqslant \ell$, we must have $q_{w_i}^{(i+1)} > 0$, hence $\mathbf{Q}_i - \mathbf{Q}_{i+1} \in \mathbb{R}^+ m_i$.*

Proof. (i) By (17),

$$m_{i,w_i} = \deg_{Q,\mathbf{s}^{(i)}}^{1\to}(w_i) + \deg_{Q,\mathbf{s}^{(i)}}^{0\leftarrow}(w_i) - 1$$

So to show $m_{i,w_i} = -1$, it is equivalent to show that the above degrees are both 0.

Since $\mathbf{s}_{w_i}^{(i)} = 0$, we must have $\mathbf{s}_j^{(i)} = 0$ for any arrow $j \to w_i$, otherwise it contradicts the assumption that $\mathbf{s}^{(i)}$ is a GCS. Thus $\deg_{Q,\mathbf{s}^{(i)}}^{1\to}(w_i) = 0$.

Similarly, since $\mathbf{s}_{w_i}^{(i-1)} = 1$, we must have $\mathbf{s}_j^{(i-1)} = 1$ for any arrow $w_i \to j$, otherwise it contradicts the assumption that $\mathbf{s}^{(i-1)}$ is a GCS. Thus $\deg_{Q,\mathbf{s}^{(i-1)}}^{0\leftarrow}(w_i) = 0$. Since $\mathbf{s}^{(i-1)}$ and $\mathbf{s}^{(i)}$ only differ in the w_i-th coordinate, we have $\deg_{Q,\mathbf{s}^{(i)}}^{0\leftarrow}(w_i) = \deg_{Q,\mathbf{s}^{(i-1)}}^{0\leftarrow}(w_i) = 0$.

(ii) Assume that $\mathbf{Q}_i - \mathbf{Q}_{i+1} = \lambda m_i$. To determine λ, it suffices to consider the w_i-th coordinate on both sides:

$$q_{w_i}^{(i)} - q_{w_i}^{(i+1)} = \lambda\, m_{i,w_i} = -\lambda$$

Then (ii) holds since $q_{w_i}^{(i)} = 0$.

(iii) We prove it by fixing i and using downward induction on i'. For $i' = \ell + 1$, $q_{w_i}^{(i')} = q_{w_i}$, hence the statement holds. For $i' < \ell + 1$, using (ii) we have

$$q_{w_i}^{(i')} - q_{w_i}^{(i'+1)} = q_{w_{i'}}^{(i'+1)} m_{i',w_i}. \tag{20}$$

Note that $w_{i'} \neq w_i$ since $i' \neq i$. We argue in two cases:

Case 1: $w_{i'} < w_i$. We have $q_{w_i}^{(i'+1)} \approx_{(1+\varepsilon)^{\ell-i'}} q_{w_i}$ and $q_{w_{i'}}^{(i'+1)} \approx_{(1+\varepsilon)^{\ell-i'}} q_{w_{i'}}$ by inductive assumption. Since $m_{i',w_i} \in \{0, \pm 1\}$, (20) implies

$$\left| \frac{q_{w_i}^{(i')}}{q_{w_i}} - 1 \right| = \left| \frac{q_{w_i}^{(i'+1)} + q_{w_{i'}}^{(i'+1)} m_{i',w_i} - q_{w_i}}{q_{w_i}} \right| \leqslant \left| \frac{q_{w_i}^{(i'+1)} - q_{w_i}}{q_{w_i}} \right| + \left| \frac{q_{w_{i'}}^{(i'+1)}}{q_{w_i}} \right|$$

$$= \left| \frac{q_{w_i}^{(i'+1)}}{q_{w_i}} - 1 \right| + \left| \frac{q_{w_{i'}}^{(i'+1)}}{q_{w_{i'}}} \right| \cdot \left| \frac{q_{w_{i'}}}{q_{w_i}} \right| \leqslant (1+\varepsilon)^{\ell-i'} - 1 + (1+\varepsilon)^{\ell-i'} \varepsilon = (1+\varepsilon)^{\ell+1-i'} - 1.$$

Case 2: $w_{i'} > w_i$. We shall show that $m_{i',w_i} = 0$ (which implies $q_{w_i}^{(i')} = q_{w_i}^{(i'+1)} \approx_{\ell+1-(i'+1)} q_{w_i}$, therefore $q_{w_i}^{(i')} \approx_{\ell+1-i'} q_{w_i}$). It suffices to show that the two degrees in the following expression are 0 and 1, respectively:

$$m_{i',w_i} = \deg_{Q,\mathbf{s}^{(i')}}^{1\to}(w_i) + \deg_{Q,\mathbf{s}^{(i')}}^{0\leftarrow}(w_i) - 1$$

Since $\mathbf{s}^{(i')}$ is GCS and $\mathbf{s}_{w_i}^{(i')} = 0$ by the construction of $\mathbf{s}^{(i')}$, there is no arrow $j \to w_i$ in Q satisfying $\mathbf{s}_j^{(i')} = 1$. Thus $\deg_{Q,\mathbf{s}^{(i')}}^{1\to}(w_i) = 0$.

Next, we prove $\deg_{Q,\mathbf{s}^{(i')}}^{0\leftarrow}(w_i) = 1$ by contradiction:

If $\deg_{Q,\mathbf{s}^{(i')}}^{0\leftarrow}(w_i) = 0$, then there is no arrow $w_i \to j$ in Q satisfying $\mathbf{s}_j^{(i')} = 0$. But then w_i is also an adjustable position in $\mathbf{s}^{(i')}$, which contradicts the property of being "smallest" in the definition of $w_{i'}$ (Definition 6.7).

If $\deg_{Q,\mathbf{s}^{(i')}}^{0\leftarrow}(w_i) \geqslant 2$, then by our assumption on Q (see (14)), there must be two arrows $w_i \to (w_i - 1)$ and $w_i \to (w_i + 1)$ in Q satisfying $\mathbf{s}_{w_i-1}^{(i')} = \mathbf{s}_{w_i+1}^{(i')} = 0$. Consider the longest path starting from w_i and going left: $j \leftarrow (j+1) \leftarrow \cdots \leftarrow (w_i-1) \leftarrow w_i$ in $Q^{\mathbf{s}^{(i')}}$ (defined in Remark 6.6). Then j is a sink in $Q^{\mathbf{s}^{(i')}}$, hence adjustable and satisfying $j < w_i < w_{i'}$, again contradicts the property of being "smallest" in the definition of $w_{i'}$. $\qquad\square$

6.4.2 The map φ is bijective

It is easy to see that φ injective, since different GCS determine different sets of w_i (i.e., the sets of walls where the broken line bend), thus determine different broken lines. Thus to show the bijectivity of φ, it suffices to show that the number of GCC is not less than the number of broken lines. To show the latter, we use [10, Theorem 7.5 (4)] which asserts that the cluster variable $x[\mathbf{a}_Q]$ is equal to the theta function $\vartheta_{\mathbf{Q},m_0}$. Since the number of GCS is equal to $x[\mathbf{a}_Q]|_{x_1=\cdots=x_{n'}=1}$, on the other hand each broken line contributes at least 1 to $x[\mathbf{a}_Q]|_{x_1=\cdots=x_{n'}=1}$, we conclude that the number of GCS is not less than the number of broken lines. Therefore φ is bijective. This completes the proof of Theorem 6.10.

Note that the above argument implies that each broken line that contributes to $\vartheta_{\mathbf{Q},m_0}$ will not bend on any outgoing walls.

6.5 A bijection between GCSs and broken lines (in the general case)

To extend Theorem 6.10 to odd ranks, we need to consider principal coefficients. We denote by $\tilde{x}[\mathbf{a}]$ the cluster variable in $\mathcal{A}_{\mathrm{prin}}$ that corresponds to the cluster variable $x[\mathbf{a}]$ in $\mathcal{A}$.

Lemma 6.12. *Let $\mathcal{A}$ be a (coefficient-free) cluster algebra of rank n, and $\mathcal{A}_{\mathrm{prin}}$ the corresponding cluster algebra with principal coefficients. Let $x[\mathbf{a}]$ be a cluster variable in $\mathcal{A}$ with g-vector $\mathbf{g} \in \mathbb{Z}^n$. Then the corresponding cluster variable $\tilde{x}[\mathbf{a}]$ in $\mathcal{A}_{\mathrm{prin}}$ has g-vector*

$$\begin{bmatrix} \mathbf{g} \\ 0 \end{bmatrix} \in \mathbb{Z}^n \times 0^n \subset \mathbb{Z}^{2n}.$$

Proof. Recall that the g-vector of $x[\mathbf{a}] \in \mathcal{A}$ is the multidegree of the corresponding cluster variable in principal coefficients. More precisely, start with matrix $\tilde{B}_{t_0} = \begin{bmatrix} B \\ I \end{bmatrix}$, and for each mutation $t \overset{k}{\text{------}} t'$, $\tilde{B}_t$ and $\tilde{B}_{t'}$ are related by the rule

$$b_{ij}^{t'} = \begin{cases} -b_{ij}^t, & \text{if } i = k \text{ or } j = k; \\ b_{ij}^t + \mathrm{sgn}(b_{ik}^t)[b_{ik}^t b_{kj}^t]_+, & \text{otherwise.} \end{cases} \tag{21}$$

the new cluster variable is determined by

$$x_{k;t'} x_{k;t} = \prod_{i=1}^{2n} x_{i;t}^{[b_{ik}^t]_+} + \prod_{i=1}^{2n} x_{i;t}^{[-b_{ik}^t]_+}. \tag{22}$$

Each cluster variable is homogeneous with the assignment that, for $1 \leqslant i \leqslant n$, $\deg(x_i) = \mathbf{e}_i \in \mathbb{Z}^n$, $\deg(x_{n+i}) = -\mathbf{b}_i \in \mathbb{Z}^n$, where $\mathbf{b}_i$ is the i-th column of B. This multidegree is the g-vector of the cluster variable.

The g-vector of $\tilde{x}[\mathbf{a}] \in \mathcal{A}_{\mathrm{prin}}$ is defined similarly as above, with B and $\tilde{B}_{t_0}$ being replaced by

$$\bar{B} = \begin{bmatrix} B & -I \\ I & 0 \end{bmatrix} \quad \text{and} \quad \tilde{\bar{B}}_{t_0} = \begin{bmatrix} B & -I \\ I & 0 \\ I & 0 \\ 0 & I \end{bmatrix}, \text{ respectively.}$$

Using (21), it can be proved by a simple induction that,

$$\tilde{\bar{B}}_t = \begin{bmatrix} B_t & -C_t \\ C_t & D_t \\ C_t & D_t \\ 0 & I \end{bmatrix}, \text{ if } \tilde{B}_t = \begin{bmatrix} B_t \\ C_t \end{bmatrix}.$$

(In fact, $D_t = 0$ if the sign-coherence conjecture is true [18, Conjecture 8.8]; in particular, it is true if B is skew-symmetric, but we do not need this fact in this paper.) Using (22), it can be proved by a simple induction that $\tilde{x}[\mathbf{a}]$ can be obtained from $x[\mathbf{a}]$ by the substitution $x_i \mapsto \tilde{x}_i$ and $x_{n+i} \mapsto \tilde{x}_{n+i}\tilde{x}_{2n+i}$, for $1 \leqslant i \leqslant n$. Since

$$\deg \tilde{x}_i = \begin{bmatrix} \mathbf{e}_i \\ 0 \end{bmatrix} = \begin{bmatrix} \deg x_i \\ 0 \end{bmatrix},$$

$$\deg(\tilde{x}_{n+i}\tilde{x}_{2n+i}) = \begin{bmatrix} 0 \\ \mathbf{e}_i \end{bmatrix} - (\text{the } i\text{-th column of } \bar{B}) = \begin{bmatrix} 0 \\ \mathbf{e}_i \end{bmatrix} - \begin{bmatrix} \mathbf{b}_i \\ \mathbf{e}_i \end{bmatrix} = \begin{bmatrix} -\mathbf{b}_i \\ 0 \end{bmatrix} = \begin{bmatrix} \deg x_{n+i} \\ 0 \end{bmatrix},$$

the multidegree (i.e., the g-vector) of $\tilde{x}[\mathbf{a}]$ must be $\begin{bmatrix} \mathbf{g} \\ 0 \end{bmatrix}$ where $\mathbf{g}$ is the multidegree (i.e., the g-vector) of $x[\mathbf{a}]$. $\qquad \square$

In the rest we show that Theorem 6.10 can be adapted to $\mathcal{A}_{\mathrm{prin}}$.

Theorem 6.13. *Assume* $\tilde{\mathbf{Q}} = (q_1, q_2, \dots, q_{2n'})$ *such that*

$$0 < q_i \ll q_1 \ll q_2 \ll \cdots \ll q_n, \text{ for each } i = n+1, \dots n'.$$

(Here $x \ll y$ means $0 < x/y \leqslant \varepsilon$, where $\varepsilon > 0$ is a fixed real number satisfying $(1+\varepsilon)^n < 2$. There is no condition on $q_{n'+1}, \dots, q_{2n'}$.) Let

$$\tilde{m}_0 = \begin{bmatrix} m_0 \\ 0 \end{bmatrix},$$

where m_0 is defined in Lemma 6.1. Then there is a bijection φ between the set of GCS and the set of broken lines for $\tilde{m}_0$ with endpoints $\tilde{\mathbf{Q}}$, such that each GCS $\mathbf{s}$ is sent to a broken line $\gamma = \varphi(\mathbf{s})$ satisfying $\mathrm{Mono}(\gamma)\big|_{x_{n'+1}=\cdots=x_{2n'}=1} = z_{\mathbf{s}}$ (defined in (10)): using notation in Definition 6.7, the broken line γ can be explicitly described as follows:
   (i) *it has $\ell + 1$ domains of linearity $L_0, L_1, \dots, L_\ell$ (L_0 is unbounded);*
   (ii) *it bends from the domain of linearity L_{i-1} to L_i at a point on the wall $\mathfrak{d}_{w_i}$;*
   (iii) *$\gamma'(t) = -\tilde{m}_i$ for $t \in L_i$, where $\tilde{m}_i = \begin{bmatrix} m_i \\ \sum_{r=1}^i \mathbf{e}_{w_r} \end{bmatrix}$.*
   (iv) *the monomial attached to L_i is $z^{\tilde{m}_i}$.*

Proof. Since $\bar{B}$ is full rank, the Fundamental Assumption in [10] is satisfied, thus the proof in §6.4 works with little change; we point out the nontrivial revision change.

Because of Lemma 6.12, the g-vector of $\tilde{x}[\mathbf{a}]$ is indeed $\tilde{m}_0$. Thus $\tilde{x}[\mathbf{a}]$ is equal to the theta function $\vartheta_{\tilde{\mathbf{Q}}, \tilde{m}_0}$ by [10, Theorem 7.5 (4)].

In Step 1, n_0 is replaced by $\tilde{n}_0 = \begin{bmatrix} n \\ 0 \end{bmatrix} \in \mathbb{R}^{2n'}$. Since the first n' coordinates of $\tilde{m}_{i-1}$ form the vector m_{i-1}, we have $\langle \tilde{n}_0, \tilde{m}_i \rangle = \langle n_0, m_i \rangle = 1$. The function attached to the wall $\mathfrak{d}_i$ is $\tilde{f}_{\mathfrak{d}_i} = 1 + z^{\tilde{v}_i}$ where $\tilde{v}_i = \begin{bmatrix} v_i \\ e_i \end{bmatrix}$. Thus

$$\tilde{v}_{w_i} = \begin{bmatrix} v_{w_i} \\ e_{w_i} \end{bmatrix} = \begin{bmatrix} m_i - m_{i-1} \\ e_{w_i} \end{bmatrix} = \tilde{m}_i - \tilde{m}_{i-1}.$$

In Step 2, since $\pi(\tilde{\mathbf{Q}}_i) = \mathbf{Q}_i$, we have $\tilde{\mathbf{Q}}_i - \tilde{\mathbf{Q}}_{i+1} = q_{w_i}^{(i+1)}\tilde{m}_i$ with the same $q_{w_i}^{(i+1)} > 0$ as in Lemma 6.11. $\qquad \square$

7 Examples

In this section, we give several examples to illustrate the computation of cluster variables and cluster monomials using methods introduced in previous sections.

Example 7.1. We compute $x[2,2,2]$, the cluster monomial with **d**-vector $(2,2,2)$ for the quiver $Q = 1 \to 2 \to 3 \to 1$ (a 3-cycle).

– Using formula (5): choose $i_0 = 1$. Then $d(1) = 0$, $d(2) = 1$, $d(3) = 2$, $\sigma_{123} = \sigma_{231} = \sigma_{312} = 1$. A GCS $\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)$ satisfies

$$(s_{1,1}, s_{2,1}), (s_{3,2}, s_{1,2}), \text{ and } (s_{2,2}, s_{3,1}) \neq (1, 0).$$

A tedious computation using (5) gives $x[2,2,2] = x_1^{-2}x_2^{-2}x_3^{-2}(x_1 + x_2 + x_3)^3$. (But at least it is easy to see that there are 27 terms; indeed, since each pair has 3 choices $(0,0), (1,1), (0,1)$, the total number of GCSs is $3 \times 3 \times 3 = 27$.)

– Using formula (6): there are three Dyck paths of size 2×2 corresponding to the three arrows $1 \to 2$, $2 \to 3$, $3 \to 1$: such that (i) we do not choose both u_1 and v_1 in each
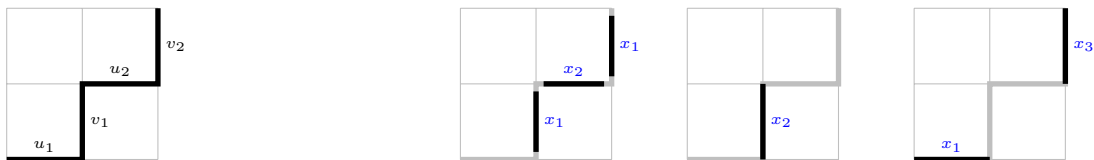

Figure 11: Left: the 2×2 maximal Dyck path   Right: An example of GCC.

Dyck path, and (ii) we choose v_r in the i-th Dyck path if and only if we do not choose u_{3-r} in the $(i+1)$-th Dyck path for $r = 1, 2$ (by convention, the 4th Dyck path is the 1st one). In the example of GCC in Figure 11 (Right), the corresponding product

$$\left(\prod_{i \to j} x_i^{\left| S_2^{(i,j)} \right|} x_j^{\left| S_1^{(i,j)} \right|} \right) \cdot \prod_{i \to j \to k \to i} x_i^{-\sigma_{jki}} = (x_1^3 x_2^2 x_3) \cdot x_1^{-1} x_2^{-1} x_3^{-1} = x_1^2 x_2.$$

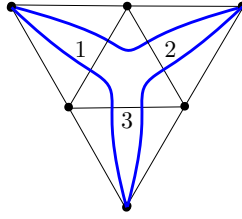Computing all possible GCCs gives $x[2,2,2] = x_1^{-2}x_2^{-2}x_3^{-2}(x_1 + x_2 + x_3)^3$.

Figure 12: Pipelines

– Using formula (8): first observe that there are 3 pipelines as shown in Figure 12.

According to §3.3, $x[2,2,2] = x[1,1,0]x[0,1,1]x[1,0,1]$. Now we compute $x[1,1,0]$ using Theorem 3.6. The pair $(|S_{1,1}|, |S_{1,2}|)$ can be $(1,0), (0,1), (0,0)$. Correspondingly, we have $(y_1, y_0, y_2) = (x_1, x_{1,0}, x_{2,0}), (x_2, x_{1,1}, x_{2,1}), (x_{1,2}, x_{1,1}, x_{2,0})$. Then

$$x[1,1,0] = (x_1^{-1}x_2^{-1}) \sum y_0 y_1 y_2 = (x_1^{-1}x_2^{-1})(x_1 x_{1,0} x_{2,0} + x_2 x_{1,1} x_{2,1} + x_{1,2} x_{1,1} x_{2,0})$$
$$= (x_1^{-1}x_2^{-1})(x_1 + x_2 + x_3)$$

where the last equality is obtained by substituting $x_{1,2} = x_3$, and $x_{1,0} = x_{1,1} = x_{2,0} = x_{2,1} = 1$. Similarly, $x[0,1,1] = (x_2^{-1}x_3^{-1})(x_1 + x_2 + x_3)$, $x[1,0,1] = (x_1^{-1}x_3^{-1})(x_1 + x_2 + x_3)$. Thus $x[2,2,2] = x_1^{-2}x_2^{-2}x_3^{-2}(x_1 + x_2 + x_3)^3$.

Example 7.2. We compute some cluster variables of $\mathcal{A}(\mathcal{Q})$ where $\mathcal{Q}$ is the following type A quiver:
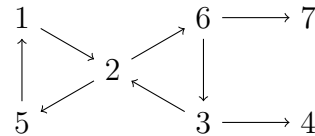
As observed in Remark 3.7, the set of non-initial cluster variables is in one-to-one correspondence with the set of **d**-vectors $\mathbf{a} = (a_1, \ldots, a_n)$, where $a_i \in \{0, 1\}$ and $\{i \,|\, a_i = 1\}$ is the vertex set of a linear full subquiver of $\mathcal{Q}$. So we can compute all cluster variables using Theorem 3.6.

For example, we consider the **d**-vector $\mathbf{a} = (1, 1, 1, 0, 0, 0, 0)$. Then the subset of vertices $\{i \,|\, a_i = 1\}$ is equal to the set of vertices Q_0 of the full linear subquiver $Q = 1 \to 2 \leftarrow 3$. It is a subquiver of an extended linear subquiver P and after completing P, we get a completely extended linear quiver Q' as shown in Figure 13.

In P, we have $v_{1,2} = 5$, $v_{2,3} = 6$, $v_{3,0} = 4$. Two 3-cycles are added and the new vertices are $v_{1,0} = 8$, $v_{1,1} = 9$ and $v_{3,1} = 10$. All GCCs are described as follows.

$$\beta_1 \quad = \quad \begin{array}{c} x_1 \\ \hline x_2 \end{array} \quad \begin{array}{c} x_3 \\ \hline x_2 \end{array} \quad , \quad x(\beta_1) = (x_5 x_6 x_9 x_{10})/(x_1 x_2 x_3)$$

$$\beta_2 \quad = \quad \begin{array}{c} x_1 \\ \hline x_2 \end{array} \quad \begin{array}{c} x_3 \\ \hline x_2 \end{array} \quad , \quad x(\beta_2) = (x_2 x_4 x_5 x_9)/(x_1 x_2 x_3)$$
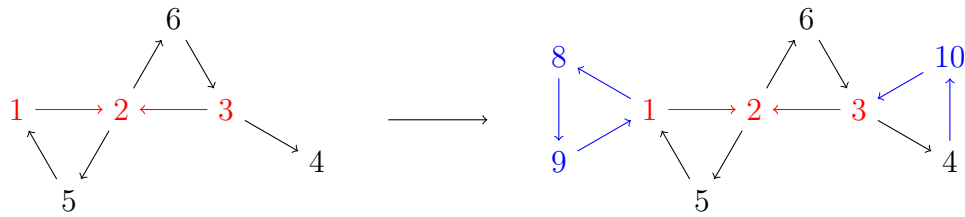
Figure 13: P and its completed version Q'

$$\beta_3 \quad = \quad \begin{array}{c} x_1 \\ \hline x_2 \end{array} \quad \begin{array}{c} x_3 \\ \hline x_2 \end{array} \quad , \quad x(\beta_3) = (x_2 x_6 x_8 x_{10})/(x_1 x_2 x_3)$$

$$\beta_4 \quad = \quad \begin{array}{c} x_1 \\ \hline x_2 \end{array} \quad \begin{array}{c} x_3 \\ \hline x_2 \end{array} \quad , \quad x(\beta_4) = (x_2^2 x_4 x_8)/(x_1 x_2 x_3)$$

$$\beta_5 \quad = \quad \begin{array}{c} x_1 \\ \hline x_2 \end{array} \quad \begin{array}{c} x_3 \\ \hline x_2 \end{array} \quad , \quad x(\beta_5) = (x_1 x_3 x_9 x_{10})/(x_1 x_2 x_3)$$

The cluster variable with **d**-vector $(1,1,1,0,0,0,0)$ is

$$x[1,1,1,0,0,0,0] = \sum_{i=1}^{5} x(\beta_i) = \frac{x_5 x_6 x_9 x_{10} + x_2 x_4 x_5 x_9 + x_2 x_6 x_8 x_{10} + x_2^2 x_4 x_8 + x_1 x_3 x_9 x_{10}}{x_1 x_2 x_3}.$$

Setting $x_8 = x_9 = x_{10} = 1$, we get the following cluster variable of $\mathcal{A}(\mathcal{Q})$:

$$x[1,1,1,0,0,0,0] = \frac{x_5 x_6 + x_2 x_4 x_5 + x_2 x_6 + x_2^2 x_4 + x_1 x_3}{x_1 x_2 x_3}.$$

The table below shows some **d**-vectors and their corresponding cluster variables of $\mathcal{A}(\mathcal{Q})$.

$(1,0,0,0,0,0,0)$	$\dfrac{x_2 + x_5}{x_1}$	$(1,1,0,0,0,0,0)$	$\dfrac{x_1 x_3 + x_2 x_6 + x_5 x_6}{x_1 x_2}$
$(0,1,0,0,0,0,0)$	$\dfrac{x_1 x_3 + x_5 x_6}{x_2}$	$(0,1,1,0,0,0,0)$	$\dfrac{x_1 x_3 + x_2 x_4 x_5 + x_5 x_6}{x_2 x_3}$
$(0,0,1,0,0,0,0)$	$\dfrac{x_2 x_4 + x_6}{x_3}$	$(0,0,1,1,0,0,0)$	$\dfrac{x_2 x_4 + x_3 x_6 + x_6}{x_3 x_4}$
$(0,0,0,1,0,0,0)$	$\dfrac{1 + x_3}{x_4}$	$(1,0,0,0,1,0,0)$	$\dfrac{x_1 + x_2 + x_5}{x_1 x_5}$
		$(0,0,1,0,0,1,0)$	$\dfrac{x_2 x_4 + x_6 + x_3 x_4 x_7}{x_3 x_6}$

(1,1,1,0,0,0,0)	$\dfrac{x_5x_6 + x_2x_4x_5 + x_2x_6 + x_2^2x_4 + x_1x_3}{x_1x_2x_3}$
(0,1,1,0,1,0,0)	$\dfrac{x_1x_3 + x_2x_3 + x_2x_4x_5 + x_5x_6}{x_2x_3x_5}$
(0,1,1,1,0,0,0)	$\dfrac{x_1x_3 + x_1x_3^2 + x_2x_4x_5 + x_5x_6 + x_3x_5x_6}{x_2x_3x_4}$
(0,0,1,1,0,1,0)	$\dfrac{x_2x_4 + x_6 + x_3x_6 + x_3x_4x_7}{x_3x_4x_6}$
(1,1,1,1,0,0,0)	$\dfrac{x_1x_3 + x_1x_3^2 + x_2^2x_4 + x_2x_4x_5 + x_2x_6 + x_2x_3x_6 + x_5x_6 + x_3x_5x_6}{x_1x_2x_3x_4}$

Example 7.3. Here we give an example to illustrate statements in §6. Let Q' be the following type A quiver:

$$
\begin{array}{ccc}
 & 4 & \\
\nearrow & & \searrow \\
1 \longleftarrow & 2 & \longrightarrow 3
\end{array}
$$
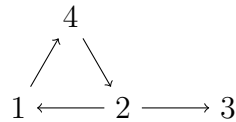
Let Q be the full linear subquiver $1 \leftarrow 2 \rightarrow 3$, and correspondingly $\mathbf{a}_Q = (1,1,1,0)$.

There are 5 GCS $\mathbf{s} = (s_1, s_2, s_3)$:

$$(0,0,0), (1,0,0), (0,0,1), (1,0,1), (1,1,1)$$

By Lemma 6.1, $m_0 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$.

Consider $\mathbf{s} = (0,0,0)$. By Definition 6.7, $\ell = 3$,

$$\mathbf{s}^{(3)} = (0,0,0),\ \mathbf{s}^{(2)} = (1,0,0),\ \mathbf{s}^{(1)} = (1,0,1),\ \mathbf{s}^{(0)} = (1,1,1),\ \ w_3 = 1,\ w_2 = 3,\ w_1 = 2.$$

By Definition 6.8, m_0 is as above,

$$
m_1 = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad
m_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad
m_3 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}.
$$

The broken line $\gamma = \varphi(\mathbf{s})$ described in Theorem 6.10 has 4 domains of linearity $L_0, \ldots, L_3$, bends on walls $\mathfrak{d}_2, \mathfrak{d}_3, \mathfrak{d}_1$ in that order. The direction vector of γ on L_i is $-m_i$ for $i = 0, \ldots, 3$. The monomials attached to $L_0, \ldots, L_3$ are x_2^{-1}, $x_1^{-1}x_2^{-1}x_3^{-1}x_4$, $x_1^{-1}x_3^{-1}x_4$, $x_1^{-1}x_2x_3^{-1}$, respectively. Thus $\mathrm{Mono}(\gamma) = x_1^{-1}x_2x_3^{-1}$.

Follow the proof in Step 2 of §6.4, we compute the coordinates of $\mathbf{Q}_i$ as follows:

$$\mathbf{Q}_4 = \mathbf{Q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}, \quad \mathbf{Q}_3 = \mathbf{Q}_4 + q_1 m_3 = \begin{bmatrix} 0 \\ q_2 + q_1 \\ q_3 - q_1 \\ q_4 \end{bmatrix},$$

$$\mathbf{Q}_2 = \mathbf{Q}_3 + (q_3 - q_1)m_2 = \begin{bmatrix} -q_3 + q_1 \\ q_2 + q_1 \\ 0 \\ q_4 + q_3 - q_1 \end{bmatrix}, \quad \mathbf{Q}_1 = \mathbf{Q}_2 + (q_2 + q_1)m_1 = \begin{bmatrix} -q_3 - q_2 \\ 0 \\ q_3 - q_2 - q_1 \\ q_4 + q_3 + q_2 \end{bmatrix}.$$

Since $q_4 \gg q_3 \gg q_2 \gg q_1$, it is clear in this example that $\mathbf{Q}_i - \mathbf{Q}_{i+1} \in \mathbb{R}^+ m_i$ for all $i = 1, 2, 3$.

8 Appendix: a bijection between T-paths and GCCs

In this section, we first recall the construction of T-paths and the formula of cluster variables using T-paths as in [19], then give a bijective proof of Theorem 3.6 via T-paths.

Let P be a convex polygon with $n + 3$ vertices. Our initial triangulation of P will consist of the set $T = \{T_1, \ldots, T_n\} \cup \{T_{1,0}, T_{1,1}, T_{n,0}, T_{n,1}\} \cup \{T_{i,i+1} : i \in [1, n-1]\}$, where the first set is the set of diagonals and the last two sets are the set of boundary edges.
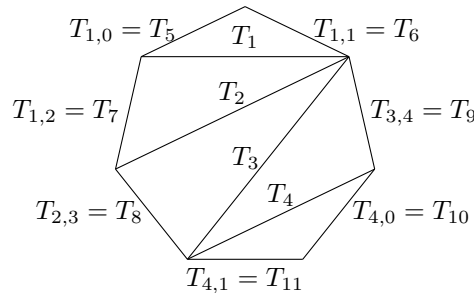

Figure 14: The initial triangulation of the quiver (Q, Q') in Example 1.1.

The process of constructing the initial triangulation starts with choosing any vertex $\mathfrak{v}$ of P, labeling its two incident boundary edges as $T_{1,0}$ and $T_{1,1}$ and letting T_1 be the diagonal such that $T_{1,0}$, $T_{1,1}$ and T_1 form a triangle in the orientation as shown in Figure 15 (the upper-left triangle).

Suppose that the diagonal T_i ($i \in [1, n-1]$) is drawn. The diagonal T_{i+1} is obtained by rotating T_i in the counterclockwise direction if $\delta_i = 0$, in the clockwise direction if $\delta_i = 1$. The boundary edge between T_i and T_{i+1} is labeled $T_{i,i+1}$.

When $i = n$ then $T_{n,1}$ is the boundary edge clockwise from T_n and $T_{n,0}$ is the boundary edge counterclockwise from T_n. Denote the common vertex of $T_{n,0}$ and $T_{n,1}$ by $\mathfrak{w}$.
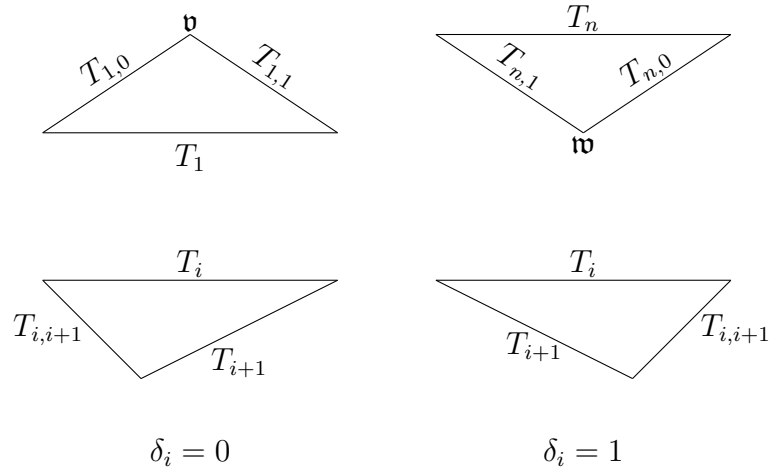
Figure 15: Boundary edges and diagonals of an initial triangulation

We can view both the snake diagram and the triangulation T as graphs. Then there is a natural graph homomorphism p between them satisfying

$$p(T_j^{(i)}) = T_j, \quad p(T_{i,j}) = T_{i,j} \tag{23}$$

(for any i, j that the equalities make sense). It is easy to check that the image of each vertex is uniquely determined using the requirement that a triangle in the snake diagram (with all the main diagonals added) must send to a triangle in T; so if the images of the three sides of a triangle are determined, the images of the three vertices are also determined (see Figures 6 and 7).

In [9], Fomin and Zelevinsky showed that the cluster variables of $\mathcal{A}(Q)$ are in bijection with the diagonals of the polygon P where the set of initial cluster variables $\{x_1, \ldots, x_n\}$ corresponds to $\{T_1, \ldots, T_n\}$.

Let $M_{\mathfrak{v},\mathfrak{w}}$ be the diagonal connecting $\mathfrak{v}$ and $\mathfrak{w}$, thus crossing the diagonals $T_1, \ldots, T_n$. For $i \in [1, n]$, let p_i be the intersection of $M_{\mathfrak{v},\mathfrak{w}}$ and T_i.

Definition 8.1. [19] A T-path α from $\mathfrak{v}$ to $\mathfrak{w}$ is the sequence

$$\alpha = w_0 \xrightarrow{T_{i_1}} w_1 \xrightarrow{T_{i_2}} \cdots \xrightarrow{T_{i_{l(\alpha)}}} w_{l(\alpha)}$$

such that
　　1) $\mathfrak{v} = w_0, w_1, \ldots, w_{l(\alpha)} = \mathfrak{w}$ are vertices of P.
　　2) $i_k \in \{0, 1, \ldots, 2n+2\}$ such that T_{i_k} connects the vertices w_{k-1} and w_k for each $k = 1, 2, \ldots, l(\alpha)$.
　　3) $i_j \neq i_k$ if $j \neq k$.
　　4) $l(\alpha)$ is odd.
　　5) T_{i_k} crosses $M_{\mathfrak{v},\mathfrak{w}}$ if k is even.
　　6) If $j < k$ and both T_{i_j} and T_{i_k} cross M then p_{i_j} is closer to $\mathfrak{v}$ than p_{i_k} is to $\mathfrak{v}$.

Let $\mathcal{P}$ be the set of all T-path from $\mathfrak{v}$ to $\mathfrak{w}$. For any $\alpha \in \mathcal{P}$, let

$$x(\alpha) = \prod_{k \text{ odd}} x_{i_k} \prod_{k \text{ even}} x_{i_k}^{-1}. \tag{24}$$

Let $\mathbf{a} = (a_1, \ldots, a_{n'}) \in \{0,1\}^{n'}$ such that $a_i = 1$ if and only if $i \in Q$. The following formula of the cluster variable $x[\mathbf{a}]$ is proved in [19]:

$$x[\mathbf{a}] = \sum_{\alpha \in \mathcal{P}} x(\alpha) \tag{25}$$

Definition 8.2. We define a map $\psi_{\mathcal{G},\mathcal{P}} : \mathcal{G} \to \mathcal{P}$ by sending $\{S_{i,r}\} \in \mathcal{G}$ to the T-path α obtained by first constructing a path $\alpha_1' \alpha_2' \ldots \alpha_{2n+1}'$ from $\mathfrak{v}$ to $\mathfrak{w}$ where $\alpha_{2i}' = T_i$ for $i \in [1, n]$,

$$\alpha_{2i+1}' = \begin{cases} T_i, & \text{if} \quad (|S_{i,1}|, |S_{i,2}|) = (\delta_i, 1 - \delta_i) \\ T_{i+1}, & \text{if} \quad (|S_{i,1}|, |S_{i,2}|) = (1 - \delta_i, \delta_i) \\ T_{i,i+1}, & \text{if} \quad (|S_{i,1}|, |S_{i,2}|) = (0, 0) \end{cases}$$

for $i \in [1, n-1]$, and

$$\alpha_1' = \begin{cases} T_{1,0} & \text{if } |S_{1,1+\delta_1}| = 1 - \delta_1, \\ T_{1,1} & \text{otherwise,} \end{cases} \qquad \alpha_{2n+1}' = \begin{cases} T_{n,0} & \text{if } |S_{n-1,2-\delta_{n-1}}| = \delta_{n-1}, \\ T_{n,1} & \text{otherwise,} \end{cases}$$

then define α to be the path obtained from α' by canceling duplicate pairs.

We define $\psi_{\mathcal{P},\mathcal{G}} := \psi_{\mathcal{G},\mathcal{P}}^{-1} : \mathcal{P} \to \mathcal{G}$. (As shown in the theorem below, $\psi_{\mathcal{G},\mathcal{P}}$ is a bijection.)

Theorem 8.3. *The maps $\psi_{\mathcal{G},\mathcal{P}}$ is a well-defined bijection. Moreover, for $\alpha = \psi_{\mathcal{G},\mathcal{P}}(\{S_{i,r}\})$,*

$$\prod_{i=1}^{n} x_i^{-1} \prod_{i=0}^{n} y_i = x(\alpha),$$

thus $\psi_{\mathcal{G},\mathcal{P}}$ induces a bijective proof of Theorem 3.6 using (25).

Proof. In order to prove Theorem 8.3, we shall show that all maps below are bijective, and that their composition is $\psi_{\mathcal{G},\mathcal{P}}$:

$$\mathcal{G} \xrightarrow{\psi_{\mathcal{G},\mathcal{M}}} \mathcal{M} \xrightarrow{\mathcal{L}} \{\text{complete } T\text{-paths from } \mathfrak{v} \text{ to } \mathfrak{w}\} \xrightarrow{\pi} \mathcal{P}.$$

(i) We first define $\mathcal{L}$, which is exactly the folding map in [16, §4.3]. As defined in [16, 21], a complete T-path α from $\mathfrak{v}$ to $\mathfrak{w}$ is similar to a T-path from $\mathfrak{v}$ to $\mathfrak{w}$ defined in Definition 8.1, in the sense that we require 1), 2), and

5') the $2j$-th edge $T_{i_{2j}} = T_j$ (i.e., $i_{2j} = j$),

6') $T_{i_1} \leqslant T_{i_2} \leqslant \cdots$,

where we use the order

$$T_{1,0} < T_{1,1} < T_1 < T_{1,2} < T_2 < T_{2,3} < \cdots < T_n < T_{n,0} < T_{n,1}. \qquad (26)$$

Note that we do not require edges in α to be distinct. It is easy to see that a complete T-path has length $2n+1$. For simplicity, we denote α using its edge sequence. For $\gamma \in \mathcal{M}$, we define (recall that p is defined in (23)):

$$\mathcal{L}(\gamma) = L_1 L_2 \cdots L_{2n+1}, \quad \text{where } L_{2j} = T_j \text{ for } j \in [1,n], \ L_{2j+1} = p(\gamma_j) \text{ for } j \in [0,n].$$
$$(27)$$

Note that the starting point of each L_i is determined by $L_1 \cdots L_{i-1}$. The union of a perfect matching γ with all the diagonals of tiles form a path α'_γ in the snake diagram. If we consider the quotient map from the snake diagram to the triangulation of P, by identifying the diagonal edge i with $T^{(i)}$ and identifying diagonal edge $i+1$ with $T^{(i)}_{i+1}$, then the image of α'_γ is the complete T-path $\mathcal{L}(\gamma)$.

(ii) We show that $\mathcal{L}$ has a well-defined inverse map $\mathcal{L}^{-1}$ (which is the unfolding map in [16, §4.5]), thus $\mathcal{L}$ is bijective. Indeed, $\mathcal{L}^{-1}$ sends a complete T-path $\theta = L_1 \cdots L_{2n+1}$ to $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, where γ_j is the unique edge in $Pl^{(j)} \cap p^{-1}(L_{2j+1})$, that is, $\gamma_1 = L_1$, $\gamma_n = L_{2n+1}$, and $\gamma_j = T^{(j)}_j$ (resp. $T^{(j)}_{j+1}$, $T_{j,j+1}$) if L_{2j+1} is T_j (resp. T_{j+1}, $T_{j,j+1}$) for $j \in [1, n-1]$.

Next we show that γ is indeed a perfect matching, it suffices to prove that the edges in γ are disjoint, because it has the correct number ($= n+1$) of edges.

For $j, j' \in [0, n-2]$ with $j < j'$, γ_j and $\gamma_{j'}$ are disjoint if $j' > j+1$ because $Pl^{(j)}$ and $PL^{(j')}$ are disjoint. So we assume $j' = j+1$. We shall only discuss the case $\delta_j = \delta_{j+1} = 0$ since other cases can be proved similarly. The subpath $L_{2j+1}L_{2j+2}L_{2j+3}L_{2j+4}$ of $\mathcal{L}(\alpha)$ is
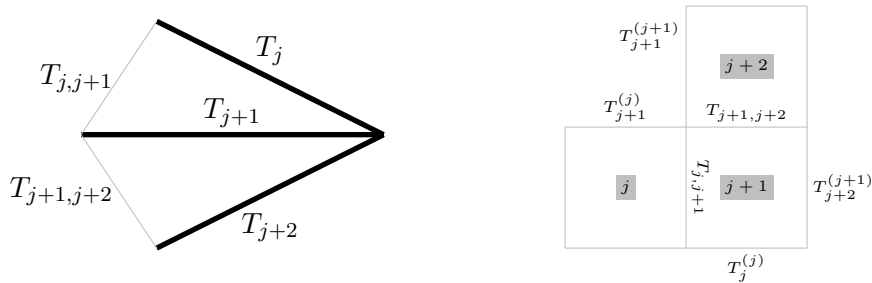

Figure 16: Parts of the polygon and the snake diagram corresponding to the subquiver $j \to j+1 \to j+2$

one of the following:

$$T_j T_{j+1} T_{j+1} T_{j+2}, \ T_j T_{j+1} T_{j+1,j+2} T_{j+2}, \ T_{j,j+1} T_{j+1} T_{j+2} T_{j+2}, \ T_{j+1} T_{j+1} T_{j+2} T_{j+2}.$$

By looking at Figure 16, we see that γ_j and γ_{j+1} are disjoint in each case.

(iii) We show π is bijective by giving its inverse π^{-1}. Suppose that $\alpha = T_{i_1} T_{i_2} \cdots T_{i_{l(\alpha)}}$ is a T-path from $\mathfrak{v}$ to $\mathfrak{w}$. If $n = 1$, then α is already a complete T-path, so we define $\pi^{-1}(\alpha) = \alpha$. Now assume $n > 1$. The sequence $\pi^{-1}(\alpha) = L = L_1 L_2 \cdots L_{2n+1}$ is obtained as a result of the following algorithm.

1. Initialize $L := \alpha$.

2. Let j run from 1 to n: if $L_{2j} \neq T_j$, then insert $T_j T_j$ to L so that the resulting L is nondecreasing with the order given in (26).

3. Define $\pi^{-1}(\alpha) := L$.

We claim that L is a complete T-path. Conditions 1) 2) 6') are obviously satisfied, and 5') can be proved by induction.

Combining (i)(ii)(iii) and Theorem 4.4, we have proved that $\psi_{\mathcal{G},\mathcal{P}}$ is bijective.

Finally, we show that $\prod_{i=1}^{n} x_i^{-1} \prod_{i=0}^{n} y_i = x(\alpha)$. By the construction of $\pi^{-1}(\alpha)$ in (iii), $x(\alpha)$ remains unchanged if we replace α by the complete T-path $\pi^{-1}(\alpha) = T_{i_1} T_{i_2} \cdots T_{i_{2n+1}}$; this is because each time we insert the pair $T_j T_j$, the extra contribution to the product (24) is $x_j x_j^{-1} = 1$. So it suffices to show

$$\prod_{i=1}^{n} x_i^{-1} \prod_{i=0}^{n} y_i = \prod_{k \text{ even}} x_{i_k}^{-1} \prod_{k \text{ odd}} x_{i_k}.$$

By 5'), $\prod_{k \text{ even}} x_{i_k}^{-1} = \prod_{i=1}^{n} x_i^{-1}$, so it suffices to show $\prod_{i=0}^{n} y_i = \prod_{k \text{ odd}} x_{i_k}$, or to show that $y_j = x_{i_{2j+1}}$ for $j \in [0, n]$. Indeed, $T_{i_{2j+1}} = L_{2j+1} = p(\gamma_j)$ by (27), thus $x_{i_{2j+1}} = w(\gamma_j)$ by the definition of the weight w in (11). Moreover, Theorem 4.4 asserts that $w(\gamma_j) = y_j$. Thus $y_j = x_{i_{2j+1}}$. $\square$

Example 8.4. With the T-path $\alpha = T_5 T_1 T_9 T_4 T_{11}$, the complete T-path is $\pi^{-1}(\alpha) = T_5 T_1 T_2 T_2 T_3 T_3 T_9 T_4 T_{11}$. Then $\psi_{\mathcal{P},\mathcal{G}}(\alpha) = ((1,0),(1,0),(0,0))$ as you can see in Figure 17.
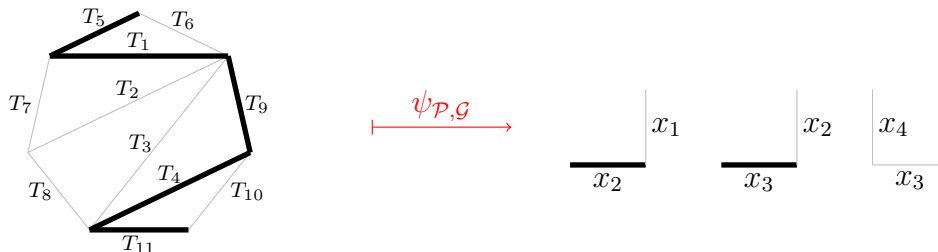

Figure 17: An example of the map $\psi_{\mathcal{P},\mathcal{G}}$

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