# Convolution estimates and the number of disjoint partitions

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**Abstract**

Let $X$ be a finite collection of sets. We count the number of ways a disjoint union of $n-1$ subsets in $X$ is a set in $X$, and estimate this number from above by $|X|^{c(n)}$ where

$$c(n) = \left(1 - \frac{(n-1)\ln(n-1)}{n \ln n}\right)^{-1}.$$

This extends the recent result of Kane–Tao, corresponding to the case $n = 3$ where $c(3) \approx 1.725$, to an arbitrary finite number of disjoint $n-1$ partitions.

## 1 Introduction

Let $\{0,1\}^m$ be the Hamming cube of dimension $m \geqslant 1$. Set $1^m := (1, 1, \ldots, 1)$ to be the corner of $\{0,1\}^m$. Take a finite number of functions $f_1, \ldots, f_n : \{0,1\}^m \to \mathbb{R}$, and define the convolution at the corner $1^m$ as

$$f_1 * f_2 * \ldots * f_n(1^m) := \sum_{x_j \in \{0,1\}^m \,:\, x_1 + \cdots + x_n = 1^m} f_1(x_1) \cdots f_n(x_n).$$

Given $f : \{0,1\}^m \to \mathbb{R}$ define its $L^p$ norm ($p \geqslant 1$) in a standard way

$$\|f\|_p := \left(\sum_{x \in \{0,1\}^m} |f(x)|^p\right)^{1/p}.$$

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For $n \in \mathbb{N}$ we set

$$p_n := \frac{\ln \frac{n^n}{(n-1)^{n-1}}}{\ln n}.$$

Our main result is the following theorem

**Theorem 1.** *For any $n, m \geqslant 1$, and any $f_1, \ldots, f_n : \{0,1\}^m \to \mathbb{R}$ we have*

$$f_1 * f_2 * \ldots * f_n(1^m) \leqslant \prod_{j=1}^{n} \|f_j\|_{p_n}. \tag{1}$$

*Moreover, for each fixed $n$ exponent $p_n$ is the best possible in the sense that it cannot be replaced by any larger number.*

As an immediate application we obtain the following corollary (see Section 2.3 below).

**Corollary 2.** *Let $X$ be a finite collection of sets. Then*

$$\left| \left\{ (A_1, \ldots, A_{n-1}, A) \in \underbrace{X \times \cdots \times X}_{n} \; : \; A = \bigsqcup_{j=1}^{n-1} A_j \right\} \right| \leqslant |X|^{\frac{n}{p_n}}, \tag{2}$$

*where $\bigsqcup$ denotes the disjoint union, and $|X|$ denotes cardinality of the set.*

The corollary extends a recent result of Kane–Tao [1], corresponding to the case $n = 3$ where $\frac{3}{p_3} \approx 1.725$, to an arbitrary finite number $n \geqslant 3$ disjoint partitions.

## 2 The proof of the theorem

Following [1] the proof goes by induction on the dimension of the cube $\{0,1\}^m$. The case $m = 1$, which is the most difficult, is the main contribution of the current paper.

### 2.1 Basis: $m = 1$

In this case, set $f_j(0) = u_j$ and $f_j(1) = v_j$ for $j = 1, \ldots, n$. Then the inequality (1) takes the form

$$\sum_{j=1}^{n} u_j \prod_{\substack{i=1 \\ i \neq j}}^{n} v_i \leqslant \prod_{j=1}^{n} \left(|u_j|^{p_n} + |v_j|^{p_n}\right)^{1/p_n}. \tag{3}$$

We do encourage the reader first to try to prove (3) in the case $n = 3$, or visit [1], to see what is the obstacle. For example, when $n = 3$ equality in (3) is attained at several points. Besides, direct differentiation of (3) reveals many "bad" critical points at which finding the values of (3) would require numerical computations [1]. The number of critical points together with equality cases increases as $n$ becomes larger, therefore one is forced to come up with a different idea. We will overcome this obstacle by looking at (3) in dual coordinates.

Without loss of generality we can assume that $u_j$ and $v_j$ are nonnegative for $j = 1, \dots, n$. Moreover, we can assume that $v_j \neq 0$ for all $j$ otherwise the inequality (3) is trivial.

Let us divide (3) by $\prod_{j=1}^n v_j$. Denoting $x_j := (u_j/v_j)^{p_n}$ we see that it is enough to prove the following lemma.

**Lemma 3.** *For any $n \geqslant 2$ and all $x_1, \dots, x_n \geqslant 0$ we have*

$$\left(\sum_{i=1}^n x_i^{1/p_n}\right)^{p_n} \leqslant \prod_{i=1}^n (1+x_i), \tag{4}$$

*where* $p_n = \frac{\ln \frac{n^n}{(n-1)^{n-1}}}{\ln n}$

*Proof.* For $n = 2$ the lemma is trivial. By induction on $n$, monotonicity of the map

$$p \to \left(\sum_{i=1}^n x_i^{1/p}\right)^p,$$

and the fact that $p_n$ is decreasing, we can assume that all $x_i$ are strictly positive. For convenience we set $p := p_n$. Introducing new variables we rewrite (4) as follows

$$p \ln\left(\sum x_i\right) \leqslant \sum \ln(1+x_i^p).$$

Concavity of the function $\ln(x)$ provides us with a simple representation of the logarithmic function

$$\ln(x) = \min_{b \in \mathbb{R}} (b + e^{-b}x - 1).$$

Therefore we are left to show that for all $x_i > 0$ and all $b_i \in \mathbb{R}$ we have

$$B(x,b) := \sum (b_i + (1+x_i^p)e^{-b_i} - 1) - p \ln\left(\sum x_i\right) \geqslant 0,$$

where $x = (x_1, \dots, x_n)$ and $b = (b_1, \dots, b_n)$. Notice that given a vector $b \in \mathbb{R}^n$, the infimum of $B(x,b)$ in $x$ cannot be reached at infinity because of the slow growth of the logarithmic function. Therefore, we look at critical points of $B$ in $x$

$$x_k^* = \frac{e^{\frac{b_k}{p-1}}}{\left(\sum_i e^{\frac{b_i}{p-1}}\right)^{\frac{1}{p}}} \quad \text{for} \quad k = 1, \dots, n.$$

Notice that $\sum x_i^* = \left(\sum_i e^{\frac{b_i}{p-1}}\right)^{\frac{p-1}{p}}$. Therefore

$$B(x^*, b) = \sum_k (b_k + e^{-b_k}) + 1 - n - (p-1) \ln\left(\sum_k e^{\frac{b_k}{p-1}}\right).$$

Setting $r := p - 1 > 0$, and introducing new variables again we are left to show that

$$f(y) := 1 - n + \sum \ln y_i^r + \sum \frac{1}{y_i^r} - r \ln \left( \sum y_i \right) \geqslant 0$$

for all $y_i > 0$. It is straightforward to check that $f(y) \geqslant 0$ on the diagonal, i.e., when $y_1 = y_2 = \ldots = y_n$.

In general, we notice that critical points of $f(y)$ satisfy the equation

$$\frac{1}{y_i} - \frac{1}{y_i^{r+1}} = \frac{1}{y_j} - \frac{1}{y_j^{r+1}} = \frac{1}{\sum y_k}. \tag{5}$$

Equation (5) gives the identity $\sum y_i^{-r} = n - 1$, and so at critical points (5) we are only left to show

$$\sum \ln y_i - \ln \left( \sum y_i \right) \geqslant 0. \tag{6}$$

Since the mapping

$$s \to \frac{1}{s} - \frac{1}{s^{r+1}}, \quad s > 0,$$

is increasing on $(0, (1+r)^{1/r})$ and decreasing on the remaining part of the ray, we can assume without loss of generality that $k$ numbers of $x_i$ equal to $u \geqslant (1+r)^{1/r}$, and the remaining $n - k$ numbers of $x_i$ equal to $v \leqslant (1+r)^{1/r}$. Moreover, we can assume that $0 < k < n$ otherwise the statement is already proved. From (5), we have

$$\frac{1}{u} - \frac{1}{u^{r+1}} = \frac{1}{v} - \frac{1}{v^{r+1}} = \frac{1}{ku + (n-k)v}. \tag{7}$$

From the equality of the first and the third expressions in (7) it follows that

$$v = \frac{u^{r+1}(1-k) + ku}{(u^r - 1)(n-k)}. \tag{8}$$

In order $v$ to be positive we assume that the numerator of (8) is non negative. If we plug the expression for $v$ from (8) into the first equality of (7) then after some simplifications we obtain the following equation in the variable $z := u^r \geqslant 1 + r$

$$\frac{(z-1)^r (n-k)^{r+1}}{(z(1-k)+k)^r} = (n-1)z - k. \tag{9}$$

It follows from (7) that $(ku + (n-k)v)^r = \left(\frac{z}{z-1}\right)^r z$, and so using (9) we obtain

$$v^r = \frac{z(n-k)}{(n-1)z - k}.$$

Therefore at critical points (6) simplifies as follows

$$(n-1)\ln z + (n-k)\ln\frac{n-k}{(n-1)z-k} - r\ln\frac{z}{z-1} \geqslant 0. \tag{10}$$

Now it is pretty straightforward to show that (10) is non negative even under the assumption $z \geqslant 1+r$ for $r = p-1 = \frac{(n-1)\ln\frac{n}{n-1}}{\ln n}$. Indeed, notice that $z > \frac{n}{n-1}$, and the map

$$k \to (n-k)\ln\frac{n-k}{(n-1)z-k}$$

is increasing on $[1, n-1]$. Therefore it is enough to check nonnegativity of (10) when $k=1$, in which case the inequality follows again using $z > \frac{n}{n-1}$, and the fact that the map

$$z \to \frac{\ln\left(1+\frac{1}{z(n-1)-1}\right)}{\ln\left(1+\frac{1}{z-1}\right)}$$

is increasing for $z \geqslant \frac{n}{n-1}$. □

*Remark* 4. Choice $x_1 = \ldots = x_n = \frac{1}{n-1}$ gives equality in (4), and this confirms the fact that $p_n$ is the best possible in Theorem 1.

### 2.2 Inductive step

Inductive step is the same as in [1] without any modifications. This is a standard argument for obtaining estimates on the Hamming cube (see for example [2]). In order to make the paper self contained we decided to repeat the argument.

Suppose (1) is true on the Hamming cube of dimension $m$. Without loss of generality assume $f_j \geqslant 0$, and set $g_j := f_j^p$ for all $j$. Define

$$B_n(y_1, \ldots, y_n) := y_1^{1/p_n} \cdots y_n^{1/p_n}.$$

For $x_j \in \{0,1\}^{m+1}$, let $x_j = (\bar{x}_j, x_j')$ where $\bar{x}_j$ is the vector consisting of the first $m$ coordinates of $x_j$, and number $x_j'$ denotes the last $m+1$ coordinate of $x_j$. Set

$$\tilde{g}_j(x_j') := \sum_{\bar{x}_j \in \{0,1\}^m} g_j(\bar{x}_j, x_j') \quad j = 1, \ldots, n.$$

We have

$$\sum_{x_j \in \{0,1\}^{m+1}\,:\, x_1+\cdots+x_n = 1^{m+1}.} B(g_1(x_1), \ldots, g_n(x_n)) =$$

$$\sum_{x_j' \in \{0,1\}\,:\, x_1'+\cdots+x_n' = 1.} \; \sum_{\bar{x}_j \in \{0,1\}^m \,:\, \bar{x}_1+\cdots+\bar{x}_n = 1^m.} B(g_1(x_1), \ldots, g_n(x_n)) \overset{\text{induction}}{\leqslant}$$

$$\sum_{x'_j\in\{0,1\}\,:\,x'_1+\dots+x'_n=1.} B(\tilde{g}_1(x'_1),\dots,\tilde{g}_n(x'_n)) \overset{\text{basis}}{\leqslant}$$

$$B\left(\sum_{x_1\in\{0,1\}^{m+1}} g_1(x_1),\dots,\sum_{x_n\in\{0,1\}^{m+1}} g_n(x_n)\right).$$

### 2.3 The proof of Corollary 2

Without loss of generality we may assume that all the sets $A$ in $X$ are subsets of $\{1,\dots,m\}$ with some natural $m \geqslant 1$ (see [1]). For $j=1,\dots,n$ define functions

$$f_j : \{0,1\}^m \to \{0,1\}$$

as follows:

$$f_1(a_1,\dots,a_m) = \ldots = f_{n-1}(a_1,\dots,a_m) = 1$$

if the set $\{1 \leqslant i \leqslant m \,:\, a_i = 1\}$ lies in $X$, and $f_j = 0$ otherwise. Finally we define

$$f_n(a_1,\dots,a_m) = 1$$

if the set $\{1 \leqslant i \leqslant m \,:\, a_i = 0\}$ lies in $X$, and $f_n = 0$ otherwise. Notice that in this case inequality (1) becomes (2).

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## References

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