Classification of Q-multiplicity-free skew Schur Q-functions

Christopher Schure

Institut für Algebra, Zahlentheorie und Diskrete Mathematik
Leibniz Universität Hannover
Welfengarten 1
D-30167 Hannover, Germany

schure@math.uni-hannover.de

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Abstract

We classify the Q-multiplicity-free skew Schur Q-functions. Towards this result, we also provide new relations between the shifted Littlewood-Richardson coefficients.

1 Introduction

Schur functions form an important basis of the algebra of symmetric functions. They appear in the study of the representations of the symmetric groups and the general linear groups. Schur P-functions and Schur Q-functions are bases of the subalgebra generated by the odd power sums. In [11], Stembridge proved a number of important properties of Schur Q-functions emphasizing that they may be viewed as shifted analogues of Schur functions. While in the classical situation Schur functions are closely related to ordinary irreducible characters of the symmetric groups, Schur Q-functions are intimately connected to irreducible spin characters of the double covers of the symmetric groups. Multiplicity-free products of Schur functions were classified by Stembridge in [10]. As a shifted analogue of Stembridge's result, Bessenrodt then classified the P-multiplicity-free products of Schur P-functions in [2]. A skew generalization of Stembridge's result was proved independently by Gutschwager in [5] and Thomas and Yong in [12]. While Gutschwager classified the multiplicity-free skew Schur functions, Thomas and Yong classified the multiplicity-free products of Schubert classes. However, what was missing was a skew analogue of Bessenrodt's result or equivalently a shifted analogue of Gutschwager's result. The main goal of this article is to provide this here, i.e., to classify the Q-multiplicity-free skew Schur Q-functions. We will heavily rely on the shifted Littlewood-Richardson rule obtained by Stembridge in [11] (another version of this rule was given by Cho [3]).

The paper is structured as follows. In the second section we provide the required definitions and some properties needed later. In the third section we prove relations between shifted Littlewood-Richardson coefficients, which will simplify proofs of the fourth section. In the fourth section we will first exclude all non-Q-multiplicity-free skew Schur Q-functions before proving the Q-multiplicity-freeness of the remaining skew Schur Q-functions to obtain our main classification result, Theorem 4.53. Note that we define some special notation for partitions with distinct parts whose shifted diagrams have at most two corners in Definition 4.19. We will use that notation in most lemmas of the fourth section.

2 Preliminaries

We will use the same notation as in [9]. Some of the tools introduced there will also be useful in the context here.

2.1 Partitions, diagrams and tableaux

We define a **partition** as a tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ where $\lambda_j \in \mathbb{N}$ for all $1 \leqslant j \leqslant n$ and $\lambda_i \geqslant \lambda_{i+1} > 0$ for all $1 \leqslant i \leqslant n-1$. The **length** of λ is $\ell(\lambda) := n$. A partition λ is called a partition of k if $|\lambda| := \lambda_1 + \lambda_2 + \ldots + \lambda_{\ell(\lambda)} = k$ where $|\lambda|$ is called the **size** of λ. A **partition with distinct parts** is a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ where $\lambda_i > \lambda_{i+1} > 0$ for all $1 \leqslant i \leqslant n-1$. The set of partitions of k with distinct parts is denoted by DP_k. By definition, the empty partition $\emptyset$ is the only element in DP_0 and it has length 0. The **set of all partitions with distinct parts** is denoted by $DP := \bigcup_k DP_k$. For $\lambda \in DP$ the **shifted diagram** D_λ is defined by $D_\lambda := \{(i,j) \mid 1 \leqslant i \leqslant \ell(\lambda), i \leqslant j \leqslant i + \lambda_i - 1\}$.

Convention. In this paper we will omit the adjective shifted. This means that whenever a diagram is mentioned it is always a shifted diagram.

For $\lambda, \mu \in DP$ with $\ell(\mu) \leqslant \ell(\lambda)$ and $\mu_i \leqslant \lambda_i$ for all $1 \leqslant i \leqslant \ell(\mu)$, we define the **skew diagram** $D_{\lambda/\mu} := D_\lambda \setminus D_\mu$. Its **size** is $|D_{\lambda/\mu}| = |D_\lambda| - |D_\mu|$.

Each edgewise connected part of a skew diagram D is called a **component**. The number of components of D is denoted by $comp(D)$. If $comp(D) = 1$, then D is called **connected**, otherwise it is called **disconnected**.

In the following, if components are numbered, the numbering is as follows: the first component is the leftmost component, the second component is the next component to the right of the first component etc.

A **corner** of a skew diagram D is a box $(x,y) \in D$ such that $(x+1,y), (x,y+1) \notin D$. An **unshifted** diagram is a skew diagram $D_{\lambda/\mu}$ with $\ell(\mu) = \ell(\lambda) - 1$. A (skew) diagram can be depicted as an arrangement of boxes where the coordinates of the boxes are interpreted in matrix notation.

Example 2.1. Let $\lambda = (6, 5, 2, 1)$ and $\mu = (4, 3)$. Then the (skew) diagram is

$$D_{\lambda/\mu} = \begin{array}{|c|c|c|c|} \cline{3-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & & \\ \cline{3-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \bullet & \times \\ \cline{1-4} & & \multicolumn{2}{c}{} \\ \cline{1-2} \multicolumn{1}{c|}{} & \times & \multicolumn{2}{c}{} \\ \cline{2-2} \end{array}.$$

We have $|D_{\lambda/\mu}| = 7$. The diagram $D_{\lambda/\mu}$ has two components where the first component consists of three boxes and the second component consists of four boxes. The corners of $D_{\lambda/\mu}$ are the boxes marked $\times$. The box marked $\bullet$ is the box $(2, 5)$

We consider the alphabet $\mathcal{A} = \{1' < 1 < 2' < 2 < \ldots\}$. The letters $1, 2, 3, \ldots$ are called **unmarked** letters and the letters denoted by $1', 2', 3', \ldots$ are called **marked** letters. For a letter x of the alphabet $\mathcal{A}$ we denote the unmarked version of this letter with $|x|$.

Definition 2.2. Let $\lambda, \mu \in DP$ with skew diagram $D_{\lambda/\mu}$. A **tableau** T of shape $D_{\lambda/\mu}$ is a map $T : D_{\lambda/\mu} \to \mathcal{A}$ such that

a) $T(i, j) \leqslant T(i + 1, j)$, $T(i, j) \leqslant T(i, j + 1)$ for all i, j,

b) each column has at most one k (for each $k = 1, 2, 3, \ldots$),

c) each row has at most one k' (for each $k' = 1', 2', 3', \ldots$).

Let $c^{(u)}(T) = (c_1^{(u)}, c_2^{(u)}, \ldots)$ where $c_i^{(u)}$ denotes the number of letters equal to i in the tableau T, for each i. Analogously, let $c^{(m)}(T) = (c_1^{(m)}, c_2^{(m)}, \ldots)$ where $c_i^{(m)}$ denotes the number of i's in the tableau T, for each i. Then the **content** of T is defined by $c(T) = (c_1, c_2, \ldots) := c^{(u)}(T) + c^{(m)}(T)$. If there is some k such that $c_k > 0$ but $c_j = 0$ for all $j > k$ then we omit all these c_j from $c(T)$.

Remark 2.3. We depict a tableau T of shape $D_{\lambda/\mu}$ by filling each box (x, y) of the diagram with the letter $T(x, y)$.

Example 2.4. Let $\lambda = (8, 6, 5, 3, 2)$ and $\mu = (5, 2, 1)$. Then a tableau of shape $D_{\lambda/\mu}$ is

$$T = \begin{array}{|c|c|c|c|c|} \cline{3-5} \multicolumn{2}{c|}{} & 1' & 1 & 2 \\ \cline{1-5} 2' & 2 & 2 & 4 & \multicolumn{1}{c}{} \\ \cline{1-4} 2 & 4 & 5 & 5 & \multicolumn{1}{c}{} \\ \cline{1-4} 4 & 6' & 6 & \multicolumn{2}{c}{} \\ \cline{1-3} \multicolumn{1}{c|}{} & 6 & 7 & \multicolumn{2}{c}{} \\ \cline{2-3} \end{array}.$$

We have $c(T) = (2, 5, 0, 3, 2, 3, 1)$.

2.2 Skew Schur Q-functions

For $\lambda, \mu \in DP$ and a countable set of independent variables $x_1, x_2, \ldots$ the **skew Schur Q-function** is defined by

$$Q_{\lambda/\mu} := \sum_{T \in T(\lambda/\mu)} x^{c(T)}$$

where $T(\lambda/\mu)$ denotes the set of all tableaux of shape $D_{\lambda/\mu}$ and $x^{(c_1,c_2,\ldots,c_\ell)} := x_1^{c_1} x_2^{c_2} \cdots$ with $c_k := 0$ for $k > \ell$. If $D_\mu \nsubseteq D_\lambda$ then $Q_{\lambda/\mu} := 0$. Since $D_{\lambda/\emptyset} = D_\lambda$, we denote $Q_{\lambda/\emptyset}$ by Q_λ.

Definition 2.5. Let a diagram D be such that the y^{th} column has no box, but there are boxes to the right of the y^{th} column and after shifting all boxes that are to the right of the y^{th} column one box to the left we obtain a diagram $D_{\alpha/\beta}$ for some $\alpha, \beta \in DP$. Then we call the y^{th} column **empty** and the diagram $D_{\alpha/\beta}$ is obtained by removing the y^{th} column. Similarly, let a diagram D be such that the x^{th} row has no box, but there are boxes below the x^{th} row and after shifting all boxes that are below the x^{th} row one box up and then all boxes of the diagram one box to the left, we obtain a diagram $D_{\alpha/\beta}$ for some $\alpha, \beta \in DP$. Then we call the x^{th} row **empty** and the diagram $D_{\alpha/\beta}$ is obtained by removing the x^{th} row.

Definition 2.6. For $\lambda, \mu \in DP$ we call the diagram $D_{\lambda/\mu}$ **basic** if it satisfies the following properties:

- $D_\mu \subseteq D_\lambda$,
- $\ell(\lambda) > \ell(\mu)$,
- $\lambda_i > \mu_i$, for all $1 \leqslant i \leqslant \ell(\mu)$,
- $\lambda_{i+1} \geqslant \mu_i - 1$, for all $1 \leqslant i \leqslant \ell(\mu)$.

This means that $D_{\lambda/\mu}$ has no empty rows or columns.

For a given diagram D, let $\bar{D}$ be the diagram obtained by removing all empty rows and columns of the diagram D. Since the tableau restrictions on each box in a diagram are unaffected by removing empty rows and columns, there is a content-preserving bijection between tableaux of a given shape and tableaux of the shape obtained by removing empty rows and columns; thus we have $Q_D = Q_{\bar{D}}$. Hence, we may restrict our considerations to partitions λ and μ such that $D_{\lambda/\mu}$ is basic.

<u>**Convention**</u>. From now on we will only consider partitions λ and μ such that $D_{\lambda/\mu}$ is basic.

For some given skew diagram D let the diagram obtained after removing empty rows and columns be $D_{\lambda/\mu}$ for some $\lambda, \mu \in DP$. Then Q_D is equal to the skew Schur Q-function $Q_{\lambda/\mu}$.

For a tableau T of shape D, the **reading word** $w = w(T)$ is the word obtained by reading the rows from left to right beginning with the bottom row and ending with the top row. The **length** $\ell(w)$ is the number of letters and, thus, the number $n = |D|$ of boxes in D. Let $(x(i), y(i))$ denote the box of the i^{th} letter of the reading word $w(T)$.

For the reading word $w = w_1 w_2 \dots w_n$ of the tableau T the statistics $m_i(j)$ are defined as follows:

- $m_i(0) = 0$ for all i.
- For $1 \leqslant j \leqslant n$ the number $m_i(j)$ is equal to the number of times i occurs in the word $w_{n-j+1} \dots w_n$.
- For $n+1 \leqslant j \leqslant 2n$ we set $m_i(j) := m_i(n) + k(i)$ where $k(i)$ is the number of times i' occurs in the word $w_1 \dots w_{j-n}$.

As Stembridge remarked [11, before Theorem 8.3], the statistics $m_i(j)$ for some given i can be calculated by taking the word $w(T)$ and scan it first from right to left while counting the letters i and afterwards scan it from left to right and adding the number of letters i'. After the j^{th} step of scanning and counting the statistic $m_i(j)$ is calculated.

Definition 2.7. Let $k \in \mathbb{N}$ and $w = w_1 w_2 \dots w_n$ be a word of length n consisting of letters from the alphabet $\mathcal{A}$. The word w is called k**-amenable** if it satisfies the following conditions:

a) if $m_k(j) = m_{k-1}(j)$ then $w_{n-j} \notin \{k, k'\}$ for all $0 \leqslant j \leqslant n-1$,

b) if $m_k(j) = m_{k-1}(j)$ then $w_{j-n+1} \notin \{k-1, k'\}$ for all $n \leqslant j \leqslant 2n-1$,

c) if j is the smallest number such that $w_j \in \{k', k\}$ then $w_j = k$,

d) if j is the smallest number such that $w_j \in \{(k-1)', k-1\}$ then $w_j = k-1$.

Note that $c_i^{(u)} = m_i(n)$ and $c_i^{(m)} = m_i(2n) - m_i(n)$.

The word w is called **amenable** if it is k-amenable for all $k > 1$. A tableau T is called (k-)amenable if $w(T)$ is (k-)amenable.

Example 2.8. Let $w = 322'24'2'1'12$. Then $m_2(9) = 3$ and $m_2(12) = 4$. The word w is not 2-amenable since $m_1(0) = m_2(0) = 0$ and $w_9 = 2$. But w is 3-amenable.

Remark 2.9. In [11] Stembridge considered words w that satisfy the lattice property. These are precisely the words we call amenable.

Lemma 2.10. *[8, Lemma 3.28] Let w be a k-amenable word for some $k \geqslant 1$. Let $n = \ell(w)$. If $m_{k-1}(n) > 0$ then $m_{k-1}(n) > m_k(n)$.*

Lemma 2.11. *Let T be an amenable tableau. Then there are no entries greater than k in the first k rows.*

Proof. Assume the opposite. Let i be the topmost row with an entry greater than i. Let this entry be x. Then, while scanning the word $w(T)$ from right to left, the letter x will be scanned before any letter $|x| - 1$ will be scanned; a contradiction to the amenability of the tableau T. □

Using Lemma 2.11, one can construct a brute force algorithm to obtain amenable tableaux of a given shape. The algorithm fills the first row with elements from $\{1', 1\}$ satisfying the conditions of Definition 2.2. Each box of the i^{th} row gets filled with entries at most i and greater or equal the entry of the box above (if there is a box above) and greater or equal the entry of the box to the left (if there is a box to the left), again in a way such that the i^{th} row satisfies Definition 2.2. After all boxes are filled the algorithm takes the reading word and uses Stembridge's scanning algorithm to check if this filling is amenable.

For the proofs of the lemmas in Section 4 we will use the following shifted Littlewood-Richardson rule by Stembridge.

Theorem 2.12. *[11, Theorem 8.3] For $\lambda, \mu \in DP$ we have*

$$Q_{\lambda/\mu} = \sum_{\nu \in DP} f^{\lambda}_{\mu\nu} Q_{\nu},$$

where $f^{\lambda}_{\mu\nu}$ is the number of amenable tableaux T of shape $D_{\lambda/\mu}$ and content ν.

For $\lambda \in DP$, the corresponding Schur P-function is defined by $P_{\lambda} := 2^{-\ell(\lambda)} Q_{\lambda}$. In [11, Section 8], Stembridge showed that the numbers $f^{\lambda}_{\mu\nu}$ above also appear in the product of P-functions:

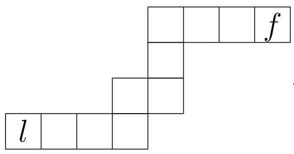
$$P_{\mu} P_{\nu} = \sum_{\lambda \in DP} f^{\lambda}_{\mu\nu} P_{\lambda} \, .$$

Using this, one easily obtains the equation $f^{\lambda}_{\mu\nu} = f^{\lambda}_{\nu\mu}$ for all $\lambda, \mu, \nu \in DP$.

Definition 2.13. A **border strip** is a connected (skew) diagram B such that for each $(x, y) \in B$ we have $(x-1, y-1) \notin B$. The box $(x, y) \in B$ such that $(x-1, y) \notin B$ and $(x, y+1) \notin B$ is called the **first box** of B. The box $(u, v) \in B$ such that $(u+1, v) \notin B$ and $(u, v-1) \notin B$ is called the **last box** of B.

Example 2.14. For $\lambda = (11, 7, 6, 4)$ and $\mu = (7, 6, 4)$ the diagram $D_{\lambda/\mu}$ is a border strip:

$D_{\lambda/\mu} =$



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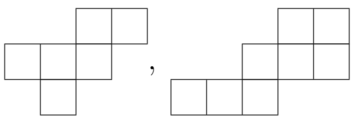
The box labeled f is the first box and the box labeled l is the last box. The following diagrams are examples of diagrams that are not border strips:



Remark 2.15. While the first component denotes the leftmost component, the first box is one of the rightmost boxes. We use this convention because the filling of the border strips that appear in Definition 2.27 can easily be derived if we fill these boxes starting at the first box and ending at the last box of the border strip.

Definition 2.16. A (possibly disconnected) diagram D where all components are border strips is called a **broken border strip**. Then the first box of the rightmost component is called the **first box of D**, and the last box of the leftmost component is called the **last box of D**.

A (p, q)**-hook** is a set of boxes

$$\{(u, v+q-1), \ldots, (u, v+1), (u, v), (u+1, v), \ldots, (u+p-1, v)\}$$

for some $u, v \in \mathbb{N}$. More precisely, we say that the previous set of boxes is a (p, q)-hook at (u, v).

Definition 2.17. Let T be a tableau of shape $D_{\lambda/\mu}$. Define $T^{(i)}$ by

$$T^{(i)} := \{(x, y) \in D_{\lambda/\mu} \mid |T(x, y)| = i\}.$$

Lemma 2.18. *[6, Remark before Theorem 13.1] Let T be a tableau of shape $D_{\lambda/\mu}$. Then $|T(x, y)| < |T(x+1, y+1)|$ for all x, y such that $(x, y), (x+1, y+1) \in D_{\lambda/\mu}$.*

As a consequence of this lemma, for a given tableau T each component of $T^{(i)}$ is a border strip. The fact that each component of $T^{(i)}$ is a border strip as well as the following lemma are derived from [7, after Corollary 8.6].

Lemma 2.19. *Let T be a tableau of shape $D_{\lambda/\mu}$. Let $T^{(i)}$ be of shape D for some diagram D. Then each component of D has two possible fillings which differ only in the last box of this component.*

We will use a criterion for k-amenability of a tableau that avoids the use of the reading word. This is provided in Lemma 2.24; to state this lemma we need the following definitions.

Definition 2.20. Let T be a tableau. If the last box of $T^{(i)}$ is filled with i we call $T^{(i)}$ **fitting**.

Definition 2.21. Let $\lambda, \mu \in DP$ and let T be a tableau of $D_{\lambda/\mu}$. Then

$$\mathcal{S}^{\boxtimes}_{\lambda/\mu}(x, y) := \{(u, v) \in D_{\lambda/\mu} \mid u \leqslant x, v \geqslant y\},$$

$$\mathcal{S}^{\boxtimes}_{T}(x, y)^{(i)} := \mathcal{S}^{\boxtimes}_{\lambda/\mu}(x, y) \cap T^{-1}(i) \text{ where } T^{-1}(i) \text{ denotes the preimage of } i,$$

$$\mathcal{B}^{(i)}_{T} := \{(x, y) \in D_{\lambda/\mu} \mid T(x, y) = i' \text{ and } T(x-1, y-1) \neq (i-1)'\},$$

$$\widehat{\mathcal{B}^{(i)}_{T}} := \{(x, y) \in D_{\lambda/\mu} \mid T(x, y) = i' \text{ and } T(x+1, y+1) \neq (i+1)'\}$$

and $b^{(i)}_T = |\mathcal{B}^{(i)}_T|$ for all i. Then let $\mathcal{B}^{(i)}_T(d)$ denote the set of the first d boxes of $\mathcal{B}^{(i)}_T$.

Remark 2.22. The set $\mathcal{S}^{\boxtimes}_{\lambda/\mu}(x, y)$ above is the set of boxes that are simultaneously weakly above and weakly to the right of the box (x, y). The set $\mathcal{S}^{\boxtimes}_{T}(x, y)^{(i)}$ is the subset of boxes (u, v) of $\mathcal{S}^{\boxtimes}_{\lambda/\mu}(x, y)$ such that $T(u, v) = i$. Note that $\mathcal{S}^{\boxtimes}_{T}(x, y)^{(i)}$ is some set of boxes which are filled only with i while $T^{(i)}$ is some set of boxes which are filled with i or i'.

Example 2.23. Let $\lambda = (11, 9, 6, 5, 4, 2, 1)$ and $\mu = (8, 6, 5, 4, 1)$ and let

$$T = \begin{array}{ccccccccccc} \times & \times & \times & \times & \times & \times & \times & \times & \mathbf{1'} & \mathbf{1} & \mathbf{1} \\ & \times & \times & \times & \times & \times & \times & \mathbf{1'} & \mathbf{2'} & \mathbf{2} & \\ & & \times & \times & \times & \times & \times & \mathbf{1} & & & \\ & & & \times & \times & \times & \times & 2' & & & \\ & & & & \times & 1' & 1 & 2 & & & \\ & & & & & 1 & 2' & & & & \\ & & & & & & 2 & & & & \end{array} .$$

Then $\mathcal{S}^{\boxtimes}_{\lambda/\mu}(3, 8)$ is the set of boxes with boldfaced entries. Also, we have $\mathcal{S}^{\boxtimes}_T(3, 8)^{(1)} = \{(1, 10), (1, 11), (3, 8)\}$, $\mathcal{B}^{(2)}_T = \{(2, 9), (4, 8)\}$ and $\widehat{\mathcal{B}^{(1)}_T} = \{(1, 9), (2, 8)\}$.

Lemma 2.24. *[9, Lemma 2.14] Let $\lambda, \mu \in DP$ and $n = |D_{\lambda/\mu}|$. Let T be a tableau of $D_{\lambda/\mu}$. Then the tableau T is k-amenable if and only if either $c(T)_{k-1} = c(T)_k = 0$ or else it satisfies the following conditions:*

(1) $c(T)^{(u)}_{k-1} > c(T)^{(u)}_k$;

(2) if $T(x, y) = k$ then $|\mathcal{S}^{\boxtimes}_T(x, y)^{(k-1)}| \geqslant |\mathcal{S}^{\boxtimes}_T(x, y)^{(k)}|$;

(3) for each $(x, y) \in \mathcal{B}^{(k)}_T$ we have $|\mathcal{S}^{\boxtimes}_T(x, y)^{(k-1)}| > |\mathcal{S}^{\boxtimes}_T(x, y)^{(k)}|$;

(4) if $d = b^{(k)}_T + c^{(u)}_k - c^{(u)}_{k-1} + 1 > 0$ then there is an injective map $\phi : \mathcal{B}^{(k)}_T(d) \to \widehat{\mathcal{B}^{(k-1)}_T}$ such that if $(x, y) \in \mathcal{B}^{(k)}_T(d)$ and $(u, v) = \phi(x, y)$ then for all $u < r < x$ we have $T(r, s) \notin \{k - 1, k'\}$ for all s such that $(r, s) \in D_{\lambda/\mu}$;

(5) $T^{(k-1)}$ is fitting;

(6) if $c(T)_k > 0$ then $T^{(k)}$ is fitting.

Corollary 2.25. *[9, Corollary 2.15] Let $\lambda, \mu \in DP$. Let T be a tableau of shape $D_{\lambda/\mu}$ such that either $c(T)_k = c(T)_{k-1} = 0$ or else it satisfies the following conditions:*

(1) there is some box (x, y) such that $T(x, y) = k - 1$ and $T(z, y) \neq k$ for all $z > x$;

(2) if $T(x, y) = k$ then there is some $z < x$ such that $T(z, y) = k - 1$;

(3) if $T(x, y) = k'$ then $T(x - 1, y - 1) = (k - 1)'$;

(4) $T^{(k-1)}$ is fitting;

(5) if $c^{(u)}_k > 0$ then $T^{(k)}$ is fitting.

Then the tableau is k-amenable.

Example 2.26. We consider again the following tableau T of shape $D_{(11,9,6,5,4,2,1)/(8,6,5,4,1)}$:

$$T = \begin{array}{ccccccccccc}
\times & \times & \times & \times & \times & \times & \times & \times & 1' & 1 & 1 \\
 & \times & \times & \times & \times & \times & \times & 1' & 2' & 2 & \\
 & & \times & \times & \times & \times & \times & 1 & & & \\
 & & & \times & \times & \times & \times & 2' & & & \\
 & & & & \times & 1' & 1 & 2 & & & \\
 & & & & & 1 & 2' & & & & \\
 & & & & & & 2 & & & &
\end{array} .$$

We want to check the conditions of Lemma 2.24 for $k = 2$. We have $c(T)_1^{(u)} = 5 > 3 = c(T)_2^{(u)}$. Since $T^{-1}(2) = \{(2,10),(5,8),(7,7)\}$ we need to check condition (2) of Lemma 2.24 for all of these boxes. We have $|\mathcal{S}_T^{\boxtimes}(2,10)^{(1)}| = 2 \geqslant 1 = |\mathcal{S}_T^{\boxtimes}(2,10)^{(2)}|$, $|\mathcal{S}_T^{\boxtimes}(5,8)^{(1)}| = 3 \geqslant 2 = |\mathcal{S}_T^{\boxtimes}(5,8)^{(2)}|$ and $|\mathcal{S}_T^{\boxtimes}(7,7)^{(1)}| = 4 \geqslant 3 = |\mathcal{S}_T^{\boxtimes}(7,7)^{(2)}|$. Since $\mathcal{B}_T^{(2)} = \{(2,9),(4,8)\}$ we need to check condition (3) of Lemma 2.24 for these boxes. We have $|\mathcal{S}_T^{\boxtimes}(2,9)^{(1)}| = 2 > 1 = |\mathcal{S}_T^{\boxtimes}(2,9)^{(2)}|$ and $|\mathcal{S}_T^{\boxtimes}(4,8)^{(1)}| = 3 > 1 = |\mathcal{S}_T^{\boxtimes}(4,8)^{(2)}|$. Since $d = 2+3-5+1 = 1$ we have to find a map as in condition (4) of Lemma 2.24 for the box $(2,9)$. We have $\mathcal{B}_T^{(2)}(1) = \{(2,9)\}$ and $\widehat{\mathcal{B}_T^{(1)}} = \{(1,9),(2,8)\}$. Both possible maps from $\mathcal{B}_T^{(2)}(1)$ to $\widehat{\mathcal{B}_T^{(1)}}$ satisfy the property of condition (4) of Lemma 2.24. Clearly, $T^{(1)}$ and $T^{(2)}$ are fitting. Hence, the tableau T is 2-amenable.

In Section 4 we will start with a specific amenable tableau for a given diagram and change some entries to obtain new tableaux. This specific tableau is obtained by an algorithm described by Salmasian in [8, Section 3.1].

Definition 2.27. Let $D_{\lambda/\mu}$ be a skew diagram. The tableau $T_{\lambda/\mu}$ is determined by the following algorithm:

(1) Set $k = 1$ and $U_1(\lambda/\mu) = D_{\lambda/\mu}$.

(2) Set $P_k = \{(x,y) \in U_k(\lambda/\mu) \mid (x-1,y-1) \notin U_k(\lambda/\mu)\}$.

(3) For each $(x,y) \in P_k$ set $T_{\lambda/\mu}(x,y) = k'$ if $(x+1,y) \in P_k$, otherwise set $T_{\lambda/\mu}(x,y) = k$.

(4) Let $U_{k+1}(\lambda/\mu) = U_k(\lambda/\mu) \setminus P_k$.

(5) Increase k by one, and go to (2).

Example 2.28. For $\lambda = (6,5,3,2)$ and $\mu = (4,1)$ we obtain

$$T_{\lambda/\mu} = \begin{array}{cccc}
 & & 1' & 1 \\
1' & 1 & 1 & 2 \\
1 & 2' & 2 & \\
 & 2 & 3 &
\end{array} .$$

The following definitions will be used in Proposition 2.31.

Definition 2.29. Let $\lambda \in DP$. Then the **border** of λ is defined by

$$B_\lambda := \{(x, y) \in D_\lambda \mid (x+1, y+1) \notin D_\lambda\}.$$

Define $B_\lambda^{(n)} := \{D_{\lambda/\mu} \mid D_{\lambda/\mu} \subseteq B_\lambda \text{ and } |D_{\lambda/\mu}| = n\}$.

Definition 2.30. Let $\lambda \in DP$. Define E_λ to be the set of all partitions whose diagram is obtained after removing a corner in D_λ.

We will restate the decomposition of $Q_{\lambda/\mu}$ where $\ell(\mu) = 1$ from [9, Proposition 3.3]. The following proposition will be used to show Q-multiplicity-freeness of some skew Schur Q-function.

Proposition 2.31. *Let $\lambda \in DP$ and n be an integer such that $1 \leqslant n \leqslant \lambda_1$. Then*

$$Q_{\lambda/(n)} = \sum_{D_{\lambda/\nu} \in B_\lambda^{(n)} \ (D_\nu \subseteq D_\lambda)} 2^{comp(D_{\lambda/\nu})-1} Q_\nu.$$

In particular, with $D_\mu = D_\lambda \setminus B_\lambda$ we have

$$Q_{\lambda/(\lambda_1-1)} = \sum_{(x,y) \in B_\lambda^\times} c_{B_\lambda}^{(x,y)} Q_{D_\mu \cup \{(x,y)\}}$$

where $B_\lambda^\times := \{(x, y) \in B_\lambda \mid (x-1, y) \notin B_\lambda$ and $(x, y-1) \notin B_\lambda\}$ and

$$c_{B_\lambda}^{(x,y)} = \begin{cases} 1 & \text{if } (x,y) \text{ is the first or last box of } B_\lambda \\ 2 & \text{otherwise,} \end{cases}$$

and

$$Q_{\lambda/(1)} = \sum_{\nu \in E_\lambda} Q_\nu.$$

2.3 Equality of skew Schur Q-functions

Before we analyze the Q-multiplicity-freeness of some given skew Schur Q-function $Q_{\lambda/\mu}$, we show that $Q_{\lambda/\mu} = Q_{\alpha/\beta}$ where $D_{\alpha/\beta}$ is a diagram obtained by certain transformations of the diagram $D_{\lambda/\mu}$. Hence it will be sufficient to analyze the skew Schur Q-function of one of these (transformed) diagrams to obtain statements for all of them. This approach significantly reduces the effort in the proofs for the lemmas that lead to the classification of Q-multiplicity-free skew Schur Q-functions.

Definition 2.32. Let D be an unshifted skew diagram. The **transpose** of D, denoted by D^t, is the unshifted skew diagram obtained by reflecting the boxes of D along the diagonal $\{(x, x) \mid x \in \mathbb{N}\}$ and moving this arrangement of boxes such that the top row with boxes is in the first row and the lowest box of the leftmost column with boxes is on the diagonal $\{(x, x) \mid x \in \mathbb{N}\}$.
The **rotation** of D, denoted by D^o, is the unshifted skew diagram obtained by rotating the boxes of D through 180° and moving this arrangement of boxes such that the topmost row with boxes is in the first row and the lowest box of the leftmost column with boxes is on the diagonal $\{(x, x) \mid x \in \mathbb{N}\}$.

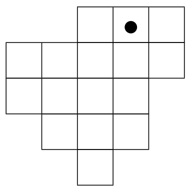
These are transformations of unshifted skew diagrams that do not change the corresponding Q-function, as is stated in the following lemma which is implied by [1, Proposition 3.3] by Barekat and van Willigenburg.

Lemma 2.33. *Let $D = D_{\lambda/\mu}$ be an unshifted skew diagram. There is a content-preserving bijection between tableaux of D and tableaux of D^t, as well as a content-preserving bijection between the tableaux of D and the tableaux of D^o. Hence*

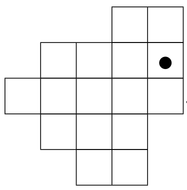
$$Q_{\lambda/\mu} = Q_{(\lambda/\mu)^t} = Q_{(\lambda/\mu)^o}.$$

Definition 2.34. Let D be a (not necessarily unshifted) skew diagram. The **orthogonal transpose** of D, denoted by D^{ot}, is obtained as follows: reflect the boxes of D along the diagonal $\{(z, -z) \mid z \in \mathbb{N}\}$. Move this arrangement of boxes such that the top row with boxes is in the first row and the lowermost box of the leftmost column with boxes is on the diagonal $\{(z, z) \mid z \in \mathbb{N}\}$.

Example 2.35. For

$D =$ 

we obtain

$D^{ot} =$ .

Note that orthogonal transposing switches the topmost row and the rightmost column as well as the leftmost column and the lowermost row. As an example we marked a box in D in the topmost row with $\bullet$ and after orthogonal transposing the respective box in the rightmost column.

As it turns out this is again a transformation on shifted skew diagrams that leaves the respective Q-function unchanged; this has been shown by DeWitt ([4, Proposition IV.13]) and independently in [9, Lemma 3.11]. DeWitt called this transformation "flip".

Lemma 2.36. *Let D be a skew diagram. There is a content-preserving bijection between the tableaux of D and the tableaux of D^{ot}.*

The diagrams U_i in the following lemma are defined by Salmasian's algorithm as in Definition 2.27.

Lemma 2.37. *Let $\lambda, \mu \in DP$, $\nu = c(T_{\lambda/\mu})$ and $n = \ell(\nu)$. Let $D^{ot}_{\lambda/\mu}$ have shape $D_{\gamma/\delta}$. Let $T' = T_{\gamma/\delta}$. If $U_i(\lambda/\mu)$ has shape $D_{\alpha/\beta}$ then $U_i(\gamma/\delta)$ has shape $D^{ot}_{\alpha/\beta}$. Particularly, for $i = n$ the set of boxes $T'^{(n)}$ has the same shape as P^{ot}_n.*

Proof. The diagram $U_i(\gamma/\delta)$ is also defined by $\{(x,y) \in D_{\gamma/\delta} \mid (x-i+1, y-i+1) \in D_{\gamma/\delta}\}$ and the image of this set of boxes after orthogonally transposing is given by the set of boxes $\{(u,v) \in D_{\lambda/\mu} \mid (u+i-1, v+i-1) \in D_{\lambda/\mu}\}$ which has the same shape as the set of boxes $\{(u,v) \in D_{\lambda/\mu} \mid (u-i+1, v-i+1) \in D_{\lambda/\mu}\} = U_i(\lambda/\mu)$. □

3 Relations between the coefficients $f^{\lambda}_{\mu\nu}$

In this section we will prove some inequalities that are satisfied by the shifted Littlewood-Richardson coefficients $f^{\lambda}_{\mu\nu}$. These inequalities will be helpful for the proofs in Section 4.

Lemma 3.1. *Let $\lambda, \mu \in DP$. Let $c(T_{\lambda/\mu}) = \nu = (\nu_1, \ldots, \nu_n)$. Let k be such that $U_k(\lambda/\mu)$ has shape $D_{\alpha/\beta}$ for some $\alpha, \beta \in DP$. Then*

$$\prod_{j=1}^{k-1} 2^{comp(P_j)-1} f^{\alpha}_{\beta\gamma} = f^{\lambda}_{\mu(\nu_1,\ldots\nu_{k-1},\gamma_1,\ldots,\gamma_{\ell(\gamma)})}.$$

In particular, we have $f^{\alpha}_{\beta\gamma} \leqslant f^{\lambda}_{\mu(\nu_1,\ldots\nu_{k-1},\gamma_1,\ldots,\gamma_{\ell(\gamma)})}$.

Proof. Let $f^{\alpha}_{\beta\gamma} = m$, i.e., there are exactly m different amenable tableaux of $D_{\alpha/\beta}$ of content $\gamma = (\gamma_1, \ldots, \gamma_{\ell(\gamma)})$. Then we can obtain $\prod_{i=1}^{k-1} 2^{comp(P_i)-1} f^{\alpha}_{\beta\gamma}$ different amenable tableaux of $D_{\lambda/\mu}$ of content $(\nu_1, \ldots \nu_{k-1}, \gamma_1, \ldots, \gamma_{\ell(\gamma)})$ as follows. For each box of $D_{\alpha/\beta}$ replace its entry i (respectively, i') by $i+k-1$ (respectively, $(i+k-1)'$). Use these as the filling of the boxes of $U_k(\lambda/\mu)$. For all $1 \leqslant j \leqslant k-1$ fill the boxes of P_j with entries from $\{j', j\}$. By Lemma 2.19, we have $2^{comp(P_j)-1}$ possible fillings of P_j such that P_j is fitting. We only need to show k-amenability for each of these tableaux, which follows straightforwardly by Corollary 2.25. Hence, $\prod_{j=1}^{k-1} 2^{comp(P_j)-1} f^{\alpha}_{\beta\gamma} \leqslant f^{\lambda}_{\mu(\nu_1,\ldots\nu_{k-1},\gamma_1,\ldots,\gamma_{\ell(\gamma)})}$.

Let T be an amenable tableaux of $D_{\lambda/\mu}$ of content $(\nu_1, \ldots \nu_{k-1}, \gamma_1, \ldots, \gamma_{\ell(\gamma)})$. Then, by Lemma 2.18, each box (x,y) of the ν_1 entries from $\{1', 1\}$ is such that $(x-1, y-1) \notin D_{\lambda/\mu}$. The set of all such boxes is P_1 and we have $|P_1| = \nu_1$. Hence, $T^{(1)} = P_1$. Then, by Lemma 2.18, each box (x,y) of the ν_2 entries from $\{2', 2\}$ satisfy the property $(x-1, y-1) \notin D_{\lambda/\mu} \setminus P_1$. The set of all such boxes is P_2 and we have $|P_2| = \nu_2$. Hence, $T^{(2)} = P_2$. Repeating this argument for all entries greater than 2, we see that $T^{(j)} = P_j$ for all $1 \leqslant j \leqslant k-1$. Hence, after removing $T^{(1)}, T^{(2)}, \ldots, T^{(k-1)}$ we obtain some tableau of $U_k(\lambda/\mu)$ of shape $D_{\alpha/\beta}$. If for each box of this tableau we replace its entry i (respectively, i') by $i-k+1$ (respectively, $(i-k+1)'$) then we obtain a tableau T' of $D_{\alpha/\beta}$ of content γ. The amenability of the tableau T' follows from the amenability of the tableau T. After removing the ribbon strips P_j for all $1 \leqslant j \leqslant k-1$, the tableau T' is independent of the different possible fillings of these P_j. By Lemma 2.19, we have $2^{comp(P_j)-1}$ possible fillings of P_j. Thus, for each of the $\prod_{j=1}^{k-1} 2^{comp(P_j)-1}$ tableaux of content $(\nu_1, \ldots \nu_{k-1}, \gamma_1, \ldots, \gamma_{\ell(\gamma)})$ with the same entries in $U_k(\lambda/\mu)$ we obtain T'. Hence, we have $\prod_{j=1}^{k-1} 2^{comp(P_j)-1} f^{\alpha}_{\beta\gamma} \geqslant f^{\lambda}_{\mu(\nu_1,\ldots\nu_{k-1},\gamma_1,\ldots,\gamma_{\ell(\gamma)})}$ and the statement follows. □

We will use the relation $f^{\alpha}_{\beta\gamma} \leqslant f^{\lambda}_{\mu(\nu_1,\dots\nu_{k-1},\gamma_1,\dots,\gamma_{\ell(\gamma)})}$ in Section 4 to show that $f^{\lambda}_{\mu\delta} \geqslant 2$ for some δ by showing that $f^{\alpha}_{\beta\gamma} \geqslant 2$ for some γ. Then by setting $\delta = (\nu_1, \dots \nu_{k-1}, \gamma_1, \dots, \gamma_{\ell(\gamma)})$ we obtain the desired assertion.

The following example illustrates how to obtain tableaux of shape $D_{\lambda/\mu}$ from the tableaux of shape $D_{\alpha/\beta}$ as explained in the proof of Lemma 3.1.

Example 3.2. Let $\lambda = (10, 8, 7, 6, 4, 1)$ and $\mu = (5, 3, 2, 1)$ and consider

$$T_{\lambda/\mu} = \begin{array}{ccccccc} & 1' & 1 & 1 & 1 & 1 & 1 \\ 1' & 1 & 2' & 2 & 2 & 2 & \\ 1' & 2' & 2 & 3' & 3 & 3 & \\ 1' & 2' & 3' & 3 & 4' & 4 & \\ 1 & 2' & 3 & 4 & 4 & & \\ & 2 & & & & & \end{array} .$$

Let $k = 3$. Then

$$U_3(\lambda/\mu) = \begin{array}{cccc} & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \end{array} .$$

Two amenable tableaux of the same content are

$$\begin{array}{cccc} & 1' & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & \end{array} , \quad \begin{array}{cccc} & 1 & 1 & 1 \\ 1' & 2 & 2 & 2 \\ 1 & 3 & 3 & \end{array} .$$

We obtain two amenable tableaux of $D_{\lambda/\mu}$ of the same content:

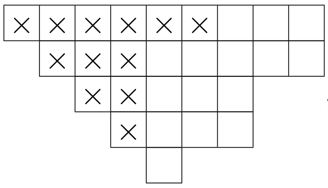
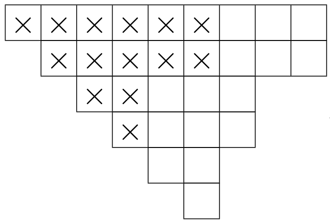
$$\begin{array}{ccccccc} & 1' & 1 & 1 & 1 & 1 & 1 \\ 1' & 1 & 2' & 2 & 2 & 2 & \\ 1' & 2' & 2 & 3' & 3 & 3 & \\ 1' & 2' & 3 & 3 & 4 & 4 & \\ 1 & 2' & 4 & 5 & 5 & & \\ & 2 & & & & & \end{array} , \quad \begin{array}{ccccccc} & 1' & 1 & 1 & 1 & 1 & 1 \\ 1' & 1 & 2' & 2 & 2 & 2 & \\ 1' & 2' & 2 & 3 & 3 & 3 & \\ 1' & 2' & 3' & 4 & 4 & 4 & \\ 1 & 2' & 3 & 5 & 5 & & \\ & 2 & & & & & \end{array} .$$

Definition 3.3. Let $\lambda, \mu \in DP$, let $2 \leqslant a \leqslant \ell(\mu) + 2$, and let $b \geqslant \ell(\lambda)$. Let $\Gamma^{\rightarrow}_{a}(D_{\lambda/\mu})$ be the diagram obtained from $D_{\lambda/\mu}$ by shifting all boxes above the a^{th} row one box to the right. Let $\Gamma^{\downarrow}_{b}(D_{\lambda/\mu})$ be the diagram obtained from $D_{\lambda/\mu}$ by shifting all boxes (x, y) such that $y < b$ one box down.

Example 3.4. For $\lambda = (8, 7, 4, 3, 1)$ and $\mu = (5, 2, 1)$ we have

$$D_{\lambda/\mu} = \begin{array}{cccccccc} \times & \times & \times & \times & \times & \square & \square & \square \\ & \times & \times & \square & \square & \square & \square & \square \\ & & \times & \square & \square & \square & & \\ & & & \square & \square & \square & & \\ & & & & \square & & & \end{array}$$

and

$\Gamma_5^{\rightarrow}(D_{\lambda/\mu}) =$  $, \Gamma_6^{\downarrow}(D_{\lambda/\mu}) =$  .

Lemma 3.5. *Let $\lambda, \mu \in DP$ and let $2 \leqslant a \leqslant \ell(\mu) + 2$ and $b \geqslant \ell(\lambda)$. Let $\Gamma_a^{\rightarrow}(D_{\lambda/\mu})$ have shape $D_{\alpha/\beta}$ and let $\Gamma_b^{\downarrow}(D_{\lambda/\mu})$ have shape $D_{\tilde{\alpha}/\tilde{\beta}}$.*

Then $f_{\mu\nu}^{\lambda} \leqslant f_{\beta\nu}^{\alpha}$ and $f_{\mu\nu}^{\lambda} \leqslant f_{\tilde{\beta}\nu}^{\tilde{\alpha}}$.

Proof. For every given amenable tableau T of shape $D_{\lambda/\mu}$ one obtains an amenable tableau $\hat{T}$ of shape $D_{\alpha/\beta}$ by setting $\hat{T}(x, y) = T(x, y-1)$ for all $1 \leqslant x \leqslant a-1$ and $\hat{T}(x, y) = T(x, y)$ for all $x \geqslant a$ such that $(x, y) \in D_{\alpha/\beta}$. The tableau $\hat{T}$ is obtained from T by shifting the boxes (x, y) for $x < a$ with their entries one box to the right. By Lemma 2.18, the tableau $\hat{T}$ is still a tableau. Since $w(\hat{T}) = w(T)$, the tableau $\hat{T}$ is amenable. Let T, T' be two amenable tableaux of shape $D_{\lambda/\mu}$ and $\hat{T}, \hat{T}'$ the tableaux obtained from T, T' as above. If $\hat{T} = \hat{T}'$ then $T(x, y) = \hat{T}(x, y+1) = \hat{T}'(x, y+1) = T'(x, y)$ for all $1 \leqslant x \leqslant a-1$ and $T(x, y) = \hat{T}(x, y) = \hat{T}'(x, y) = T'(x, y)$ for all $x \geqslant a$ and, hence, $T = T'$. Thus, the statement $f_{\mu\nu}^{\lambda} \leqslant f_{\beta\nu}^{\alpha}$ follows.

For every given amenable tableau T of shape $D_{\lambda/\mu}$ one obtains an amenable tableau $\tilde{T}$ of shape $D_{\tilde{\alpha}/\tilde{\beta}}$ by setting $\tilde{T}(x, y) = T(x-1, y)$ for all $1 \leqslant y \leqslant b-1$ and $\tilde{T}(x, y) = T(x, y)$ for all $y \geqslant b$ such that $(x, y) \in D_{\tilde{\alpha}/\tilde{\beta}}$. The tableau $\tilde{T}$ is obtained from T by shifting the boxes (x, y) for $x < b$ with their entries one box down. By Lemma 2.18, the tableau $\tilde{T}$ is still a tableau. By Lemma 2.24, the tableau $\tilde{T}$ is amenable. Let T, T' be two amenable tableaux of shape $D_{\lambda/\mu}$ and $\tilde{T}, \tilde{T}'$ the tableaux obtained from T, T' as above. If $\tilde{T} = \tilde{T}'$ then $T(x, y) = \tilde{T}(x+1, y) = \tilde{T}'(x+1, y) = T'(x, y)$ for all $1 \leqslant y \leqslant b-1$ and $T(x, y) = \tilde{T}(x, y) = \tilde{T}'(x, y) = T'(x, y)$ for all $y \geqslant b$ and, hence, $T = T'$. Thus, the statement $f_{\mu\nu}^{\lambda} \leqslant f_{\tilde{\beta}\nu}^{\tilde{\alpha}}$ follows. □

Remark 3.6. The statement $f_{\mu\nu}^{\lambda} \leqslant f_{\beta\nu}^{\alpha}$ in the sense of Lemma 3.5 appeared in the proof of [2, Theorem 2.2] and is, hence, due to Bessenrodt. In the same proof the statement $f_{\mu\nu}^{\lambda} \leqslant f_{\tilde{\beta}\nu}^{\tilde{\alpha}}$ for $b = \mu_1 + 2$ can be found (without explicitly stating that $\mu_1 + 2 \geqslant \ell(\lambda)$ is required).

Lemma 3.7. *Let w be an amenable word. Let $\tilde{w}$ be a word such that after removing one letter 1 the word obtained is w (this means that $\tilde{w}$ can be obtained from w by adding a letter 1). Then $\tilde{w}$ is amenable.*

Proof. The number of letters equal to 1 in $\tilde{w}$ is greater than the number of letters equal to 1 in w. Then the word $\tilde{w}$ is not amenable only if there is some $j \geqslant n = \ell(\tilde{w})$ such that $m_1(j) = m_2(j)$ and w_{j-n+1} is this added 1. This is a contradiction to Definition 2.7 b) and, hence, a contradiction to the amenability of the word w. □

Definition 3.8. Let $\alpha \in DP$ and $a \in \mathbb{N}$ such that $a \leqslant \ell(\alpha)+1$. Then

$$\alpha + (1^a) := (\alpha_1 + 1, \alpha_2 + 1, \ldots, \alpha_a + 1, \alpha_{a+1}, \alpha_{a+2}, \ldots, \alpha_{\ell(\alpha)}).$$

Lemma 3.9. *Let $\lambda, \mu \in DP$ and let $1 \leqslant a \leqslant \ell(\mu)+1$. Then $f^{\lambda}_{\mu\nu} \leqslant f^{\lambda+(1^a)}_{\mu+(1^{a-1}),\nu+(1)}$.*

Proof. For this proof we will assume that for a tableau of shape $D_{\lambda/\mu}$ the boxes of D_μ are not removed but instead are filled with 0. Given an amenable tableau T of shape $D_{\lambda/\mu}$ we obtain an amenable tableau $\bar{T}$ of shape $D_{(\lambda+(1^a))/(\mu+(1^{a-1}))}$ as follows. Insert a box with entry 0 into each of the first $a-1$ rows such that the rows are weakly increasing from left to right and insert a box with entry 1 into the a^{th} row such that this row is weakly increasing from left to right.

The word $w(\bar{T})$ differs from $w(T)$ only by one added 1. By Lemma 3.7, the word $w(\bar{T})$ is amenable. Clearly, if $T \neq T'$ for some tableaux $T, T' \in T(\lambda/\mu)$ then $\bar{T} \neq \bar{T}'$. □

Remark 3.10. Note that $\Gamma^{\rightarrow}_{a+1}(D_{\lambda/\mu}) \cup \{(a, a+\mu_a)\}$ has shape $D_{(\lambda+(1^a))/(\mu+(1^{a-1}))}$.

The proof of Lemma 3.9 is inspired by the proof of [5, Theorem 3.1] where Gutschwager proved a similar statement for classical Littlewood-Richardson coefficients.

Lemma 3.11. *Let $\lambda, \mu \in DP$ and let $b \geqslant \ell(\lambda)$. Let $(a, b-1)$ be the uppermost box of $D_{\lambda/\mu}$ in the $(b-1)^{th}$ column. Let $\Gamma^{\downarrow}_b(D_{\lambda/\mu}) \cup \{(a, b-1)\}$ have shape $D_{\alpha/\beta}$.*

Then $f^{\lambda}_{\mu\nu} \leqslant f^{\alpha}_{\beta,\nu+(1)}$.

Proof. Again, we assume that for a tableau of shape $D_{\lambda/\mu}$ the boxes of D_μ are not removed but instead are filled with 0. Given an amenable tableau T of shape $D_{\lambda/\mu}$ we obtain an amenable tableau $\bar{T}$ of shape $D_{\alpha/\beta}$ as follows. Insert a box with entry 0 into each of the first $b-2$ columns such that the columns are weakly increasing from top to bottom. If there is no $1'$ or 1 in the $(b-1)^{\text{th}}$ column then insert a box with entry 1 into the $(b-1)^{\text{th}}$ column such that this column is weakly increasing from top to bottom. If there is a $1'$ or a 1 in the $(b-1)^{\text{th}}$ column then insert a box with entry $1'$ into the $(b-1)^{\text{th}}$ column such that this column is weakly increasing from top to bottom.

Let $\hat{T}$ be the tableau defined by $\hat{T}(x,y) := T(x-1,y)$ for all $1 \leqslant y \leqslant b-1$ and $\hat{T}(x,y) = T(x,y)$ for all $y \geqslant b$ such that $(x,y) \in \Gamma^{\downarrow}_b(D_{\lambda/\mu})$. By Lemma 3.5, the tableau $\hat{T}$ is amenable. The word $w = w(\bar{T})$ differs from $w(\hat{T})$ only by an added $1'$ or an added 1. If a $1'$ is added then clearly, the tableau $\bar{T}$ is amenable. If a 1 is added then, by Lemma 3.7, the word $w(\bar{T})$ is amenable. Clearly, if $T \neq T'$ for some tableaux $T, T' \in T(\lambda/\mu)$ then $\bar{T} \neq \bar{T}'$. □

4 Q-multiplicity-free skew Schur Q-functions

With the tools provided in Section 3 we can finally start to prove results towards our desired classification.

Definition 4.1. A symmetric function $f \in span(Q_\lambda \mid \lambda \in DP)$ is called **Q-multiplicity-free** if the coefficients in the decomposition of f into Schur Q-functions are from $\{0,1\}$. In particular, a skew Schur Q-function $Q_{\lambda/\mu}$ is called Q-multiplicity-free if $f^{\lambda}_{\mu\nu} \leqslant 1$ for all $\nu \in DP$.

Lemma 3.1 will be crucial in this chapter. It allows us to consider subdiagrams consisting of the boxes of $T_{\lambda/\mu}$ with entries bigger than some given k and simplifies the proof of non-Q-multiplicity-freeness. This follows from the fact that if the skew Schur Q-function to a diagram is not Q-multiplicity-free then the same must hold for each diagram that contains this diagram as a subdiagram of $T_{\lambda/\mu}$ of boxes with entries greater than k for some k.

We will analyze diagrams $D_{\lambda/\mu}$ and show that the corresponding $Q_{\lambda/\mu}$ are not Q-multiplicity-free by finding two different amenable tableaux of the same content, derived by changing entries in the tableau $T_{\lambda/\mu}$ obtained in Definition 2.27. Usually, we will change only few entries in $T_{\lambda/\mu}$ to obtain new tableaux. We will almost always use Lemma 2.24 or Corollary 2.25 to prove the amenability of these obtained tableaux. We will discuss the technique of applying Lemma 2.24 and Corollary 2.25 to tableaux that differ from $T_{\lambda/\mu}$ only in few entries in the following.

If we change the entry in a box from k (marked or unmarked) to i (marked or unmarked) then this does only effect the j-amenability for $j \in \{k, k+1, i, i+1\}$. Hence, we only need to consider and prove j-amenability for $j \in \{k, k+1, i, i+1\}$. As an example, the change (marked in boldface) in

$$T_{\lambda/\mu} = \begin{array}{ccccccc} & 1' & 1 & 1 & 1 & 1 & 1 \\ & 1' & 2' & 2 & 2 & 2 & 2 \\ & 1' & 2' & 3' & 3 & 3 & 3 \\ & 1' & 2 & 3 & 4 & 4 & 4 \\ 1 & \mathbf{1} & & & & & \end{array} \to T_1 = \begin{array}{ccccccc} & 1' & 1 & 1 & 1 & 1 & 1 \\ & 1' & 2' & 2 & 2 & 2 & 2 \\ & 1' & 2' & 3' & 3 & 3 & 3 \\ & 1' & 2 & 3 & 4 & 4 & 4 \\ 1 & \mathbf{2} & & & & & \end{array}$$

has no effect on the 4-amenability. Often, we change some k (marked or unmarked) to $k+1$ (marked or unmarked) and actually just have to prove j-amenability for $j \in \{k, k+1, k+2\}$. If the new entry i is unmarked then, by Lemma 3.7, the $(i+1)$-amenability follows directly from the $(i+1)$-amenability of tableau $T_{\lambda/\mu}$ and will not be mentioned.

If we say that the k-amenability follows from Corollary 2.25 then the conditions (2) - (5) leave no room for interpretation. For every box (x, y) of T that contains a k or k' there is a $k-1$ in the y^{th} column or there is a $(k-1)'$ in the box $(x-1, y-1)$, respectively. Also the last box of $T^{(k)}$ and $T^{(k-1)}$ is unmarked. But condition (1) is a little bit different. In $T_{\lambda/\mu}$ the last box of P_{k-1} and the last box of P_k satisfy this condition. Often, the same boxes satisfy this condition in the new tableau. In that case we will not mention condition (1) and just say that k-amenability follows from Corollary 2.25. If the last box of P_{k-1} or the last box of P_k do not satisfy this condition anymore then we will give another box that satisfies this condition. For example, in the previous change of entries the last box of P_1 still satisfies condition (1) and we can state that 2-amenability follows from Corollary 2.25. As mentioned above the added 2 does not change the 3-amenability of T_1 and the 4-amenability is not effected at all. So, in this case we can say, that the amenability follows from Corollary 2.25. On the other hand, for

$$T_{\lambda/\mu} = \begin{array}{|c|c|c|c|c|c|c|} \hline 1' & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & 2 & 2 & 2 & \\ \hline 1' & 2' & 3' & 3 & 3 & 3 & \\ \hline 1' & 2 & 3 & 4 & 4 & 4 & \\ \hline \mathbf{1} & & & & & & \\ \hline \end{array} \to T_2 = \begin{array}{|c|c|c|c|c|c|c|} \hline 1' & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & 2 & 2 & 2 & \\ \hline 1' & 2' & 3' & 3 & 3 & 3 & \\ \hline 1 & 2 & 3 & 4 & 4 & 4 & \\ \hline \mathbf{2} & & & & & & \\ \hline \end{array}$$

the last box of P_1 is not the reason why condition (1) is satisfied anymore. In that case we say that the first box of P_1 has 1 as entry and there is no box with 2 below in the column of that box such condition (1) is still satisfied.

If we use Lemma 2.24 to prove k-amenability then some of the conditions of Corollary 2.25 are not satisfied anymore. Since condition (4) and condition (5) of Corollary 2.25 are condition (5) and condition (6) of Lemma 2.24, respectively, these conditions must still be satisfied. They can easily be proved and will not be mentioned. Condition (1) of Corollary 2.25 follows from condition (1) of Lemma 2.24 and, thus, must be satisfied. We stick to the same way of mentioning or omitting it as for Corollary 2.25.

If condition (2) of Corollary 2.25 is not satisfied anymore then this means that there is some box (x, y) with entry k and in the y^{th} column there is no box filled with a $k-1$. In that case we will give a box (u, v) weakly above and to the right that is filled with a $k-1$ such that there is no box filled with k in the v^{th} column. If several boxes are the reason why condition (2) is not satisfied we will give the appropriate number of such boxes (u, v) such that the boxes causing the problem can injectively be mapped to them. Then again we still need another box (u, v) that is filled with a $k-1$ and there is no box filled with k in the v^{th} column. Hence, we will never map to the last box of $T^{(k-1)}$. For example in

$$T_{\lambda/\mu} = \begin{array}{|c|c|c|c|c|} \hline 1' & 1 & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & 2 & 2 \\ \hline 1' & 2' & 3 & 3 & 3 \\ \hline 1 & 2 & & & \\ \hline \end{array} \to T_3 = \begin{array}{|c|c|c|c|c|} \hline 1' & 1 & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & 2 & 2 \\ \hline 1 & 2 & 3 & 3 & 3 \\ \hline 4 & 4 & & & \\ \hline \end{array}$$

the boxes filled with 4 are the reason why condition (2) of Corollary 2.25 is not satisfied. But these boxes can injectively be mapped to the first two boxes of P_3. Since the last box of P_3 is none of these boxes, this is enough to show the 4-amenability.

If condition (3) of Corollary 2.25 is not satisfied anymore then this means that there is some box (x, y) with entry k' and the box $(x-1, y-1)$ is either not part of the diagram or this box is not filled with $(k-1)'$. In that case we will give either a box (u, v) weakly to the right that is filled with a $(k-1)'$ such that $(u+1, v+1)$ is not filled with k' or a box (u, v) filled with $k-1$ such there is no box filled with k in the v^{th} column. Note that in the second case the mentioned box must be different from any box that are mapped to for the case that condition (2) of Corollary 2.25 is also not satisfied. Also, an additional box filled with $k-1$ with no box filled with k in that column will be given if the last box of P_{k-1} is not the reason why condition (1) is satisfied anymore. If several boxes are the reason why condition (3) is not satisfied we will give the appropriate number of such boxes (u, v) such that the boxes causing the problem can injectively be mapped to them.

For example in

$$T_{\lambda/\mu} = \begin{array}{ccccc} & & 1' & 1 & 1 \\ & & 1' & 2 & \\ & & 1' & & \\ 1' & 1 & 1 & & \\ 1 & 2 & & & \end{array} \quad \to T_4 = \begin{array}{ccccc} & & 1' & 1 & \mathbf{1} \\ & & 1' & 2 & \\ & & \mathbf{2'} & & \\ \mathbf{1'} & 1 & \mathbf{2'} & & \\ 1 & 2 & & & \end{array}$$

both boxes filled with the boldfaced entries $2'$ are the reason why condition (2) is not satisfied. But these boxes can injectively be mapped to the box filled with the boldface $1'$ and to the first box of P_1.

Using these methods of changing few entries to obtain two different amenable tableaux with the same content, we are able to find many types of diagrams $D_{\lambda/\mu}$ such that the corresponding $Q_{\lambda/\mu}$ are not Q-multiplicity-free; we will prove afterwards that all remaining diagrams do give Q-multiplicity-free skew Schur Q-functions, hence then our classification is complete.

Let $\lambda, \mu \in DP$ and $\nu = c(T_{\lambda/\mu})$. Lemma 2.19 and the definition of amenability, which requires P_i to be fitting, state that $f^{\lambda}_{\mu\nu} = 1$ is only possible if each P_i is connected.

Hypothesis 4.2. *From now on we will consider only diagrams that satisfy the property that each P_i is connected.*

The following lemmas give restrictions for the border strips P_i; they will enable us to prove Lemma 4.16 and Corollary 4.17 which lower the number of families of partitions λ we have to consider to find Q-multiplicity-free skew Schur Q-functions.

Lemma 4.3. *Let $\lambda, \mu \in DP$. Let $\nu = c(T_{\lambda/\mu})$ and $n = \ell(\nu)$. If P_n is neither a hook nor a rotated hook then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. By Lemma 3.1, it is enough to find two amenable tableaux of P_n of the same content. Assume that the diagram P_n is neither a hook nor a rotated hook. Then we can find a subset of boxes of P_n such that all but one of the boxes form a (p, q)-hook where $p, q \geqslant 2$ and there is either a single box above the rightmost box of the hook, or a single box to the left of the lowermost box of the hook. By Lemmas 2.33, 3.5, 3.9 and 3.11, it is enough to assume that P_n has shape $D_{(4,2)/(2)}$. Since $Q_{(4,2)/(2)} = Q_{(4)} + 2Q_{(3,1)}$, the statement follows. □

Now that we know that P_n must be a (possibly rotated) hook we want to take a look at the case that this hook is "big enough" in some way and how this effects P_{n-1}.

Example 4.4. For $\lambda = (7, 6, 3, 2)$ and $\mu = (3, 2, 1)$ we have $P_n = P_2$ and it is a $(3, 3)$-hook:

$$T_{\lambda/\mu} = \begin{array}{cccc} 1' & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 \\ 1' & 2' & & \\ 1 & 2 & & \end{array} .$$

Since $Q_{(7,6,3,2)/(3,2,1)} = Q_{(7,5)} + Q_{(7,4,1)} + Q_{(7,3,2)} + Q_{(6,5,1)} + 2Q_{(6,4,2)} + Q_{(6,3,2,1)} + Q_{(5,4,3)} + Q_{(5,4,2,1)}$, the two tableaux

$$T = \begin{array}{cccc} 1' & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 \\ 1 & 2 & & \\ 3 & 3 & & \end{array}, \quad T' = \begin{array}{cccc} 1' & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 \\ 1 & 3' & & \\ 2 & 3 & & \end{array}$$

with content $(6,4,2)$ are the reason why $Q_{(7,6,3,2)/(3,2,1)}$ is not multiplicity-free. Since the last box of P_1 is not in the row above the last box of P_2 we can obtain these two tableaux from $T_{\lambda/\mu}$ by changing the entries of the last boxes of P_1 and P_2 (and adjust some markings) for T and additionally switch the entry of the second-to-last box of P_2 (and adjust some markings) for T'. We will prove that this holds in general in the following lemma.

Lemma 4.5. *Let $\lambda, \mu \in DP$. Let $\nu = c(T_{\lambda/\mu})$ and $n = \ell(\nu) > 1$. Let P_n be a (p,q)-hook or a rotated (p,q)-hook where $p, q \geqslant 3$. Suppose the last box of P_{n-1} is not in the row directly above the row of the last box of P_n. Then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. We may assume that P_n is a (p,q)-hook where $p, q \geqslant 3$. Otherwise, P_n is a rotated (p,q)-hook where $p, q \geqslant 3$ and we may consider $D^{ot}_{\lambda/\mu}$ since if $D^{ot}_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ then, by Lemma 2.37, the set of boxes $T^{(n)}_{\alpha/\beta}$ is a (q,p)-hook where $p, q \geqslant 3$.

By Lemma 3.1, we may assume that $n = 2$. Let (x,y) be the last box of P_2. By Lemmas 3.5, 3.9 and 3.11, we may assume that $(x, y-1)$ is the last box of P_1. We get a new tableau T if we set $T(x,y-1) = 3$, $T(x-1,y-1) = 1$, $T(x,y) = 3$, $T(x-1,y) = 2$ and $T(r,s) = T_{\lambda/\mu}(r,s)$ for every other box $(r,s) \in D_{\lambda/\mu}$.

By Corollary 2.25, this tableau is m-amenable for $m \neq 3$. We have $T(x,y-1) = 3$ but there is no 2 in the $(y-1)^{\text{th}}$ column. However, there are at least two 2s with no 3 below them in the first two boxes of P_2. Hence, by Lemma 2.24, this tableau is amenable.

We get another tableau T' if we set $T'(x,y) = 3$, $T'(x-1,y) = 3'$, $T'(x,y-1) = 2$, $T'(x-1,y-1) = 1$ and $T'(r,s) = T_{\lambda/\mu}(r,s)$ for every other box $(r,s) \in D_{\lambda/\mu}$.

By Corollary 2.25, this tableau is m-amenable for $m \neq 2, 3$. Since there is a 1 but no 2 in the y^{th} column, 2-amenability follows. We have $T'(x,y) = 3$ but there is no 2 in the y^{th} column. Also, we have $T'(x-1,y) = 3'$ and $T'(x-2,y-1) \neq 2'$. However, in the first two boxes of P_n are 2s with no 3 below. Additionally, there is another 2 with no 3 below in the $(y-1)^{\text{th}}$ column. Thus, by Lemma 2.24, 3-amenability follows. ☐

Now we want to take a look at tableaux where the last box of P_k for some k is lower than the last box of P_n and the first box of P_i for some i is to the right of the first box of P_n.

Example 4.6. For $\lambda = (7,5,4,1)$ and $\mu = (2,1)$ we have $P_n = P_3$, $P_k = P_2$ and $P_i = P_1$:

$$T_{\lambda/\mu} = \begin{array}{ccccc} 1' & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & \\ 1 & 2' & 3 & 3 & \\ & 2 & & & \end{array}.$$

Since $Q_{(7,5,4,1)/(2,1)} = Q_{(7,5,2)} + Q_{(7,4,3)} + Q_{(7,4,2,1)} + 2Q_{(6,5,3)} + Q_{(6,5,2,1)} + Q_{(6,4,3,1)}$, the two tableaux

$$T = \begin{array}{|c|c|c|c|c|}\hline 1' & 1 & 1 & 1 & 1 \\ \hline 1 & 2' & 2 & 2 & \\ \cline{1-4} 2 & 2 & 3 & 3 & \\ \cline{1-4} & 3 & & & \\ \cline{2-2} \end{array}, \quad T' = \begin{array}{|c|c|c|c|c|}\hline 1' & 1 & 1 & 1 & 1 \\ \hline 1 & 2' & 2 & 2 & \\ \cline{1-4} 2 & 2 & 3' & 3 & \\ \cline{1-4} & 3 & & & \\ \cline{2-2} \end{array}$$

with content $(6,5,3)$ are the reason why $Q_{(7,5,4,1)/(2,1)}$ is not multiplicity-free. We can obtain T from $T_{\lambda/\mu}$ by removing the last box of P_1 from P_1, naming this set of boxes P_1', adjust the marking of the second-to-last box of P_1 and filling $D_{\lambda/\mu} \setminus (P_1')$ as in the algorithm of Definition 2.27 starting by $k = 2$ (k like in Definition 2.27) does. The tableau T' is then obtained from T by changing the marking of the last box of P_3 since $T^{(3)}$ is disconnected. We will prove that this holds in general in the following lemma.

Lemma 4.7. *Let $\lambda, \mu \in DP$. Let $\nu = c(T_{\lambda/\mu})$ and $n = \ell(\nu) \geqslant 2$. Let there be some $k < n$ such that the last box of P_k is in a row strictly lower than the last box of P_n and some $i < n$ such that the first box of P_i is in a column strictly to the right of the first box of P_n. Then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. Let k, i be maximal with respect to these conditions and let $j = \min\{k, i\}$. If we can find two amenable tableaux of $U_j(\lambda/\mu)$ then, by Lemma 3.1, we can find two amenable tableaux of $D_{\lambda/\mu}$ Hence, we only need to consider the case $j = 1$. First, we assume that $i \leqslant k$. Then let $\bar{k}$ be minimal such that the last box of $P_{\bar{k}}$ is in a row strictly lower than the last box of P_n. Consider the columns where $P_{\bar{k}}$ has at least a box. Then consider the subset of such columns where the lowermost box of $P_{\bar{k}}$ in that column is in a row strictly lower than the last box of $P_{\bar{k}+1}$. Let the rightmost such column be the v^{th} column. Then let (u, v) be the lowermost box of $P_{\bar{k}}$ in that column.

In the following example the box (u, v) is the box $(5, 7)$ and has its entry written in boldface:

$$T_1 = \begin{array}{ccccccc} \cline{5-7} & & & & \multicolumn{1}{|c|}{1'} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \cline{4-7} & & & \multicolumn{1}{|c|}{1'} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2'} & \multicolumn{1}{c|}{2} \\ \cline{3-7} & & \multicolumn{1}{|c|}{1'} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2'} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3'} \\ \cline{2-7} & \multicolumn{1}{|c|}{1'} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} \\ \cline{1-7} \multicolumn{1}{|c|}{1'} & \multicolumn{1}{c|}{\mathbf{1}} & & & & & \\ \cline{1-2} \multicolumn{1}{|c|}{1} & & & & & & \\ \cline{1-1} \end{array}.$$

Let $x = u - \bar{k} + i$ and $y = v - \bar{k} + i$. Then (x, y) is the lowermost box of P_i in the y^{th} column. We get a new tableau T if after the $(i-1)^{\text{th}}$ step of the algorithm of Definition 2.27 we use $P_i' := P_i \setminus \{(x, y)\}$ instead of P_i.

Let $P_z' = T^{(z)}$. Then for $i + 1 \leqslant r \leqslant \bar{k}$ if $(x + r - i, y + r - i) \in P_r$ then we have $(x + r - i - 1, y + r - i - 1) \in P_r'$. Hence, $(x, y) \in P_{i+1}'$. Clearly, by Corollary 2.25, this tableau is m-amenable for $m \neq i + 1$. We possibly have $T(x, y) = (i + 1)'$ and $T(x - 1, y - 1) \neq i'$. But there is an i with no $i + 1$ below in the column of the first box of P_i. Thus, by Lemma 2.24, $(i + 1)$-amenability follows.

Let (c, d) be the last box of $P_{\bar{k}+1}$. We get another tableau T' with the same content if we set $T'(c, d) = (\bar{k} + 1)'$ and $T'(e, f) = T_{\lambda/\mu}(e, f)$ for every other box $(e, f) \in D_{\lambda/\mu}$

By Corollary 2.25, it is clear that T' is amenable if T is and we have $c(T') = c(T) = (\nu_1, \ldots, \nu_{i-1}, \nu_i - 1, \nu_{i+1}, \ldots, \nu_{\bar{k}}, \nu_{\bar{k}+1} + 1, \nu_{\bar{k}+2}, \ldots \nu_n)$.

If $k \leqslant i$ then $U_k(\lambda/\mu)$ is an unshifted diagram and we showed that two amenable tableaux of $U_k(\lambda/\mu)^t$ with the same content exist. By Lemma 2.33, the statement follows. □

We will consider tableaux where there is some k such that there is some corner in P_k that lies above all boxes of P_n and there is some $i \leqslant k$ such that the first box of P_i is "very high" in some way.

Example 4.8. For $\lambda = (7,6,5,4,2,1)$ and $\mu = (6)$ we have $P_n = P_5$, $P_k = P_3$ and $P_i = P_1$:

$$T_{\lambda/\mu} = \begin{array}{cccccc} & & & & & 1' \\ 1 & 1 & 1 & 1 & 1 & 1 \\ & 2 & 2 & 2 & 2 & 2 \\ & & 3 & 3 & 3 & 3 \\ & & & 4 & 4 & \\ & & & & 5 & \end{array}.$$

Since $Q_{(7,6,5,4,2,1)/(6)} = Q_{(7,5,4,2,1)} + 2Q_{(6,5,4,3,1)}$, the two tableaux

$$T = \begin{array}{cccccc} & & & & & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 \\ & 2 & 2 & 2 & 2 & 3 \\ & & 3 & 3 & 3 & 4 \\ & & & 4 & 4 & \\ & & & & 5 & \end{array}, \quad T' = \begin{array}{cccccc} & & & & & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 \\ & 2 & 2 & 2 & 2 & 3 \\ & & 3 & 3 & 3 & 4' \\ & & & 4 & 4 & \\ & & & & 5 & \end{array}$$

with content $(6,5,4,3,1)$ are the reason why $Q_{(7,6,5,4,2,1)/(6)}$ is not multiplicity-free. We can obtain T from $T_{\lambda/\mu}$ by shifting the entries of the rightmost column up and add a $k+1 = 4$ in the first box of $P_k = P_3$. The tableau T' is then obtained from T by changing the marking of the first box of P_3 since $T^{(4)}$ is disconnected. Note that $i = k$ is possible. An example of that case is added after the proof of Lemma 4.9.

Lemma 4.9. *Let $\lambda, \mu \in DP$. Let $\nu = c(T_{\lambda/\mu})$ and $n = \ell(\nu) > 1$. Let there be some $k < n$ such that there is a corner, (x,y) say, in P_k above all the boxes of P_n and let there be some $i \leqslant k$ such that the first box of P_i is above the $(x-k+i)^{th}$ row. Then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. Let k be minimal and i be maximal with respect to these conditions. Then for all $i+1 \leqslant a \leqslant k$ the first box of P_a has no box of P_a below. Let $(x-k+a, y)$ be the first box of P_a for $i+1 \leqslant a \leqslant k-1$ and let $(x-k+i, y)$ be the rightmost box of P_i in the $(x-k+i)^{\text{th}}$ row. We get a new tableau T if we set $T(x-k+i, y) = i+1$, $T(x-k+i-1, y) = i$, for all $i+1 \leqslant a \leqslant k$ set $T(x-k+a, y) = a+1$, $T(x,y) = k+1$ and $T(u,v) = T_{\lambda/\mu}(u,v)$ for every other box $(u,v) \in D_{\lambda/\mu}$. By Corollary 2.25, this tableau is amenable.

We get a new tableau T' if we set $T'(x,y) = (k+1)'$ and $T'(u,v) = T(u,v)$ for every other box $(u,v) \in D_{\lambda/\mu}$. We have $T'(x,y) = (k+1)'$ and $T'(x-1, y-1) \neq k'$.

However, we have $T'(x-1,y)=k$ and there is no $k+1$ in the y^{th} column. Hence, by Lemma 2.24, T' is m-amenable for all m. It is easy to see that we have $c(T)=c(T')=(\nu_1,\dots,\nu_{i-1},\nu_i-1,\nu_{i+1}\dots,\nu_k,\nu_{k+1}+1,\nu_{k+2},\dots,\nu_n)$. ☐

Example 4.10. For

$$T_{\lambda/\mu}=\begin{array}{|c|c|c|c|}\cline{4-4} \multicolumn{3}{c|}{} & 1' \\ \hline 1 & 1 & 1 & 1 \\ \hline \multicolumn{1}{c|}{} & 2 & 2 & \multicolumn{1}{c}{} \\ \cline{2-3}\end{array}$$

we get

$$T=\begin{array}{|c|c|c|c|}\cline{4-4} \multicolumn{3}{c|}{} & 1 \\ \hline 1 & 1 & 1 & 2 \\ \hline \multicolumn{1}{c|}{} & 2 & 2 & \multicolumn{1}{c}{} \\ \cline{2-3}\end{array},\quad T'=\begin{array}{|c|c|c|c|}\cline{4-4} \multicolumn{3}{c|}{} & 1 \\ \hline 1 & 1 & 1 & 2' \\ \hline \multicolumn{1}{c|}{} & 2 & 2 & \multicolumn{1}{c}{} \\ \cline{2-3}\end{array}.$$

We have $Q_{(5,4,2)/(4)}=Q_{(5,2)}+2Q_{(4,3)}+Q_{(4,2,1)}$.

If there is some k such that the first box of P_{k-1} is to the right of the first box of P_k and P_{k-1} is not a hook then the respective Schur Q-function is also not multiplicity-free.

Example 4.11. For $\lambda=(8,6,5)$ and $\mu=(3,2)$ we have $P_k=P_2$:

$$T_{\lambda/\mu}=\begin{array}{|c|c|c|c|c|c|}\cline{2-6} \multicolumn{1}{c|}{} & 1' & 1 & 1 & 1 & 1 \\ \cline{2-6} \multicolumn{1}{c|}{} & 1' & 2' & 2 & 2 & \multicolumn{1}{c}{} \\ \hline 1 & 1 & 2 & 3 & 3 & \multicolumn{1}{c}{} \\ \cline{1-5}\end{array}.$$

We have $Q_{(8,6,5)/(3,2)}=Q_{(8,4,2)}+2Q_{(7,5,2)}+2Q_{(7,4,3)}+2Q_{(6,5,3)}$. We take a look at the two tableaux

$$T=\begin{array}{|c|c|c|c|c|c|}\cline{2-6} \multicolumn{1}{c|}{} & 1' & 1 & 1 & 1 & 1 \\ \cline{2-6} \multicolumn{1}{c|}{} & 1 & 2' & 2 & 2 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 2 & 3 & 3 & \multicolumn{1}{c}{} \\ \cline{1-5}\end{array},\quad T'=\begin{array}{|c|c|c|c|c|c|}\cline{2-6} \multicolumn{1}{c|}{} & 1' & 1 & 1 & 1 & 1 \\ \cline{2-6} \multicolumn{1}{c|}{} & 1' & 2' & 2 & 2 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 2 & 3 & 3 & \multicolumn{1}{c}{} \\ \cline{1-5}\end{array}$$

with content $(7,5,2)$. We can obtain T from $T_{\lambda/\mu}$ by removing the lowest box of P_1 in the column to the left of the last box of P_2, naming this set of boxes P_1', adjust the marking of the box above the removed box and filling $D_{\lambda/\mu}\setminus(P_1')$ as the algorithm of Definition 2.27 starting by $k=2$ (k like in Definition 2.27) does. The tableau T' is then obtained from T by changing the marking of the box above the removed box. We will prove this in the following lemma.

Lemma 4.12. *Let $\lambda,\mu\in DP$. Let $\nu=c(T_{\lambda/\mu})$ and $n=\ell(\nu)>1$. Let there be some $k>1$ such that the first box of P_{k-1} is to the right of the column of first box of P_k, and P_{k-1} is not a hook. Then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. Let k be maximal with respect to this property. By Lemma 3.1, we may assume that $k=2$. If the first box of P_1 is not a corner then there is a box of P_1 below the first box of P_1 which is the second box of P_1. Since the first box of P_1 is to the right of the column of the first box of P_2, so is the second box. The lowermost box of the column of the first box of P_1 is a corner and, by Lemma 4.9 with $i=k=1$, $Q_{\lambda/\mu}$ is not Q-multiplicity-free. Thus, consider that the first box of P_1 is a corner. If the first box of P_1 is not in the row above the first box of P_2 then an orthogonally transposed version of

Lemma 4.9 states that $Q_{\lambda/\mu}$ is not Q-multiplicity-free. Since P_1 is not a hook, there are v, w such that the boxes $(v-1, w), (v, w), (v, w-1) \in P_1$ and the first box of P_1 is not in the w^{th} column. Let v be maximal with respect to this property.

We get a new tableau T if we use $P_1' := P_1 \setminus \{(v, w)\}$ instead of P_1 in the algorithm of Definition 2.27. By Corollary 2.25, it is clear that T is i-amenable for $i \neq 2$. We possibly have $T(v, w) = 2'$ and $T(v-1, w-1) \neq 1'$. However, in the column containing the first box of P_1 there is a 1 and no 2. Thus, by Lemma 2.24, this tableau is amenable.

We get another tableau T' if we set $T'(v-1, w) = 1'$ and $T'(r, s) = T(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. By Corollary 2.25, T' is i-amenable for $i \neq 2$. There is a 2 but no 1 in the w^{th} column. However, in the column containing the first box of P_1 there is a 1 and no 2. We possibly have $T(v, w) = 2'$ and $T(v-1, w-1) \neq 1'$. However, we have $T(v-1, w) = 1'$. Thus, by Lemma 2.24, T' is amenable.

It is easy to see that $c(T) = c(T')$. □

If P_n is a "real" hook (i.e. a hook that is not only a single line or a single box) and there is some k such that P_k has more than one box in a column i weakly to the right of the first box of P_n then the respective Schur Q-function is also not multiplicity-free.

Example 4.13. For $\lambda = (6, 5, 4, 3, 1)$ and $\mu = (5, 1)$ we have $k = 1$ and the i^{th} column is the column of the first box of $P_n = P_3$:

$$T_{\lambda/\mu} = \begin{array}{cccc} & & & 1' \\ 1' & 1 & 1 & 1 \\ 1 & 2' & 2 & 2 \\ & 2 & 3' & 3 \\ & & 3 & \end{array}.$$

Since $Q_{(6,5,4,3,1)/(5,1)} = Q_{(6,4,3)} + Q_{(6,4,2,1)} + 2Q_{(5,4,3,1)}$, the two tableaux

$$T = \begin{array}{cccc} & & & 1 \\ 1' & 1 & 1 & 2 \\ 1 & 2' & 2 & 3 \\ & 2 & 3' & 4 \\ & & 3 & \end{array}, \quad T' = \begin{array}{cccc} & & & 1 \\ 1' & 1 & 1 & 2 \\ 1 & 2' & 2 & 3' \\ & 2 & 3 & 3 \\ & & 4 & \end{array}$$

with content $(5, 4, 3, 1)$ are the reason why $Q_{(7,6,5,4,2,1)/(6)}$ is not multiplicity-free. We can obtain T from $T_{\lambda/\mu}$ by incrementing the entries of the first boxes of all P_i except for P_1 where the lowest box of P_1 in the i^{th} column is filled with 2 and the box above with 1. The tableau T' is then obtained from T be switching the entries of the first and last box of P_n and adjusting some markings.

Lemma 4.14. *Let $\lambda, \mu \in DP$. Let $\nu = c(T_{\lambda/\mu})$ and $n = \ell(\nu) > 1$. Let P_n be a (p, q)-hook where $p, q \geqslant 2$ and let (x, y) be the first box of P_n. Let there be some $k < n$ and some $i \geqslant y$ such that there are at least two boxes of P_k in the i^{th} column. Then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. Let k be maximal with respect to this property. Let (u, v) be the lowermost box of P_k in the i^{th} column and let (a_r, b_r) be the first box of P_r for all r. We get a new tableau T if we set $T(u, v) = k+1$, $T(u-1, v) = k$, for all $k+1 \leqslant r \leqslant n$ set $T(a_r, b_r) = r+1$ and $T(c, d) = T_{\lambda/\mu}(c, d)$ for every other box $(c, d) \in D_{\lambda/\mu}$. By Corollary 2.25, T is amenable.

Let (e, f) be the last box of P_n and let $(x-1, z)$ be the rightmost box of P_{n-1} in the $(x-1)^{\text{th}}$ row. We get another tableau T' if we set $T'(e, f) = n+1$, $T'(e-1, f) = n$, $T'(a_n, b_n) = n$, $T'(x-1, z) = n'$ and $T'(c, d) = T(c, d)$ for every other box $(c, d) \in D_{\lambda/\mu}$. By Corollary 2.25, T' is m-amenable for $m \neq n$. We have $T'(x-1, z) = n'$ and $T'(x-2, z-1) \neq (n-1)'$. However, if (g, h) is the last box of P_n then we have $T'(g-2, h-1) = (n-1)'$ and $T'(g-1, h) \neq n'$. Thus, by Lemma 2.24, amenability follows. □

Corollary 4.15. *Let $\lambda, \mu \in DP$. Let $\nu = c(T_{\lambda/\mu})$ and $n = \ell(\nu) > 1$. Let P_n be a (p, q)-hook where $p, q \geqslant 2$ and let (x, y) be the first box of P_n. Let there be some $k < n$ and some $i \geqslant x$ such that there are at least two boxes of P_k in the i^{th} row. Then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. The diagram $U_k(\lambda/\mu)$ is unshifted. Then we may transpose $U_k(\lambda/\mu)$ and use Lemma 4.14. □

Now we are able to show an intermediate result that bounds the number of corners of $D_{\lambda/\mu}$ in the case of Q-multiplicity-freeness and, hence, of D_λ for $\mu \neq \emptyset, (1)$. The number of corners of D_μ is then also bounded for most $D_{\lambda/\mu}$ because of orthogonal transposition. This restricts the number of cases we have to analyze.

Lemma 4.16. *Let $\lambda, \mu \in DP$ where $\mu \neq \emptyset, (1)$. If D_λ has more than two corners then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. Assume D_λ has more than two corners, $\mu \neq \emptyset, (1)$, and $Q_{\lambda/\mu}$ is Q-multiplicity-free. We will construct two amenable tableaux of shape $D_{\lambda/\mu}$ and of the same content to arrive at a contradiction. Let $\nu = c(T_{\lambda/\mu})$ and $n = \ell(\nu)$. Let k be maximal such that $U_k(\lambda/\mu)$ has at least three corners. Thus, at least one corner is in P_k. By Lemma 4.3 P_n must be a hook or a rotated hook, so P_n can have at most two corners and, hence, $k < n$. By Lemma 4.7 either the uppermost or the lowermost corner must be in P_n, so we only consider diagrams such that the uppermost or the lowermost corner is in P_n. Without loss of generality we may assume that the lowermost corner of $U_k(\lambda/\mu)$ is in P_n, otherwise $U_k(\lambda/\mu)$ is an unshifted diagram and we may transpose $U_k(\lambda/\mu)$. Thus, the uppermost corner is in P_k. By Lemma 4.9, which forbids to having boxes of P_k to the left and above a corner in P_k at the same time, the uppermost corner is the first box of P_k and it is the only corner of the diagram $U_k(\lambda/\mu)$ that is in P_k.

Case 1: two corners are in P_n.

Then P_n is a (p, q)-hook where $p \geqslant 2$ and $q \geqslant 2$. In this case, Lemma 4.14 forbids having more than one box of P_i in the column of the first box for all $k \leqslant i \leqslant n-1$. Additionally, Corollary 4.15 forbids having more than one box of P_i in the row of the last box of P_n for all $k \leqslant i \leqslant n-1$. Hence, all P_i are hooks.

Case 1.1: the last box of P_{n-1} is in the same row as the last box of P_n.

Let (u_a, v_a) be the last box of P_a for all a. We get a new tableau T_1 if for all $k \leqslant a \leqslant n$ we set $T_1(u_a, v_a) = a+1$, $T_1(u_a - 1, v_a) = a$ and $T_1(r,s) = T_{\lambda/\mu}(r,s)$ for every other box $(r,s) \in D_{\lambda/\mu}$. By Corollary 2.25, T_1 is m-amenable for $m \neq k+1$. Also by Corollary 2.25, the tableau T_1 is also $(k+1)$-amenable because in the column of the first box of P_k there is a k and no $k+1$.

We get another tableau T_1' if we set $T_1'(u_n - 1, v_n) = n'$ and $T_1'(r,s) = T_1(r,s)$ for every other box $(r,s) \in D_{\lambda/\mu}$. By Corollary 2.25, T_1' is m-amenable for $m \neq n+1$. We have $T_1'(u_n, v_n) = n+1$ and $T_1'(u_n - 1, v_n) < n$, however, there is an n with no $n+1$ below in the first box of P_n, and we have $T_1'(u_{n-1}, v_{n-1}) = n$. Thus, by Lemma 2.24, $(n+1)$-amenability follows.

We have $c(T_1) = c(T_1')$.

As an example, for

$$T_{\lambda/\mu} = \begin{array}{ccccccc} 1' & 1 & 1 & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & 2 & 2 & \\ 1' & 2' & 3' & 3 & 3 & 3 & \\ 1 & 2' & 3' & 4' & 4 & 4 & \\ & 2 & 3' & 4' & 5' & 5 & \\ & & 3 & 4 & 5 & & \end{array}$$

we obtain

$$T_1 = \begin{array}{ccccccc} 1' & 1 & 1 & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & 2 & 2 & \\ 1 & 2' & 3' & 3 & 3 & 3 & \\ 2 & 2 & 3' & 4' & 4 & 4 & \\ & 3 & 3 & 4 & 5 & 5 & \\ & & 4 & 5 & 6 & & \end{array}, \quad T_1' = \begin{array}{ccccccc} 1' & 1 & 1 & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & 2 & 2 & \\ 1 & 2' & 3' & 3 & 3 & 3 & \\ 2 & 2 & 3' & 4' & 4 & 4 & \\ & 3 & 3 & 4 & 5' & 5 & \\ & & 4 & 5 & 6 & & \end{array}.$$

We obtain $Q_{(10,8,7,6,5,3)/(3,2,1)} = Q_{(10,8,7,5,3)} + Q_{(10,8,7,5,2,1)} + Q_{(10,8,7,4,3,1)} + Q_{(10,8,6,5,3,1)} + Q_{(9,8,7,6,3)} + Q_{(9,8,7,6,2,1)} + Q_{(9,8,7,5,4)} + 3Q_{(9,8,7,5,3,1)} + Q_{(9,8,7,4,3,2)} + Q_{(9,8,6,5,4,1)} + Q_{(9,8,6,5,3,2)}$.

Case 1.2: the last box of P_{n-1} is in the row above the row of the last box of P_n.

For $p = 2$ we get $\mu = (1)$, which is a contradiction. Thus, we have $p > 2$. Let (u_a, v_a) be the last box of P_a for all a. We get a new tableau T_2 if we set $T_2(u_n, v_n) = n+1$, $T_2(u_n - 1, v_n) = (n+1)'$, for all $k \leqslant a \leqslant n-1$ set $T_2(u_a, v_a) = a+1$, $T_1(u_a - 1, v_a) = a$ and $T_2(r,s) = T_{\lambda/\mu}(r,s)$ for every other box $(r,s) \in D_{\lambda/\mu}$. By Corollary 2.25, T_2 is m-amenable for $m \neq n+1$. We have $T_2(u_n - 1, v_n) = (n+1)'$ and $T_2(u_n - 2, v_n - 1) \neq n'$. However, we have $T_2(u_n - 2, v_n) = n'$. Thus, by Lemma 2.24, $(n+1)$-amenability follows.

We get another tableau T_2' if we set $T_2'(u_n - 2, v_n) = n$ and $T_2'(r,s) = T_2(r,s)$ for every other box $(r,s) \in D_{\lambda/\mu}$. By Lemma 2.24, it is clear that T_2' is amenable if T_2 is amenable. We have $c(T_2) = c(T_2')$.

For example, for

$$T_{\lambda/\mu} = \begin{array}{ccccc} 1' & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & \\ 1 & 2' & 3' & 3 & \\ & 2 & 3' & & \\ & & 3 & & \end{array}$$

we obtain

$$T_2 = \begin{array}{|c|c|c|c|c|}\hline 1' & 1 & 1 & 1 & 1 \\ \hline 1 & 2' & 2 & 2 & \\ \cline{1-4} 2 & 2 & 3' & 3 & \\ \cline{1-4} \multicolumn{1}{c|}{} & 3 & 4' & \multicolumn{2}{c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & 4 & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array} \quad , \quad T_2' = \begin{array}{|c|c|c|c|c|}\hline 1' & 1 & 1 & 1 & 1 \\ \hline 1 & 2' & 2 & 2 & \\ \cline{1-4} 2 & 2 & 3 & 3 & \\ \cline{1-4} \multicolumn{1}{c|}{} & 3 & 4' & \multicolumn{2}{c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & 4 & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array} \quad .$$

We have $Q_{(7,5,4,2,1)/(2,1)} = Q_{(7,5,4)} + Q_{(7,5,3,1)} + Q_{(6,4,3,2,1)} + 2Q_{(6,5,3,2)} + Q_{(7,4,3,2)} + Q_{(6,5,4,1)}$.

Case 2: only one corner is in P_n.

Let the second uppermost corner be in P_i. Then by Lemma 4.9, the second uppermost corner is the first box of P_i and the uppermost corner is the first box of P_k. If P_i has all boxes in a row then $\mu = \emptyset$; a contradiction. Thus, the diagram P_i has at least two corners. By Lemma 4.12, P_i is a hook. Then for all $i \leqslant j < n$ each P_j is a (p, q)-hook for some $p, q \geqslant 2$.

Case 2.1: The last box of P_{i-1} is in the same row as the last box of P_i.

Let (g, h) be the last box of P_i and (c_a, d_a) be the rightmost box of P_a in the lowermost row with boxes from P_a for all $k \leqslant a \leqslant i-1$. We get a new tableau T_3 if for all $k \leqslant a \leqslant i-1$ we set $T_3(c_a, d_a) = a+1$ if $(c_a+1, d_a) \notin D_{\lambda/\mu}$ or else set $T_3(c_a, d_a) = (a+1)'$ if $(c_a+1, d_a) \in D_{\lambda/\mu}$, set $T_3(c_a-1, d_a) = a$, $T_3(g, h) = i+1$ and $T_3(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. By Corollary 2.25, the tableau T_3 is m-amenable for $m \neq k+1, i+1$. We possibly have $T_3(c_k, d_k) = (k+1)'$ and $T_3(c_k-1, d_k-1) \neq k'$. If not, then there is possibly a $k+1$ in the ${d_k}^{\text{th}}$ column. Anyway, there is a k with no $k+1$ below in the first box of P_k. Thus, by Lemma 2.24, $(k+1)$-amenability follows. We have $T_3(g, h) = i+1$ and $T_3(g-1, h) < i$. However, there is an i with no $i+1$ below in the first box of P_i. Thus, by Lemma 2.24, $(i+1)$-amenability follows.

We get another tableau T_3' if we set $T_3'(g-1, h) = i$ and $T_3'(r, s) = T_3(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. Clearly, T_3' is amenable if T_3 is and we have $c(T_3) = c(T_3')$.

As example, for

$$T_{\lambda/\mu} = \begin{array}{|c|c|c|c|c|}\hline 1' & 1 & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & 2 & \\ \cline{1-4} 1' & 2' & 3' & \multicolumn{2}{c}{} \\ \cline{1-3} 1 & 2 & 3 & \multicolumn{2}{c}{} \\ \cline{1-3} \end{array}$$

we obtain

$$T_3 = \begin{array}{|c|c|c|c|c|}\hline 1' & 1 & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & 2 & \\ \cline{1-4} 1 & 2' & 3' & \multicolumn{2}{c}{} \\ \cline{1-3} 2 & 3 & 3 & \multicolumn{2}{c}{} \\ \cline{1-3} \end{array} \quad , \quad T_3' = \begin{array}{|c|c|c|c|c|}\hline 1' & 1 & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & 2 & \\ \cline{1-4} 1 & 2 & 3' & \multicolumn{2}{c}{} \\ \cline{1-3} 2 & 3 & 3 & \multicolumn{2}{c}{} \\ \cline{1-3} \end{array} \quad .$$

We have $Q_{(8,6,4,3)/(3,2,1)} = Q_{(8,5,2)} + Q_{(8,4,3)} + Q_{(7,6,2)} + Q_{(8,4,2,1)} + 2Q_{(7,5,3)} + Q_{(6,4,3,2)} + 2Q_{(6,5,3,1)} + Q_{(6,5,4)} + 2Q_{(7,4,3,1)} + 2Q_{(7,5,2,1)}$.

Case 2.2: The last box of P_{i-1} is in the row above the row of the last box of P_i.

If in the column of the last box of P_i there are only two boxes of P_i then we have $\mu = (1)$, which is a contradiction. Thus, there are at least three boxes of P_i in the column of the last box of P_i. Let (c_a, d_a) be the last box of P_a for all $k \leqslant a \leqslant i+1$. We get a new tableau T_4 if for all $k \leqslant a \leqslant i-1$ we set $T_4(c_a, d_a) = a+1$, $T_4(c_a-1, d_a) = a$,

$T_4(c_i, d_i) = i+1$, $T_4(c_i - 1, d_i) = (i+1)'$, $T_4(c_{i+1}, d_{i+1}) = i+2$, $T_4(c_{i+1}-1, d_{i+1}) = i+1$ and $T_4(r,s) = T_{\lambda/\mu}(r,s)$ for every other box $(r,s) \in D_{\lambda/\mu}$.

By Corollary 2.25, the tableau T_4 is m-amenable for $m \neq k+1, i+1$. There is a k with no $k+1$ below in the first box of P_k. Thus, by Corollary 2.25, $(k+1)$-amenability follows. We have $T_4(c_i, d_i) = i+1$ and there is no i in the d_i^{th} column. However, there is an i with no $i+1$ below in the first box of P_i. We have $T_4(c_i - 1, d_i) = (i+1)'$ and $T_4(c_i - 2, d_i - 1) \neq i'$. However, we have $T_4(c_i - 2, d_i) = i'$. Thus, by Lemma 2.24, $(i+1)$-amenability follows.

We get another tableau T_4' if we set $T_4'(c_i - 2, d_i) = i$ and $T_4'(r,s) = T_4(r,s)$ for every other box $(r,s) \in D_{\lambda/\mu}$.

The tableau T_4' is m-amenable for $m \neq i+1$. We have $T_4'(c_i - 1, d_i) = (i+1)'$ and $T_4'(c_i - 2, d_i - 1) \neq i'$. However, there is an i with no $i+1$ below in the first box of P_i. Thus, by Lemma 2.24, $(i+1)$-amenability follows. We have $c(T_4) = c(T_4')$.

As a final example, for

$$T_{\lambda/\mu} = \begin{array}{|c|c|c|c|c|} \hline 1' & 1 & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & 2 & \\ \cline{1-4} 1 & 2' & 3' & & \\ \cline{1-3} & 2 & 3 & & \\ \cline{2-3} \end{array}$$

we obtain

$$T_4 = \begin{array}{|c|c|c|c|c|} \hline 1' & 1 & 1 & 1 & 1 \\ \hline 1 & 2' & 2 & 2 & \\ \cline{1-4} 2 & 3' & 3 & & \\ \cline{1-3} & 3 & 4 & & \\ \cline{2-3} \end{array}\,, \quad T_4' = \begin{array}{|c|c|c|c|c|} \hline 1' & 1 & 1 & 1 & 1 \\ \hline 1 & 2 & 2 & 2 & \\ \cline{1-4} 2 & 3' & 3 & & \\ \cline{1-3} & 3 & 4 & & \\ \cline{2-3} \end{array}\,.$$

We have $Q_{(7,5,3,2)/(2,1)} = Q_{(7,5,2)} + Q_{(7,4,3)} + Q_{(7,4,2,1)} + Q_{(6,5,3)} + Q_{(6,5,2,1)} + 2Q_{(6,4,3,1)} + Q_{(5,4,3,2)}$. □

Corollary 4.17. *Let $\lambda, \mu \in DP$. Let $\nu = c(T_{\lambda/\mu})$ and $n = \ell(\nu) > 1$. If $D^{ot}_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ where $\beta \neq \emptyset, (1)$ and D_α has more than two corners then $Q_{\lambda/\mu}$ is not Q-multiplicity-free. If $D_{\lambda/\mu}$ is an unshifted diagram and $D^{o}_{\lambda/\mu}$ has more than two corners then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Remark 4.18. As it will turn out (and will be proved in Lemma 4.35), for $\mu = \emptyset$ or $\mu = (1)$ the skew Schur Q-function $Q_{\lambda/\mu}$ is Q-multiplicity-free. Thus, we will only consider the case $\mu \neq \emptyset, (1)$. Since we want to find all λ, μ such that $Q_{\lambda/\mu}$ is Q-multiplicity-free, by Lemma 4.16 from now on we can assume that λ has at most two corners.

The case that the diagram D_λ or the diagram D_μ has at most two corners also occurs in the classical setting of Schur functions $s_{\lambda/\mu}$. Gutschwager proved [5, Theorem 3.5] where the cases in condition (2) of said theorem have this property. However, this property is not enough in the classical case, where further restrictions need to be imposed for the classification of (s-)multiplicity-free skew Schur functions. For the classification of Q-multiplicity-free skew Schur Q-functions we also need to find further restrictions since the properties from Lemma 4.16 and Corollary 4.17 are not sufficient.

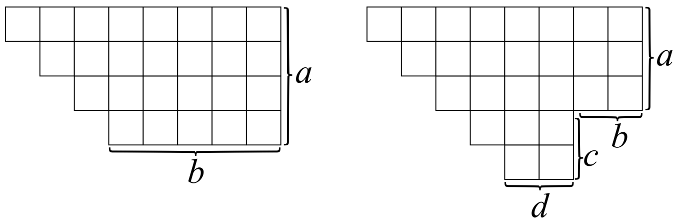
We will introduce some new notation for partitions in DP with at most two corners and then obtain restrictions until we can exclude all non-Q-multiplicity-free skew Schur Q-functions in Proposition 4.33.

Definition 4.19. Let $DP^{\leqslant 2} \subseteq DP$ be the set of partitions λ with distinct parts such that D_λ has at most two corners. For $\lambda \in DP^{\leqslant 2}$ the **shape path** π_λ is defined as follows. Let a be the row of the upper corner of D_λ. Let

$$b = \begin{cases} \lambda_a \text{ if } a = \ell(\lambda); \\ \lambda_a - \lambda_{a+1} - 1 \text{ otherwise.} \end{cases}$$

If there is a lower corner in D_λ let $c = \ell(\lambda) - a$ and $d = \lambda_{\ell(\lambda)}$. Then the shape path to λ is $\pi_\lambda = [a, b]$ if D_λ has one corner and $\pi_\lambda = [a, b, c, d]$ if D_λ has two corners.

Remark 4.20. The numbers a, b, c, d of the shape path can be visualized as follows.



In particular, we see that for $\lambda = (8, 7, 6, 5)$ we have $\pi_\lambda = [4, 5]$, and for $\lambda = (8, 7, 6, 3, 2)$ we have $\pi_\lambda = [3, 2, 2, 2]$

Remark 4.21. For a given $\lambda \in DP^{\leqslant 2}$ the cardinality of the border B_λ can be derived from the shape path. If $\lambda = [a, b]$ then $|B_\lambda| = a + b - 1$. If $\lambda = [a, b, c, d]$ then $|B_\lambda| = a + b + c + d - 1$.

Lemma 4.22. *The map* $DP^{\leqslant 2} \to \mathbb{N}^4 \cup \mathbb{N}^2 : \lambda \mapsto \pi_\lambda$ *is a bijection.*

Proof. For a given $[a, b, c, d]$, its unique preimage is $\lambda = (a + b + c + d - 1, a + b + c + d - 2, \ldots, b + c + d + 1, b + c + d, c + d - 1, c + d - 2, \ldots, d)$. For a given $[a, b]$, its unique preimage is $\lambda = (a + b - 1, a + b - 2, \ldots, b)$. Hence the map sending a partition in $DP^{\leqslant 2}$ to its shape path is bijective. □

Notation. From now on we will identify a partition $\lambda \in DP^{\leqslant 2}$ with at most two corners with its shape path π_λ.

We already showed that for $\mu \neq \emptyset, (1)$ the partition λ has at most two corners if $Q_{\lambda/\mu}$ is Q-multiplicity-free. We can use this to prove that for $\lambda \neq [a, 1], [a, 2]$ for some a the partition μ must have at most two corners if $Q_{\lambda/\mu}$ is Q-multiplicity-free. We will show this in the following lemma and afterwards restrict λ and μ further to three families.

Lemma 4.23. *Let* $\mu \in DP$, $\lambda \in DP^{\leqslant 2}$, *and suppose* $\lambda \neq [a, b]$ *where* $b \leqslant 2$. *If* D_μ *has more than two corners then* $Q_{\lambda/\mu}$ *is not* Q*-multiplicity-free.*

Proof. For each corner (x, y) of D_μ except for the lowermost, the box $(x+1, y) \notin D_\mu$ and, hence, $(x + 1, y) \in D_{\lambda/\mu}$. Let the leftmost box of $D_{\lambda/\mu}$ in the $(x + 1)^{\text{th}}$ row be $(x + 1, z)$. Since $(x + 1, z)$ is the leftmost box of $D_{\lambda/\mu}$ and (x, y) is not the lowermost corner of D_μ, the box $(x+1, z-1) \in D_\mu$. Since $(x, y) \in D_\mu$ and $z < y$, the box $(x+1, z-1) \in D_\mu$. So we have $(x+1, z) \in D_{\lambda/\mu}$ and $(x, z), (x+1, z-1) \notin D_{\lambda/\mu}$. Also there is a box $(1, w) \in D_{\lambda/\mu}$ such that $(1, w-1) \notin D_{\lambda/\mu}$ and there is no box above because $(1, w)$ is in the first row. After transposing this diagram orthogonally, the image of these boxes are corners of $D^{ot}_{\lambda/\mu}$. The diagram $D^{ot}_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ where $\beta \neq \emptyset, (1)$ and D_α has more than two corners. By Corollary 4.17, $Q_{\lambda/\mu}$ is not Q-multiplicity-free. □

Lemma 4.24. *Let $\mu \in DP$, $\lambda \in DP^{\leqslant 2}$, and suppose $\lambda \neq [a, b]$ where $b \leqslant 2$. If $\mu = [w, x, y, z]$ where $z > 1$ then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. The leftmost box of the first row of $D_{\lambda/\mu}$, which is $(1, w+x+y+z)$, has no box to the left or above. Also, the leftmost box of the $(w+1)^{\text{th}}$ row of $D_{\lambda/\mu}$, which is $(w+1, w+y+z)$, has no box to the left or above. In addition, the leftmost box of the $(w+y+1)^{\text{th}}$ row of $D_{\lambda/\mu}$, which is $(w+y+1, w+y+1)$, has no box to the left or above. After transposing this diagram orthogonally, the images of these boxes are corners of $D^{ot}_{\lambda/\mu}$. Then the diagram $D^{ot}_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ where $\beta \neq \emptyset, (1)$ and D_α has more than two corners. By Corollary 4.17, $Q_{\lambda/\mu}$ is not Q-multiplicity-free. □

Lemma 4.25. *Suppose $\lambda = [a, b, c, d]$ and $\mu = [w, x]$ where $x > 1$ or $\mu = [w, x, y, 1]$. Then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. Let k be such that $U_k(\lambda/\mu)$ has only one box in the diagonal $\{(s, t) \mid t - s = x - 1\}$ for the case $\mu = [w, x]$ (the diagonal contains the corner $(w, w+x-1)$ of D_μ) or in the diagonal $\{(s, t) \mid t - s = x + y\}$ for the case $\mu = [w, x, y, 1]$ (the diagonal contains the upper corner $(w, w+x+y)$ of D_μ). Since we consider basic connected diagrams, the box $(w+1, w+x) \in D_{\lambda/\mu}$ for the case $\mu = [w, x]$. Thus, the diagonal defined above is not empty and, hence, there exists such a k which is also unique. For the case $\mu = [w, x, y, 1]$ the box $(w+1, w+x+y+1) \in D_{\lambda/\mu}$. Thus, in this case the diagonal defined above is not empty and there exists such a k which is also unique. Let the box in the diagonal defined above be (p, q). Then $(p, q) \in P_k$ and also $(p-1, q), (p, q-1) \in P_k$, so P_k is not a hook. Let $n = \ell(c(T_{\lambda/\mu}))$.

Case 1: $k = n$.

If P_n is not a rotated hook, then by Lemma 4.3, $Q_{\lambda/\mu}$ is not Q-multiplicity-free. If P_n is a rotated (l, m)-hook where $l, m \geqslant 2$ then, since $\lambda = [a, b, c, d]$, there is some $j < n$ such that either the first box of P_j is in a column to the right of the boxes of P_n or the last box of P_j is in a row below the boxes of P_n. Let j be maximal with respect to this condition.

We may assume that the first box of P_j is in a column to the right of the boxes of P_n, otherwise $U_j(\lambda/\mu)$ is unshifted and we may consider $U_j(\lambda/\mu)^t$. By Lemma 2.37, if $D^{ot}_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ then $T^{(n)}_{\alpha/\beta}$ is an (m, l)-hook where $l, m \geqslant 2$ and the diagram $U_j(\alpha/\beta)$ satisfies the conditions of Lemma 4.14. By Lemma 3.1, it follows that $Q_{\lambda/\mu}$ is not Q-multiplicity-free.

Case 2: $k \neq n$.

If $U_{k+1}(\lambda/\mu)$ has at least two components then the last box of the second component can be filled with $(k+1)'$ or $k+1$ and, by Lemma 2.24, $Q_{\lambda/\mu}$ is not Q-multiplicity-free. Thus, we may consider that all boxes of $U_{k+1}(\lambda/\mu)$ are either above or below the diagonal $\{(s,t) \mid t-s = x-1\}$ for the case $\mu = [w,x]$ or are above or below the diagonal $\{(s,t) \mid t-s = x+y\}$ for the case $\mu = [w,x,y,1]$.

Case 2.1: P_n is an (l,m)-hook where $l, m \geqslant 2$.

Then either $U_k(\lambda/\mu)$ or $U_k(\lambda/\mu)^t$ satisfies the conditions of Lemma 4.14 and $Q_{\lambda/\mu}$ is not Q-multiplicity-free.

Case 2.2: only one corner is in P_n.

Let (f,g) be this corner. Then there is some e such that there are two boxes of P_e either in a row weakly below the f^{th} row or in a column weakly to the right of g^{th} column. There is also some h such that either the first box of P_h is to the right of the g^{th} column or the last box of P_h is below the f^{th} row. Let e, h be maximal with respect to these conditions.

By orthogonal transposition, transposition or rotation of $U_{\min\{e,h\}}(\lambda/\mu)$, we may assume that $h \leqslant e$ and that the first box of P_h is to the right of the g^{th} column. By Lemma 4.9, if $h = e$ then $Q_{\lambda/\mu}$ is not Q-multiplicity-free. Hence, we assume $h < e$.

There is a box $(r,u) \in P_h$ in the diagonal $\{(s,t) \mid t-s = x-1\}$ for the case $\mu = [w,x]$ or in the diagonal $\{(t,s) \mid t-s = x+y\}$ for the case $\mu = [w,x,y,1]$.

We get a tableau T if after the $(h-1)^{\text{th}}$ step of the algorithm of Definition 2.27 we use $P'_h := P_h \setminus \{(r,u)\}$ instead of P_h. By Corollary 2.25, this tableau is m-amenable for $m \neq h+1$. We have $T(r,u) = (h+1)'$ and $T(r-1,u-1) \neq h'$. However, there is an h with no $(h+1)$ below in the first box of P_h. Thus, by Lemma 2.24, this tableau is amenable.

We get another tableau T' with the same content if we set $T'(r-1,u) = h'$ and $T'(f,g) = T(f,g)$ for every other box $(f,g) \in D_{\lambda/\mu}$. By Corollary 2.25, this tableau is m-amenable for $m \neq h+1$.

We have $T'(r,u) = (h+1)'$ and $T'(r-1,u-1) \neq h'$. However, we have $T'(r-1,u) = h'$. In the u^{th} column is an $h+1$ but no h. However, there are hs with no $(h+1)$s below in the first box and in the last box of P_h. Thus, by Lemma 2.24, this tableau is amenable.

By Lemma 3.1, $Q_{\lambda/\mu}$ is not Q-multiplicity-free. □

Example 4.26. For $\lambda = [1,1,4,1]$ and $\mu = [1,1,1,1]$ we have $T_{\lambda/\mu} = \begin{array}{cccc} & 1' & 1 & 1 \\ 1' & 1 & 2' & \\ 1 & 2' & 2 & \\ & 2 & 3' & \\ & & 3 & \end{array}$.

Then we obtain $T = \begin{array}{cccc} & 1 & 1 & 1 \\ 1' & 2' & 2 & \\ 1 & 2' & 3' & \\ & 2 & 3' & \\ & & 3 & \end{array}$, $T' = \begin{array}{cccc} & 1' & 1 & 1 \\ 1' & 2' & 2 & \\ 1 & 2' & 3' & \\ & 2 & 3' & \\ & & 3 & \end{array}$.

We have $Q_{(6,4,3,2,1)/(3,1)} = Q_{(6,4,2)} + 2Q_{(5,4,3)} + Q_{(5,4,2,1)}$.

Now for Q-multiplicity-free skew Schur Q-functions $Q_{\lambda/\mu}$, for a given λ, the partition μ is restricted to certain families of partitions. The following two lemmas and their corollaries restrict λ and μ further until Proposition 4.33 can be proved.

Lemma 4.27. *Let $\lambda = [a, b, c, 1]$ and $\mu = [w, 1]$. If $a \geqslant 3$, $b \geqslant 3$, $c \geqslant 3$ and $4 \leqslant w \leqslant a + c - 2$ then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. We will show that for case $a = 3$ and for case $w = a + c - 2$ the statement holds. Afterwards we will explain case $a > 3$ and $w < a + c - 2$ by these two cases.

Case 1: $a = 3$.

Let $b \geqslant 3$, $c \geqslant 3$ and $4 \leqslant w \leqslant a + c - 2$. The lowermost box in the leftmost column of the diagram is $(w + 1, w + 1)$. Since $w < a + c - 1$, we have $(w, w + 2) \in D_{\lambda/\mu}$.

We get a new tableau T_1 as follows: In the algorithm of Definition 2.27 use $P'_1 := P_1 \setminus \{(w + 1, w + 1)\}$, $P'_2 := P_2 \setminus \{(w + 1, w + 2), (w + 2, w + 2)\}$ and $P'_3 := P_3 \setminus \{(w, w + 3), (w + 1, w + 3), (w + 2, w + 3), (w + 3, w + 3)\}$ (for $w = a + c - 2$ this means $P'_3 = P_3$) instead of P_1, P_2 and P_3, respectively, and stop after the third step in the algorithm. Then replace the entry 3 in the last box of P'_3 with $3'$ and set $T_1(w + 1, w + 1) = 3$. Afterwards fill the remaining boxes using the algorithm of Definition 2.27 starting with $k = 4$. By Corollary 2.25, it is clear that T_1 is m-amenable for $m \neq 3, 4$. There is a 3 but no 2 in the $(w + 1)^{\text{th}}$ column. However, there is a 2 and no 3 in the column of the last box of P'_3 and there is a 2 and no 3 in the column to the left of it. Thus, by Lemma 2.24, this tableau is 3-amenable. In the $(w + 2)^{\text{th}}$ column and possibly in the $(w + 3)^{\text{th}}$ column, there are 4s and no 3s. However, there are 3s and no 4s in the columns of the first two boxes of P'_3. We have $T_1(w+1, w+2) = 4'$ and $T_1(w, w+1) \neq 3'$. However, if (y, z) is the third box of P'_3 then we either have $T_1(y, z) = 3$ and there is no 4 in the z^{th} column or if $w = a + c - 2$ we have $T_1(y, z) = 3'$ and $T_1(y + 1, z + 1) \neq 4'$. If $w < a + c - 2$ then we have $T_1(w, w+3) = 4'$ and $T_1(w-1, w+2) \neq 3'$. However, we have $T_1(w-1, w+3) = 3'$. Thus, by Lemma 2.24, this tableau is 4-amenable.

We get another tableau T'_1 of the same content if we set $T'_1(w+1, w+1) = 3$, $T'_1(w, w+2) = 2$ and $T'_1(r, s) = T_1(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. It is easy to see that, by Corollary 2.25, T'_1 is m-amenable for $m \neq 2, 3, 4$. There is a 1 with no 2 below in the $(w + 2)^{\text{th}}$ column. Thus, by Lemma 2.24, 2-amenability follows. There is a 3 with no 2 above in the $(w + 2)^{\text{th}}$ column. However, there is a 2 with no 3 below in the column of the last box of P_3. Thus, by Lemma 2.24, this tableau is 3-amenable. By Lemma 2.24, it is clear that T'_1 is 4-amenable if T_1 is.

For example, for $\lambda = [3, 3, 6, 1]$ and $\mu = [5, 1]$ the tableaux are

$$T_1 = \begin{array}{|c|c|c|c|c|c|c|}
\hline
1' & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
1' & 2' & 2 & 2 & 2 & 2 & 2 \\ \hline
1' & 2' & 3' & 3 & 3 & 3 & 3 \\ \hline
1' & 2' & 3' & 4' \\ \cline{1-4}
1 & 2 & 4' & 4 \\ \cline{1-4}
3 & 4' & 4 & 5' \\ \cline{1-4}
\multicolumn{1}{c|}{} & 4 & 5' & 5 \\ \cline{2-4}
\multicolumn{2}{c|}{} & 5 & 6' \\ \cline{3-4}
\multicolumn{3}{c|}{} & 6 \\ \cline{4-4}
\end{array}\ , \quad T_1' = \begin{array}{|c|c|c|c|c|c|c|}
\hline
1' & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline
1' & 2' & 2 & 2 & 2 & 2 & 2 \\ \hline
1' & 2' & 3' & 3 & 3 & 3 & 3 \\ \hline
1' & 2' & 3' & 4' \\ \cline{1-4}
1 & 3 & 4' & 4 \\ \cline{1-4}
2 & 4' & 4 & 5' \\ \cline{1-4}
\multicolumn{1}{c|}{} & 4 & 5' & 5 \\ \cline{2-4}
\multicolumn{2}{c|}{} & 5 & 6' \\ \cline{3-4}
\multicolumn{3}{c|}{} & 6 \\ \cline{4-4}
\end{array}\ .$$

Case 2: $w = a + c - 2$.

By Case 1, we may assume $a > 3$. The lowermost box in the leftmost column of the diagram is $(w + 1, w + 1)$. Since $w < a + c - 1$, we have $(w, w + 2) \in D_{\lambda/\mu}$.

Let (y, z) be the last box of P_3. We get a new tableau T_2 if we set $T_2(w+1, w+1) = 3$, $T_2(w, w+1) = 1$, $T_2(w, w+2) = 2$, $T_2(w+1, w+2) = 4$, $T_2(w+2, w+2) = 5$, $T_2(y, z) = 3'$, $T_2(y, z+1) = 4'$, for the case $P_5 \neq \emptyset$ set $T_2(y, z+2) = 5'$ (in this case $(y, z+2)$ is the last box of P_5), and set $T_2(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$.

By Corollary 2.25, T_2 is m-amenable for $m \neq 3, 4, 5$. There is a 3 and no 2 in the $(w+1)^{\text{th}}$ column. However, there are 2s and no 3s in the z^{th} and in the $(w+2)^{\text{th}}$ column. Thus, by Lemma 2.24, this tableau is 3-amenable. There is a 4 with no 3 above in the $(w + 2)^{\text{th}}$ column. However, there are 3s and no 4s in the $(w + 1)^{\text{th}}$ column and in the $(z+1)^{\text{th}}$ column. Thus, by Lemma 2.24, 4-amenability follows. The 5-amenability is clear for $P_5 = \emptyset$. If $P_5 \neq \emptyset$ then there is a 4 and no 5 in the $(z + 2)^{\text{th}}$ column. Thus, by Lemma 2.24, this tableau is 5-amenable.

We get another tableau T_2' of the same content if we set $T_2'(w+1, w+1) = 2$, $T_2'(w, w+2) = 3$ and $T_2'(r, s) = T_2(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$. By Corollary 2.25, T_2' is m-amenable for $m \neq 2, 3, 4$. There is a 1 and no 2 in the $(w + 2)^{\text{th}}$ column. Thus, by Lemma 2.24, 2-amenability follows. There is a 3 and no 2 in the $(w + 2)^{\text{th}}$ column. However, there is a 2 with no 3 below in the z^{th} column. Thus, by Lemma 2.24, this tableau is 3-amenable. By Lemma 2.24, it is clear that T_2' is 4-amenable if T_2 is.

Case 3: $a > 3$ and $w < a + c - 2$.

The diagram $U_2(\lambda/\mu)$ has shape $D_{\lambda'/\mu}$ where $\lambda' = [a', b, c, 1]$ where $a' = a - 1$. Either we have $a' = a-1 = 3$ or $w = a'+c-2$ or else there is some j such that $U_j(\lambda/\mu)$ has shape $D_{\lambda''/\mu}$ where $\lambda'' = [a'', b, c, 1]$ where $a'' = a - j$ such that either $a'' = 3$ or $w = a'' + c - 2$. Then, by Case 1 and Case 2, we find two different amenable tableaux of the same content and, by Lemma 3.1, $Q_{\lambda/\mu}$ is not Q-multiplicity-free.

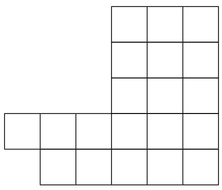
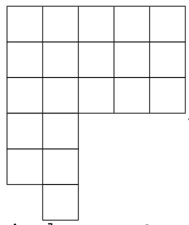
As example, for $\lambda = [4,5,3,1]$ and $\mu = [5,1]$ the tableaux are

$$T_2 = \begin{array}{|c|c|c|c|c|c|c|} \hline 1' & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & 2 & 2 & 2 & 2 \\ \hline 1' & 2' & 3' & 3 & 3 & 3 & 3 \\ \hline 1' & 2' & 3' & 4' & 4 & 4 & 4 \\ \hline 1 & 2 \\ \cline{1-2} 3 & 4 \\ \cline{1-2} \multicolumn{1}{c|}{} & 5 \\ \cline{2-2} \end{array}, \quad T'_2 = \begin{array}{|c|c|c|c|c|c|c|} \hline 1' & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & 2 & 2 & 2 & 2 \\ \hline 1' & 2' & 3' & 3 & 3 & 3 & 3 \\ \hline 1' & 2' & 3' & 4' & 4 & 4 & 4 \\ \hline 1 & 3 \\ \cline{1-2} 2 & 4 \\ \cline{1-2} \multicolumn{1}{c|}{} & 5 \\ \cline{2-2} \end{array}. \qquad \square$$

Corollary 4.28. *Let $\lambda = [a,b]$ and $\mu = [w,x]$. If $w \geqslant 3$, $x \geqslant 4$, $a \geqslant w+2$, $b \geqslant 5$ and $a+b-w-x \geqslant 3$ then $Q_{\lambda/\mu}$ is not multiplicity-free.*

Proof. If λ, μ satisfy these properties then the diagram $D^{ot}_{\lambda/\mu}$ is equal to $D_{\alpha/\beta}$ where $\alpha = [a', b', c', 1]$ and $\beta = [w', 1]$. Then $b' = w \geqslant 3$ and $c' = x - 1 \geqslant 3$. The number a' is the number of boxes of the first row of $D_{\lambda/\mu}$ and can be calculated by $a' = \lambda_1 - \mu_1 = |B_\lambda| - |B_\mu| = a + b - w - x \geqslant 3$. Since $a \geqslant w + 2$, we have $a - w - 2 \geqslant 0$ and, hence, $b \leqslant a + b - w - 2$. Then we get $4 \leqslant b - 1 = w' = b - 1 \leqslant a + b - w - 2 - 1 = a + b - w - x + x - 1 - 2 = a' + c' - 2$. By Lemma 4.27, $Q_{D^{ot}_{\lambda/\mu}}$ is not Q-multiplicity-free and, thus, $Q_{\lambda/\mu}$ is not Q-multiplicity-free. $\square$

Example 4.29. The smallest diagram satisfying the properties of Corollary 4.28 is $D_{(9,8,7,6,5)/(6,5,4)}$ (that is $w = 3, x = 4, a = 5, b = 5$ and, hence $a + b - w - x = 3$). We have

$D_{(9,8,7,6,5)/(6,5,4)} =$  and $D^{ot}_{(9,8,7,6,5)/(6,5,4)} =$ .

We see that $D^{ot}_{(9,8,7,6,5)/(6,5,4)} = D_{\alpha/\beta}$ where $\alpha = [a', b', c', 1]$ and $\beta = [w', 1]$. Since the lowest row and the leftmost column switch after orthogonally transposing, the number $w' = b - 1 = 4$. Similarly, we obtain $b' = w = 3$ and $c' = x - 1 = 3$. The number a' is the number of boxes in the first row, which is equal to $a + b - w - x = 3$. Hence, the diagram $D^{ot}_{(9,8,7,6,5)/(6,5,4)}$ satisfies the properties of Lemma 4.27.

We have $Q_{(9,8,7,6,5)/(6,5,4)} = Q_{(9,8,3)} + Q_{(9,7,4)} + Q_{(9,7,3,1)} + Q_{(9,6,4,1)} + Q_{(9,6,3,2)} + Q_{(9,5,4,2)} + Q_{(8,7,5)} + Q_{(8,7,4,1)} + Q_{(8,7,3,2)} + Q_{(8,6,5,1)} + 2Q_{(8,6,4,2)} + Q_{(8,6,3,2,1)} + Q_{(8,5,4,3)} + Q_{(8,5,4,2,1)} + Q_{(7,6,5,2)} + Q_{(7,6,4,3)} + Q_{(7,6,4,2,1)} + Q_{(7,5,4,3,1)}$.

Lemma 4.30. *Let $\lambda = [a,b,c,d]$ and $\mu = [w,1]$. If $a,b,c,d \geqslant 2$ and $3 \leqslant w \leqslant a + c - 1$ then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. Let $n = \ell(c(T_{\lambda/\mu}))$. First, we assume $w = a + c - 1$ and prove the result when $a = 2$ or $d = 2$. Then, we show that the case where $a, d \geqslant 3$ can be explained by the cases where $a = 2$ or $d = 2$. Afterwards, we tackle the case $w < a + c - 1$ using the case $w = a + c - 1$ where we first prove the subcase $d = 2$ and then show how to add boxes with entries to obtain diagrams such that $d > 2$.

Case 1: $w = a + c - 1$ and $2 \in \{a, d\}$.

We may assume $a = 2$, otherwise we transpose the diagram. If $d = 2$ then P_n is a $(b+1, c+1)$-hook and, by Lemma 4.5, which in this case forbids having a box directly to the left of the last box of P_n, $Q_{\lambda/\mu}$ is not Q-multiplicity-free. Thus, we may now assume $d \geqslant 3$.

The box $(w + 1, w + 1)$ is the last box of P_1. We get a new tableau T_1 if we set $T_1(w, w + 1) = 1$, $T_1(w + 1, w + 1) = 3$, $T_1(w, w + 2) = 2$, $T_1(w + 1, w + 2) = 3$, $T_1(w, w + 3) = 3$, $T_1(w + 1, w + 3) = 4$ and $T_1(r, s) = T_{\lambda/\mu}(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$.

By Corollary 2.25, T_1 is m-amenable for $m \neq 3$. There is a 3 and no 2 in the $(w+1)^{\text{th}}$ column. However, there are 2s and no 3s in the columns of the first two boxes of P_2. Thus, by Lemma 2.24, T_1 is amenable.

We get another tableau T_1' if we set $T_1'(w + 1, w + 1) = 2$, $T_1'(w, w + 2) = 3'$ and $T_1'(r, s) = T_1(r, s)$ for every other box $(r, s) \in D_{\lambda/\mu}$.

By Corollary 2.25, T_1' is m-amenable for $m \neq 2, 3$. In the $(w+2)^{\text{th}}$ column is a 1 with no 2 below. Thus, by Corollary 2.25, 2-amenability follows. We have $T_1'(w, w + 2) = 3'$ and $T_1'(w - 1, w + 1) \neq 2'$ and there is a 3 and no 2 in the $(w + 2)^{\text{th}}$ column. However, there are two 2s and no 3s in the columns of the first two boxes of P_2. Thus, by Lemma 2.24, 3-amenability follows. It is clear that T_1' has the same content as T_1. Hence, $Q_{\lambda/\mu}$ is not Q-multiplicity-free.

Case 2: $w = a + c - 1$ and $a, d \geqslant 3$.

The diagram $U_2(\lambda/\mu)$ has shape $D_{\alpha/\beta}$ where $\alpha = [a', b, c, d']$ and $\beta = [a' + c - 1, 1]$, and $a' = a - 1$ and $d' = d - 1$. If $a' = 2$ or $d' = 2$ then Case 1 proves the statement. Otherwise, there is some j such that $U_j(\lambda/\mu)$ has shape $D_{\alpha'/\beta'}$ where $\alpha = [a'', b, c, d'']$ and $\beta = [a'' + c - 1, 1]$, and $a'' = 2$ or $d'' = 2$. By Lemma 3.1 and Case 1, $Q_{\lambda/\mu}$ is not Q-multiplicity-free.

Case 3: $3 \leqslant w < a + c - 1$.

Assume $a > 2$. Let (x, y) be the lower corner. Since $w < a + c - 1$, the last box of P_1 is not in the x^{th} row. Then the diagram $U_2(\lambda/\mu)$ has shape $D_{\lambda/\mu}$ where $\lambda' = [a - 1, b, c, d]$ and $\mu' = [w, 1]$. Then there is some j such that $U_j(\lambda/\mu)$ has shape $D_{\alpha'/\beta'}$ where either $\alpha' = [2, b, c, d]$ and $\beta' = [w, 1]$ or where $\alpha' = [e, b, c, 2]$ and $\beta' = [w', 1]$ where $a > e \geqslant 3$ and $w' = e + c - 1$. In the latter case the transpose of the diagram is covered in Case 2. Thus, it suffices to consider the case $D_{\alpha/\beta}$ where $\alpha = [2, b, c, d]$ and $\beta = [w, 1]$ and $3 \leqslant w < 2 + c - 1 = c + 1$.

Case 3.1: $d = 2$.

The box $(w + 1, w + 1)$ is the last box of P_1. We get a new tableau T_2 as follows: In the algorithm of Definition 2.27 use $P_1' := P_1 \setminus \{(w+1, w+1)\}$ and $P_2' := P_2 \setminus \{(w+1, w+2), (w+2, w+2)\}$ instead of P_1 and P_2, respectively. By Corollary 2.25, T_2 is m-amenable for $m \neq 3$. There is a 3 and no 2 in the $(w+1)^{\text{th}}$ column. However, there are 2s and no 3s in the columns of the first two boxes of P_2. Thus, by Lemma 2.24, 3-amenability follows.

We get another tableau T_2' as follows:

- Set $T_2'(r, s) = T_2(r, s)$ for every $(r, s) \in P_1' \cup (P_2' \setminus \{(w, w + 2)\})$ where P_1' and P_2' as above.

- Set $T'_2(w+1, w+1) = 2$.
- Fill the remaining boxes using the algorithm of Definition 2.27 starting with $k = 3$.

By Corollary 2.25, T'_2 is m-amenable for $m \neq 2, 3$. There is a 1 and no 2 in the $(w+2)^{\text{th}}$ column. Thus, by Corollary 2.25, 2-amenability follows. There is a 3 and no 2 in the $(w+2)^{\text{th}}$ column. However, there is a 2 and no 3 in the column of the first box of P_2. We have $T'_2(w+1, w+2) = 3'$ and $T'_2(w, w+1) \neq 2'$. However, there is a 2 and no 3 in the column of the second box of P_2. We have $T'_2(w, w+2) = 3'$ and $T'_2(w-1, w+1) \neq 2'$. However, we have $T'_2(w-1, w+2) = 2'$. Thus, by Lemma 2.24, 3-amenability follows.

We have $|T_2(w+1+j, w+1+j)| = j+3$ and $|T_2(w+j, w+2+j)| = j+2$ for $0 \leqslant j \leqslant n-2$ and we have $|T'_2(w+1+j, w+1+j)| = j+2$ and $|T'_2(w+j, w+2+j)| = j+3$ for $0 \leqslant j \leqslant n-2$. The entries of the other boxes in T_2 and T'_2 can only differ by markings. Thus, T'_2 has the same content as T_2.

Case 3.2: $d > 2$.

Let (x, y) be the lower corner. We get two tableaux $\tilde{T}_2$ and $\tilde{T}'_2$ of shape $D_{\alpha/\beta}$ where $\alpha = [2, b, c, d]$ and $\beta = [w, 1]$ if we take the two tableaux from Case 3.1 of shape $D_{\alpha'/\beta'}$ where $\alpha' = [2, b, c, 2]$ and $\beta' = [w, 1]$ and add $d-2$ columns using the following algorithm (we provide an example right after the algorithm):

1. Set $\tilde{T}_2(e, f) = T_2(e, f)$ and $\tilde{T}'_2(e, f) = T'_2(e, f)$ for all $f \leqslant y$ and for all e such that $(e, f) \in D_{\lambda/\mu}$.
2. Set $\tilde{T}_2(p, q) = T_2(p, q-d+2)$ and $\tilde{T}'_2(p, q) = T'_2(p, q-d+2)$ for all $q > y$ and for all p such that $(p, q) \in D_{\lambda/\mu}$.
3. For $1 \leqslant j \leqslant n$ set $\tilde{T}_2(j, y+1) = \tilde{T}'_2(j, y+1) = j$.
4. For $n+1 \leqslant r \leqslant x-2$ set $\tilde{T}_2(r, y+1) = \tilde{T}'_2(r, y+1) = (n+1)'$.
5. Set $\tilde{T}_2(x-1, y+1) = \tilde{T}'_2(x-1, y+1) = n+1$ and set $\tilde{T}_2(x, y+1) = \tilde{T}'_2(x, y+1) = n+2$.
6. Do the following algorithm:
   (i) Set $i = y+2$:
   (ii) Scan the $(i-1)^{\text{th}}$ column of $\tilde{T}_2$ from top to bottom and find the uppermost marked letter, z say. If there is no marked letter in the $(i-1)^{\text{th}}$ column then set $z = 2+c$.
   (iii) For $1 \leqslant r \leqslant |z|$ set $\tilde{T}_2(r, i) = \tilde{T}'_2(r, i) = r$.
   (iv) For $|z|+1 \leqslant s \leqslant 2+c$ set $\tilde{T}_2(s, i) = \tilde{T}'_2(s, i) = t+1$ if $\tilde{T}_2(s-1, i) = \tilde{T}_2(s-1, i) = t$ or else set $\tilde{T}_2(s, i) = \tilde{T}'_2(s, i) = (t+1)'$ if $\tilde{T}_2(s-1, i) = \tilde{T}_2(s-1, i) = t'$.
   (v) Increment i.
   (vi) If $i \leqslant d-2$ go to (ii) or else stop.

For example, let $a = 2, b = 2, c = 6, d = 4$ and $w = 5$. For $\lambda' = [2, 2, 6, 2]$ and $\mu' = [5, 1]$ obtain the tableaux

$$T_2 = \begin{array}{cccccc} 1' & 1 & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & 2 & 2 \\ 1' & 2' & 3' & 3 & & \\ 1' & 2' & 3' & 4' & & \\ 1 & 2 & 3' & 4' & & \\ 3 & 3 & 3 & 4' & & \\ & 4 & 4 & 4 & & \\ & & 5 & 5 & & \end{array} \quad , \quad T_2' = \begin{array}{cccccc} 1' & 1 & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & 2 & 2 \\ 1' & 2' & 3' & 3 & & \\ 1' & 2' & 3' & 4' & & \\ 1 & 3' & 3 & 4' & & \\ 2 & 3' & 4' & 4 & & \\ & 3 & 4' & 5' & & \\ & & 4 & 5 & & \end{array} \; .$$

By Step 1 and Step 2, we fill the following boxes according to the tableaux for the case $d = 2$:

$$\tilde{T}_2 = \begin{array}{cccccccc} 1' & 1 & 1 & 1 & \square & \square & 1 & 1 \\ 1' & 2' & 2 & 2 & \square & \square & 2 & 2 \\ 1' & 2' & 3' & 3 & \square & \square & & \\ 1' & 2' & 3' & 4' & \square & \square & & \\ 1 & 2 & 3' & 4' & \square & \square & & \\ 3 & 3 & 3 & 4' & \square & \square & & \\ & 4 & 4 & 4 & \square & \square & & \\ & & 5 & 5 & \square & \square & & \end{array} \quad , \quad \tilde{T}_2' = \begin{array}{cccccccc} 1' & 1 & 1 & 1 & \square & \square & 1 & 1 \\ 1' & 2' & 2 & 2 & \square & \square & 2 & 2 \\ 1' & 2' & 3' & 3 & \square & \square & & \\ 1' & 2' & 3' & 4' & \square & \square & & \\ 1 & 3' & 3 & 4' & \square & \square & & \\ 2 & 3' & 4' & 4 & \square & \square & & \\ & 3 & 4' & 5' & \square & \square & & \\ & & 4 & 5 & \square & \square & & \end{array} \; .$$

By Step 3, Step 4 and Step 5 and since $n = 4$ and $x = 8$, we fill the leftmost empty column as follows:

$$\tilde{T}_2 = \begin{array}{cccccccc} 1' & 1 & 1 & 1 & 1 & \square & 1 & 1 \\ 1' & 2' & 2 & 2 & 2 & \square & 2 & 2 \\ 1' & 2' & 3' & 3 & 3 & \square & & \\ 1' & 2' & 3' & 4' & 4 & \square & & \\ 1 & 2 & 3' & 4' & 5' & \square & & \\ 3 & 3 & 3 & 4' & 5' & \square & & \\ & 4 & 4 & 4 & 5 & \square & & \\ & & 5 & 5 & 6 & \square & & \end{array} \quad , \quad \tilde{T}_2' = \begin{array}{cccccccc} 1' & 1 & 1 & 1 & 1 & \square & 1 & 1 \\ 1' & 2' & 2 & 2 & 2 & \square & 2 & 2 \\ 1' & 2' & 3' & 3 & 3 & \square & & \\ 1' & 2' & 3' & 4' & 4 & \square & & \\ 1 & 3' & 3 & 4' & 5' & \square & & \\ 2 & 3' & 4' & 4 & 5' & \square & & \\ & 3 & 4' & 5' & 5 & \square & & \\ & & 4 & 5 & 6 & \square & & \end{array} \; .$$

Note that this new column is filled in the same way in both tableaux. By 6. and since $z = 5'$ we fill the next (and last) empty column as follows:

$$\tilde{T}_2 = \begin{array}{cccccccc} 1' & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & 2 & 2 & 2 & 2 \\ 1' & 2' & 3' & 3 & 3 & 3 & & \\ 1' & 2' & 3' & 4' & 4 & 4 & & \\ 1 & 2 & 3' & 4' & 5' & 5 & & \\ 3 & 3 & 3 & 4' & 5' & 6' & & \\ & 4 & 4 & 4 & 5 & 6 & & \\ & & 5 & 5 & 6 & 7 & & \end{array} \quad , \quad \tilde{T}_2' = \begin{array}{cccccccc} 1' & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1' & 2' & 2 & 2 & 2 & 2 & 2 & 2 \\ 1' & 2' & 3' & 3 & 3 & 3 & & \\ 1' & 2' & 3' & 4' & 4 & 4 & & \\ 1 & 3' & 3 & 4' & 5' & 5 & & \\ 2 & 3' & 4' & 4 & 5' & 6' & & \\ & 3 & 4' & 5' & 5 & 6 & & \\ & & 4 & 5 & 6 & 7 & & \end{array} \; .$$

Note that this new column is filled in the same way since its filling algorithm just relates to the column directly to the left which is also filled in the same way.

It is easy to see that these tableaux are amenable if the tableaux for $d = 3$ are amenable. By definition of the algorithm, if we have $T_2(u, y+1) = T_2'(u, y+1) = (n+1)'$ then $T_2(u-1, y) = T_2'(u-1, y) = n'$. Hence, by Lemma 2.24, these tableaux are amenable.

For $d > 3$, since the $(y+1)^{\text{th}}$ column has the same entries in both tableaux, the algorithm fills the other $d-3$ columns in the same amenable way. Clearly, the contents of $\tilde{T}_2$ and $\tilde{T}_2'$ are equal. □

Example 4.31. For $\lambda = [2, 2, 3, 5]$ and $\mu = [4, 1]$ the diagram $D_{\lambda/\mu}$ satisfies the properties of Case 1 of Lemma 4.30. The two in the proof of this case obtained tableaux are

$$T_1 = \begin{array}{|c|c|c|c|c|c|c|} \hline 1' & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & 2 & 2 & 2 & 2 \\ \hline 1' & 2' & 3' & 3 & 3 \\ \cline{1-5} 1 & 2 & 3 & 4' & 4 \\ \cline{1-5} 3 & 3 & 4 & 4 & 5 \\ \cline{1-5} \end{array}, \quad T_1' = \begin{array}{|c|c|c|c|c|c|c|} \hline 1' & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 1' & 2' & 2 & 2 & 2 & 2 & 2 \\ \hline 1' & 2' & 3' & 3 & 3 \\ \cline{1-5} 1 & 3' & 3 & 4' & 4 \\ \cline{1-5} 2 & 3 & 4 & 4 & 5 \\ \cline{1-5} \end{array}.$$

Corollary 4.32. *Let $\lambda = [a, b]$ and $\mu = [w, x, y, 1]$. If $w \geqslant 2$, $x \geqslant 2$, $b \geqslant 4$ and $a + b - 1 - w - x - y \geqslant 2$ then $Q_{\lambda/\mu}$ is not Q-multiplicity-free.*

Proof. If λ, μ satisfy these properties then the diagram $D^{ot}_{\lambda/\mu}$ is equal to $D_{\alpha/\beta}$ where $\alpha = [a', b', c', d']$ and $\beta = [w', 1]$ where $b' = w \geqslant 2$, $c' = x \geqslant 2$, $d' = y + 1 \geqslant 2$ and additionally $a' + c' - 1 \geqslant w' = b - 1 \geqslant 3$. The number a' is the number of boxes of the first row of $D_{\lambda/\mu}$ and can be calculated by $a' = \lambda_1 - \mu_1 = |B_\lambda| - |B_\mu| = a + b - 1 - w - x - y \geqslant 2$. By Lemma 4.30, $Q_{D^{ot}_{\lambda/\mu}}$ is not Q-multiplicity-free and, thus, $Q_{\lambda/\mu}$ is not Q-multiplicity-free. □

As we will see soon we have already determined all non-Q-multiplicity-free skew Schur Q-functions. The following proposition gives a list of all skew Schur Q-functions that are possibly Q-multiplicity-free. This is half of the classification of Q-multiplicity-free skew Schur Q-functions.

Proposition 4.33. *Let $\lambda, \mu \in DP$ such that $D_{\lambda/\mu}$ is basic. Let $a, b, c, d, w, x, y \in \mathbb{N}$. If $Q_{\lambda/\mu}$ is Q-multiplicity-free then λ and μ satisfy one of the following conditions:*

(i) λ *is arbitrary and* $\mu \in \{\emptyset, (1)\}$,

(ii) $\lambda = [a, b]$ *where* $b \in \{1, 2\}$ *and* μ *is arbitrary,*

(iii) $\lambda = [a, b]$ *and* $\mu = [w, x, y, 1]$ *where* $a + b - w - x - y - 1 = 1$ *or* $w = 1$ *or* $x = 1$ *or* $b \leqslant 3$,

(iv) $\lambda = [a, b, c, d]$ *where* $d \neq 1$ *and* $\mu = [w, 1]$ *where* $1 \in \{a, b, c\}$ *or* $w \leqslant 2$,

(v) $\lambda = [a, b, c, 1]$ *and* $\mu = [w, 1]$ *where* $a \leqslant 2$ *or* $b \leqslant 2$ *or* $c \leqslant 2$ *or* $w \leqslant 3$ *or* $w = a+c-1$.

(vi) $\lambda = [a, b]$ *and* $\mu = [w, x]$ *where* $2 \leqslant b \leqslant 4$ *or* $w \leqslant 2$ *or* $x \leqslant 3$ *or* $a = w + 1$ *or* $a + b - w - x \leqslant 2$.

Some of these cases overlap.

The cases (iii) - (vi) are depicted as diagrams in the remark after the proof of this proposition.

We want to note that Case (i) is the orthogonal transposition of Case (ii). Also, Case (iii) is the orthogonal transposition of Case (iv). Case (v) is the orthogonal transposition of Case (vi) for $x > 1$. The orthogonal transposition of Case (vi) for $x = 1$ is also covered in Case (vi).

Proof. If $\mu = \emptyset, (1)$ we have no restrictions for λ. We also have no restrictions for μ if $\lambda = [a, b]$ where $b \in \{1, 2\}$.

Now assume that $\mu \notin \{\emptyset, (1)\}$ and if $\lambda = [a, b]$, then assume that $b \geqslant 3$. Then by Lemma 4.23, Lemma 4.24 and Lemma 4.25, if $Q_{\lambda/\mu}$ is Q-multiplicity-free then λ and μ satisfy one of the following cases:

- $\lambda = [a, b]$ and $\mu = [w, x]$
- $\lambda = [a, b]$ and $\mu = [w, x, y, 1]$
- $\lambda = [a, b, c, d]$ and $\mu = [w, 1]$

for some $a, b, c, d, w, x, y \in \mathbb{N}$. Note that in the last case if $w \geqslant a + c$ then $\ell(\mu) \geqslant \ell(\lambda)$ and the diagram $D_{\lambda/\mu}$ is either not defined or is not basic since it has an empty column. Hence, we will only consider $w \leqslant a + c - 1$.

By Corollary 4.28, for the case $\lambda = [a, b]$ and $\mu = [w, x]$, we have the restriction $b \leqslant 4$ or $w \leqslant 2$ or $x \leqslant 3$ or $a = w + 1$ or $a + b - w - x \leqslant 2$.

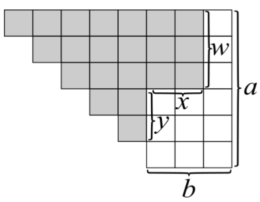
By Corollary 4.32, for the case $\lambda = [a, b]$ and $\mu = [w, x, y, 1]$, we have the restriction $w = 1$ or $x = 1$ or $b \leqslant 3$ or $a + b - w - x - y - 1 = 1$.

By Lemma 4.30, for the case $\lambda = [a, b, c, d]$ where $d \neq 1$ and $\mu = [w, 1]$, we have the restriction $1 \in \{a, b, c\}$ or $w \leqslant 2$.

By Lemma 4.27, for the case $\lambda = [a, b, c, 1]$ and $\mu = [w, 1]$, we have the restriction $a \leqslant 2$ or $b \leqslant 2$ or $c \leqslant 2$ or $w \leqslant 3$ or $w = a + c - 1$. □

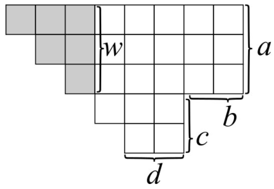
Remark 4.34. The following shows the diagrams in cases (iii) - (vi) of Proposition 4.33; here all boxes of a diagram belong to the diagram of λ and the gray boxes belong to the diagram of μ:

Case 4.33 (iii):



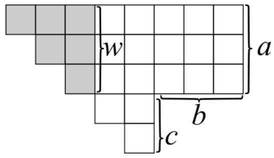
$$a + b - w - x - y - 1 = 1 \text{ or } w = 1 \text{ or } x = 1 \text{ or } b \leqslant 3.$$

Case 4.33 (iv):



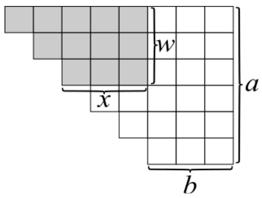
If $d \geqslant 2$ then $1 \in \{a, b, c\}$ or $w \leqslant 2$.

Case 4.33 (v):



$$a \leqslant 2 \text{ or } b \leqslant 2 \text{ or } c \leqslant 2 \text{ or } w \leqslant 3 \text{ or } w = a + c - 1.$$

Case 4.33 (vi):



$$2 \leqslant b \leqslant 4 \text{ or } w \leqslant 2 \text{ or } x \leqslant 3 \text{ or } a = w + 1 \text{ or } a + b - w - x \leqslant 2.$$

To show that the list in Proposition 4.33 gives the classification of Q-multiplicity-free skew Schur Q-functions we have to prove the Q-multiplicity-freeness in each of these cases. We will do this in the following until we are able to state the classification result as Theorem 4.53.

The next lemma shows the Q-multiplicity-freeness of 4.33 (i).

Lemma 4.35. *If λ is arbitrary and $\mu = \emptyset$ then $Q_{\lambda/\mu} = Q_\lambda$ and, thus, $Q_{\lambda/\mu}$ is Q-multiplicity-free.*

If λ is arbitrary and $\mu = (1)$ then

$$Q_{\lambda/\mu} = \sum_{\nu \in E_\lambda} Q_\nu,$$

where E_λ is the set from Definition 2.30. In particular, $Q_{\lambda/\mu}$ is Q-multiplicity-free.

Proof. For $\mu = \emptyset$ we have $Q_{\lambda/\emptyset} = Q_\lambda$. Thus, $f^\lambda_{\emptyset\lambda} = 1$ and $f^\lambda_{\emptyset\nu} = 0$ for $\nu \neq \lambda$. Hence, $Q_{\lambda/\emptyset}$ is Q-multiplicity-free.

The case $\mu = (1)$ is a case of Proposition 2.31. □

Example 4.36. Since $E_{(8,6,5,1)} = \{(7, 6, 5, 1), (8, 6, 4, 1), (8, 6, 5)\}$ we have

$$Q_{(8,6,5,1)/(1)} = Q_{(7,6,5,1)} + Q_{(8,6,4,1)} + Q_{(8,6,5)}.$$

Before showing the Q-multiplicity-freeness of 4.33 (ii) we will define a notation in order to describe the decomposition for a subcase of 4.33 (ii).

Definition 4.37. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}) \in DP$. Let $\mu = (\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_{\ell(\mu)}})$ such that $\{i_1, i_2, \ldots, i_{\ell(\mu)}\} \subseteq \{1, 2, \ldots, \ell(\lambda)\}$. Then $\lambda \setminus \mu$ is defined as the partition obtained by removing the parts of μ from λ.

Example 4.38. For $\lambda = (9, 7, 5, 4, 3, 1)$ and $\mu = (5, 3, 1)$ we obtain $\lambda \setminus \mu = (9, 7, 4)$.

Lemma 4.39. *If $\lambda = [a, b]$ where $b \in \{1, 2\}$ and μ is arbitrary then $Q_{\lambda/\mu}$ is Q-multiplicity-free. In particular, if $\lambda = [a, 1]$ then $Q_{\lambda/\mu} = Q_{\lambda\setminus\mu}$.*

Proof. Case 1: $b = 2$.
Then $D^{ot}_{\lambda/\mu}$ has shape $D_{\alpha/(1)}$ for an $\alpha \in DP$. So, by Lemma 2.36, $Q_{\lambda/\mu} = Q_{D^{ot}_{\lambda/\mu}} = Q_{\alpha/(1)}$. By Lemma 4.35, $Q_{\alpha/(1)}$ is Q-multiplicity-free.
Case 2: $b = 1$.
The Q-multiplicity-freeness as well as the decomposition of $Q_{\lambda/\mu}$ is is proved in [9, Lemma 4.19]. □

Remark 4.40. In [8] there is already a proof for the statement that $Q_{[a,1]/\mu} = Q_\nu$ for some ν. Using Lemma 2.36, this statement follows immediately and Lemma 4.39 helps to obtain this ν.

We postpone to prove the Q-multiplicity-freeness of 4.33 (iii). We will first show the Q-multiplicity-freeness of 4.33 (iv) and then prove that 4.33 (iii) is the orthogonally transposed version of 4.33 (iv).

Lemma 4.41. *Let D be a basic diagram of shape $D_{\lambda/[s,1]}$ for some s. If the first a rows of D form a diagram $D_{\alpha/\beta}$ where $\alpha = [a, b]$ and $\beta = [w, 1]$ then the filling of the boxes of the first a rows of D in any amenable tableau T of D is unique up to marks.*

Proof. Let the diagram be shifted such that the uppermost leftmost box is $(1, 1)$, the uppermost rightmost box is $(1, a+b-w-1)$ and the lowermost rightmost box is the box $(a, a+b-w-1)$. Let T be an amenable tableau of D.

Case 1: $w = a - 1$.

Then the uppermost leftmost box is $(1, 1)$, the uppermost rightmost box is $(1, b)$, the lowermost leftmost box is $(a, 1)$ and the lowermost rightmost box is (a, b). Let $T_{(j)}$ be the subtableau of T consisting of the boxes with their entries in the first j rows. We need to show that $T_{(j)} \cap T^{(i)}$ is a $(j+1-i, b+1-i)$-hook at (i, i) for $1 \leqslant i \leqslant \min\{b, j\}$ where $T^{(i)}$ is as in Definition 2.17.

Case 1.1: $T_{(j)} \cap T^{(1)}$ is not a (j, b)-hook at $(1, 1)$ but $T_{(j-1)} \cap T^{(i)}$ is a $(j-i, b+1-i)$-hook at (i, i) for all $1 \leqslant i \leqslant \min\{b, j-1\}$ for some j.

Then we have $T(j, 1) > 1$. Let $t = T(j, 1)$. For $t \in \{j', j\}$, by Lemma 2.11, all boxes in the j^{th} row are then filled with entries from $\{j', j\}$. The remark after Definition 2.7 implies that $c^{(u)}(T)_j = b \geqslant c^{(u)}(T)_{j-1}$; a contradiction to Lemma 2.10.Thus, we have $1 < t < j'$. Then the last box of $T_{(j-1)} \cap T^{(t)}$ contains a $|t|'$, for otherwise, by the remark after Definition 2.7, we have at least as many $|t|$s as $(|t|-1)$s, which contradicts Lemma 2.10. We have $|T(j,2)| > |t|$. Otherwise, we would have at least as many $|t|$s as $(|t|-1)$s, which contradicts Lemma 2.10.

Repeating this argument, we get $|T(j,s)| > |T(j,s-1)|$ for $2 \leqslant s \leqslant r$ where r is such that $T(j,r+1)$ is the leftmost box with an entry that does not appear in the first $(j-1)^{\text{th}}$ rows.

By Lemma 2.11, $T(j,r+1) \in \{j',j\}$ and, hence $T(j,k) \in \{j',j\}$ for $r+1 \leqslant k \leqslant b$. If $T(j-1,r+1) \notin \{(j-1)',j-1\}$ then the remark after Definition 2.7 implies that $c^{(u)}(T)_j > c^{(u)}(T)_{j-1}$; a contradiction to Lemma 2.10. Hence, we have $T(j-1,r+1) \in \{(j-1)',j-1\}$. If $T(j,r) \notin \{(j-1)',j-1\}$ then, again, the remark after Definition 2.7 implies that $c^{(u)}(T)_j \geqslant c^{(u)}(T)_{j-1}$; a contradiction to Lemma 2.10. If $T(j,r) \in \{(j-1)',j-1\}$ then $T(j-1,r+1) = (j-1)'$. Let $(j,r+1)$ be the box of the l^{th} letter of the reading word $w(T)$. Then $m_{j-1}(n-l) = m_j(n-l)$ and $w(T)_l \in \{j',j\}$, contradicting Definition 2.7 *a*).

Case 1.2: $T_{(j)} \cap T^{(v)}$ is not a $(j+1-v, b+1-v)$-hook at (v,v) but $T_{(j-1)} \cap T^{(i)}$ is a $(j+1-i, b-i)$-hook at (i,i) for all $1 \leqslant i \leqslant \min\{b-1,j\}$ for some j and some minimal $v \leqslant j-1$.

Let j be minimal with respect to this property. By Case 1.1, we may assume that $v > 1$. Let v be minimal with respect to this property. Then we may take $T_{(j)}$, remove $P_1, P_2, \ldots, P_{v-1}$, and replace each entry x by $x-v+1$ for all $x \geqslant v$. In this way, we get a tableau U of shape $D_{\alpha'/\beta'}$ where $\alpha' = [a-v+1, b-v+1]$ and $\beta' = [(a-v+1)-1, 1]$ such that $U_{(j-v+1)} \cap U^{(1)}$ is not a $(j-v+1, b-v+1)$-hook at $(1,1)$; a contradiction to the proven fact that $T_{(j)} \cap T^{(1)}$ is a (j,b)-hook at $(1,1)$ for each $1 \leqslant j \leqslant \min\{a,b\}$ if T is of shape $D_{[a,b]/[a-1,1]}$.

Case 2: $w < a-1$.

The tableau $T_{(w+1)}$ is a tableau of shape $D_{\alpha'/\beta'}$ where $\alpha' = [w+1, a+b-w-1]$ and $\beta' = [w,1]$. Thus, P_1 is a $(a+b-w-1, b)$-hook at $(1,1)$. After removing P_1 and replacing each entry x by $x-1$ and x' by $(x-1)'$ for all $2 \leqslant x \leqslant \ell(c(T))$, we get a tableau of shape $D_{\alpha''/\beta''}$ where $\alpha'' = [a-1, b]$ and $\beta'' = [w,1]$ where $w \leqslant a-2$. Using the same argument, P_2 is a $(w+1, a+b-w-2)$-hook at $(2,2)$.

Repeating this argument, we find that all non-empty P_is are hooks at (i,i) and, therefore, the filling of the boxes of the first k rows of D in any amenable tableau T is unique up to marks. □

Remark 4.42. Since, by the remark after Definition 2.7, each hook $T^{(i)}$ must be fitting, this shows that there is only one amenable filling for diagrams of shape $D_{\lambda/\mu}$ where $\lambda = [a,b]$ and $\mu = [w,1]$. Different proofs of this fact were given by Salmasian [8, Proposition 3.29] and DeWitt [4, Theorem IV.3].

Lemma 4.43. *Let $\lambda = [a,b,1,d]$ and $\mu = [w,1]$. Then $Q_{\lambda/\mu}$ is Q-multiplicity-free.*

Proof. Let the diagram $D = D_{\lambda/\mu}$ be shifted such that the uppermost leftmost box is $(1,1)$. Since case $w=1$ is shown in Lemma 4.35, we only have to show case $w \geqslant 2$. The subdiagram consisting of the first a rows is $D_{\alpha/\beta}$ where $\alpha = [a,p]$ and $\beta = [q,1]$ for some p,q. By Lemma 4.41, it has a unique filling up to marks in the a^{th} row.

Suppose there are two amenable tableaux T_1 and T_2 of D of the same content. Then the difference between these two tableaux are marks since the content of the $(a+1)^{\text{th}}$ row and, therefore, the filling of this row up to marks is determined. Thus, there is a minimal k such that an entry k is in the lowermost row and there is a box (a, k) with entry k' in T_1, say, and with entry k in T_2. Since the k in the $(a+1)^{\text{th}}$ row must be in a column to the left of the k^{th} column, we have $k > 1$. In T_2, if there is no $k-1$ in the $(a+1)^{\text{th}}$ row then $c^{(u)}(T_2)_k = b = c^{(u)}(T_2)_{k-1}$, which is a contradiction to Lemma 2.10. Thus, there is a $k-1$ in the $(a+1)^{\text{th}}$ row in a box to the left of the $(k-1)^{\text{th}}$ column. If there is no $k-2$ in the $(a+1)^{\text{th}}$ row then $c^{(u)}(T_2)_{k-1} = b = c^{(u)}(T_2)_{k-2}$, which is a contradiction to Lemma 2.10. Thus, there is a $k-2$ in the $(a+1)^{\text{th}}$ row in a box to the left of the $(k-2)^{\text{th}}$ column.

Repeating this argument for $k-3, k-4, \ldots 1$, there must be a 1 in a box to the left of the first column; a contradiction. Thus, there are no two amenable tableaux T_1 and T_2 of D of the same content. □

Corollary 4.44. *Let $\lambda = [1, b, c, d]$ and $\mu = [w, 1]$. Then $Q_{\lambda/\mu}$ is Q-multiplicity-free.*

Proof. For each tableau T of shape $D_{\lambda/\mu}$ let R_T be the diagram of the tableau after removing the boxes of $T^{(1)}$. By Lemma 2.11, the boxes of the first row of $D_{\lambda/\mu}$ must have entries from $\{1', 1\}$ and, hence, these boxes (and possibly other ones as well) are removed to obtain R_T. Let T_1 and T_2 be two amenable tableaux of shape $D_{\lambda/\mu}$ such that $R_{T_1} \neq R_{T_2}$. Then T_1 and T_2 cannot have the same content because then $c(T_1)_1 \neq c(T_2)_1$. Thus, $R_T = R_{T_1} = R_{T_2}$ has shape $D_{\alpha/\beta}$ where $\alpha = [c, y]$ and $\beta \in \{[v, 1], [v, 2], [z, 1, v, 1]\}$ for some v and z. If for all T the diagram R_T has no two amenable tableaux of the same content then $Q_{\lambda/\mu}$ is Q-multiplicity-free.

We have $R_T^{ot} = D_{\alpha'/\beta'}$ where $\alpha' = [c+y-v-1, v+1]$ and $\beta' = [y-1, 1]$ for $\alpha = [c, y]$ and $\beta = [v, 1]$. We have $R_T^{ot} = D_{\alpha'/\beta'}$ where $\alpha' = [c+y-v-2, v, 1, 1]$ and $\beta' = [y-1, 1]$ for $\alpha = [c, y]$ and $\beta = [v, 2]$. In addition, we have $R_T^{ot} = D_{\alpha'/\beta'}$ where $\alpha' = [c+y-z-v-2, z, 1, v+1]$ and $\beta' = [y-1, 1]$ for $\alpha = [c, y]$ and $\beta = [z, 1, v, 1]$.

By Lemmas 4.41 and 4.43, in each of these cases R_T^{ot} does not have two amenable tableaux of the same content. Thus, $Q_{\lambda/\mu}$ is Q-multiplicity-free. □

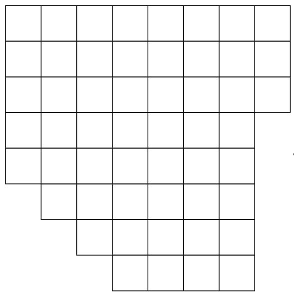
Lemma 4.45. *Let $\lambda = [a, 1, c, d]$, $d \neq 1$ and $\mu = [w, 1]$. Then $Q_{\lambda/\mu}$ is Q-multiplicity-free.*

Proof. Consider $D^{ot}_{\lambda/\mu} = D_{\lambda'/\mu'}$ where $\lambda' = [a+c+d-w, w+1]$ and $\mu' = [1, c, d-1, 1]$. Note that λ' is a partition such that $D_{\lambda'}$ has only one corner. Thus, we have $\lambda' = (a+c+d, a+c+d-1, \ldots, w+1)$ and $\mu' = (c+d, d-1, d-2, \ldots, 1)$.

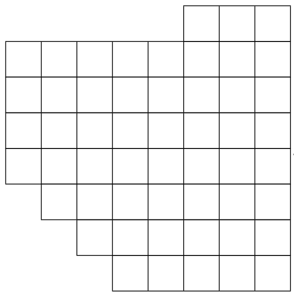
Since $f^{\lambda'}_{\mu'\nu} = f^{\lambda'}_{\nu\mu'}$, we can look at tableaux of shape $D_{\lambda'/\nu}$ and content μ'.

Let T and T' be two different amenable tableaux of shape $D_{\lambda'/\nu}$ and content μ'. By Lemma 2.10, all $2, 3, \ldots, d = \ell(\mu')$ are unmarked. Since d is the largest entry of both tableaux, it must be in a corner in each of the tableaux. Since there is only one corner, say (x, y), we must have $T(x, y) = T'(x, y) = d$. Now we want to find out which boxes can have $d-1$ or $(d-1)'$ as entry. As already mentioned, both $(d-1)$s must be unmarked. Also, in each amenable tableaux at least one $d-1$ must be in the y^{th} column, otherwise the d is scanned before any $d-1$ is scanned and the tableau is not amenable. Thus, we have $T(x-1, y) = T'(x-1, y) = d-1$. The other $d-1$ cannot be in the y^{th} column and there is no other choice for any amenable tableaux than to put it in the lowermost box in the $(y-1)^{\text{th}}$ column. Repeating this argument, we see that the numbers $2, 3, \ldots, d$ are distributed as follows: For $0 \leqslant i \leqslant d-2$ in the $(y-i)^{\text{th}}$ column the lowermost boxes are filled from bottom to top with $d-i, d-i-1, \ldots, 2$. This is fixed for all amenable tableaux of the given shape. To get an amenable tableau there must be an unmarked 1 in each column with a 2 and in at least one column with no 2.

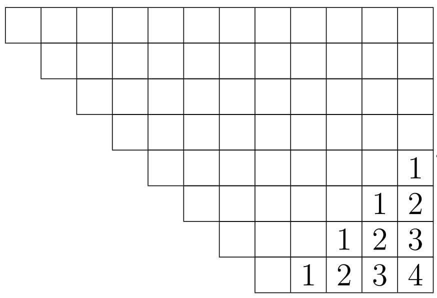
For example, let $\lambda = (12, 11, 10, 8, 7, 6, 5, 4)$ and $\mu = (4, 3, 2, 1)$. Then we have

$D_{\lambda/\mu} =$  .

Since $f^{\lambda}_{\mu\nu} = f^{\lambda'}_{\mu'\nu}$ and $D^{ot}_{\lambda/\mu} = D_{\lambda'/\mu'}$ where $\lambda'/\mu' = (12, 11, 10, 9, 8, 7, 6, 5)/(9, 3, 2, 1)$ we can consider $D_{\lambda'/\mu'}$:

$D_{\lambda'/\mu'} =$  .

Since $f^{\lambda'}_{\mu'\nu} = f^{\lambda'}_{\nu\mu'}$ we can consider amenable tableaux of shape $D_{(12,11,10,9,8,7,6,5)/\nu}$ and content $(9, 3, 2, 1)$. We know fixed entries:

$\tilde{T} =$  .

Now we have five entries from $\{1', 1\}$ left to put into boxes such that we get an amenable tableau. For example we obtain

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
\, & \, & \, & \, & \, & \, & \, & \, & \, & \, & \, & \, \\ \hline
 & \, & \, & \, & \, & \, & \, & \, & \, & \, & \, & \, \\ \hline
 & & \, & \, & \, & \, & \, & \, & \, & \, & \, & 1' \\ \hline
 & & & \, & \, & \, & \, & \, & \, & \, & \, & 1' \\ \hline
 & & & & \, & \, & \, & \, & \, & \, & 1' & 1 \\ \hline
 & & & & & \, & \, & \, & \, & \, & 1 & 2 \\ \hline
 & & & & & & \, & \, & 1' & 1 & 2 & 3 \\ \hline
 & & & & & & & 1 & 1 & 2 & 3 & 4 \\ \hline
\end{array},$$

which is the only tableau of shape $D_{(12,11,10,9,8,7,6,5)/(12,11,9,8,6,5,2)}$ and content $(9,3,2,1)$ which we will prove below. Thus, we have

$$f^{(12,11,10,8,7,6,5,4)}_{(4,3,2,1)(12,11,9,8,6,5,2)} = f^{(12,11,10,9,8,7,6,5)}_{(9,3,2,1)(12,11,9,8,6,5,2)} = f^{(12,11,10,9,8,7,6,5)}_{(12,11,9,8,6,5,2)(9,3,2,1)} = 1.$$

If there are two amenable tableaux of the same shape then they differ only by markings on some 1s. Let (u, v) be such that $T(u, v) = 1'$ and $T'(u, v) = 1$ or vice versa. Then $T(u + 1, v), T'(u + 1, v), T(u, v - 1), T'(u, v - 1) \notin \{1', 1\}$. Thus, either (u, v) is in the lowermost row of the v^{th} column or $T(u + 1, v) = T'(u + 1, v) = 2$. If $T(u + 1, v) = T'(u + 1, v) = 2$ then $T(u, v) = T'(u, v) = 1$ as mentioned above. By the remark after Definition 2.7, the leftmost box of the lowermost row with boxes that are filled with entry from $\{1', 1\}$ must contain a 1. Thus, there is no such box (u, v) and, therefore, there are no two amenable tableaux of the same shape. □

Lemma 4.46. *Let $\lambda = [a, b, c, d]$ and $\mu = [w, 1]$ where $w \leqslant 2$. Then $Q_{\lambda/\mu}$ is Q-multiplicity-free.*

Proof. Case $w = 1$ follows from Lemma 4.35. Thus, consider case $w = 2$. Since $f^{\lambda}_{\mu\nu} = f^{\lambda}_{\nu\mu}$, we may consider tableaux of shape $D_{\lambda/\nu}$ and content $(2, 1)$. There are two amenable words with content $(2, 1)$, namely $w^{(1)} = 121$ and $w^{(2)} = 211$. If $Q_{\lambda/\mu}$ is not Q-multiplicity-free then there must be some ν such that $D_{\lambda/\nu}$ is a diagram with two tableaux T_1 and T_2 where $c(T_1) = w^{(1)}$ and $c(T_2) = w^{(2)}$. If $(x(2), y(2)) = (x(3), y(3) - 1)$ (where $(x(i), y(i))$ is from the part after Definition 2.6) then $T_1(x(2), y(2)) = 2$ and $T_1(x(3), y(3)) = 1$ and T_1 is not a tableau; a contradiction. If $(x(2), y(2)) = (x(3) + 1, y(3))$ then $T_2(x(2), y(2)) = 1$ and $T_2(x(3), y(3)) = 1$ and T_2 is not a tableau; a contradiction. Similarly, we have $(x(1), y(1)) \neq (x(2), y(2) - 1)$ and $(x(1), y(1)) \neq (x(2) + 1, y(2))$. Thus, these three boxes are all in different components consisting of one box. Each component of a diagram has a corner, hence, λ has at least three corners; a contradiction to $\lambda = [a, b, c, d]$. □

Lemma 4.43, Corollary 4.44, Lemma 4.45 and Lemma 4.46 together prove that 4.33 (iv) is Q-multiplicity-free.

Lemma 4.47. *Let $\lambda = [a, b]$ and $\mu = [w, x, y, 1]$ where $w = 1$ or $x = 1$ or $2 \leqslant b \leqslant 3$ or $a + b - w - x - y - 1 = 1$. Then $Q_{\lambda/\mu}$ is Q-multiplicity-free.*

Proof. Let $D = D_{\lambda/\mu}$, where $\lambda = [a, b]$ and $\mu = [w, x, y, 1]$. Then D^{ot} has shape $D_{\alpha/\beta}$ where $\alpha = [a+b-w-x-y-1, w, x, y+1]$ and $\beta = [b-1, 1]$. For each of the given restrictions we have one of the following cases.

Case $w = 1$: Then we have $\alpha = [a+b-x-y-2, 1, x, y+1]$ and Lemma 4.45 proves Q-multiplicity-freeness.

Case $x = 1$: Then we have $\alpha = [a+b-w-y-2, w, 1, y+1]$ and Lemma 4.43 proves Q-multiplicity-freeness.

Case $2 \leqslant b \leqslant 3$: Then we have $\beta = [z, 1]$ where $1 \leqslant z \leqslant 2$ and Lemma 4.46 proves Q-multiplicity-freeness.

Case $a+b-w-x-y-1 = 1$: Then we have $\alpha = [1, w, x, y+1]$ and Corollary 4.44 proves Q-multiplicity-freeness. □

Thus, we have shown that 4.33 (iii) is Q-multiplicity-free by showing that 4.33 (iii) is the orthogonal transpose of 4.33 (iv). Now we will prove the Q-multiplicity-freeness of 4.33 (vi) and afterwards we will show that the orthogonal transpose of 4.33 (v) is included in 4.33 (vi) which means that the last remaining case of Proposition 4.33 is proved to be Q-multiplicity-free.

Lemma 4.48. *Let $\lambda = [a, b, c, 1]$ and $\mu = [w, 1]$ where $a \leqslant 2$. Then $Q_{\lambda/\mu}$ is Q-multiplicity-free.*

Proof. Since case $a = 1$ is shown in Corollary 4.44, we only have to show case $a = 2$. For each tableau T of shape $D_{\lambda/\mu}$ let R_T be the diagram of the remaining tableau after removing the boxes with entry from $\{1', 1, 2', 2\}$. By Lemma 2.11, the first two rows only have entries from $\{1', 1, 2', 2\}$. The boxes with entry from $\{1', 1\}$ form a hook. If the boxes with entry from $\{2', 2\}$ form a border strip all the marks of the entries are determined. If the boxes with entry from $\{2', 2\}$ form a diagram with more than one component then it must have precisely two components. The first component has boxes only in the $(w+1)^{\text{th}}$ column and the second component has boxes in all other columns. In this case the last box of the second component must contain a $2'$ by the remark after Definition 2.7 and by Lemma 2.10. Thus, there are no two tableaux differing just by marks on the entries from $\{1', 1, 2', 2\}$.

If no R_T for any T has two amenable tableaux of the same content then $Q_{\lambda/\mu}$ is Q-multiplicity-free. R_T^{ot} is a diagram of shape $D_{\alpha'}$ for some $\alpha' \in DP$. Such a diagram has only one amenable tableau, namely the one that has just is in the i^{th} row for $1 \leqslant i \leqslant \ell(\alpha')$. Thus, $Q_{\lambda/\mu}$ is Q-multiplicity-free. □

Lemma 4.49. *Let $\lambda = [a, b, c, 1]$ and $\mu = [w, 1]$ where $b \leqslant 2$. Then $Q_{\lambda/\mu}$ is Q-multiplicity-free.*

Proof. Case 1: $b = 1$.

The diagram $D^{ot}_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ where $\alpha = [a+c+1-w, w+1]$ and $\beta = [1, c+1]$. Thus, $\alpha = (a+c+1, a+c, \dots, w+1)$ and $\beta = (c+1)$. Then B_α is a rotated hook and every diagram from $B_\alpha^{(n)}$ is connected. By Proposition 2.31, $Q_{\alpha/\beta} = Q_{\lambda/\mu}$ is Q-multiplicity-free.

Case 2: $b = 2$.

The diagram $D^{ot}_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ where $\alpha = [a+c-w+2, w+1]$ and $\beta = [2, c+1]$. Thus, $\alpha = (a+c+2, a+c+1, \ldots, w+1)$ and $\beta = (c+2, c+1)$. Since $f^{\alpha}_{\beta\nu} = f^{\alpha}_{\nu\beta}$, we can look at amenable tableaux of shape $D_{\alpha/\nu}$ and content $(c+2, c+1)$. The boxes with an entry from $\{2', 2\}$ form a border strip (in fact a rotated hook) where marks are determined. In every column with a box of this border strip there is a box filled with 2. To obtain an amenable tableau in each of these columns there must be a box filled with a 1. Above the uppermost box filled with a 1 there cannot be a box filled with a $1'$. Otherwise, if w is the reading word of this tableau and the uppermost box filled with 1 is $(x(j), y(j))$ then $c+1 = m_2(\ell(w)+j-1) \geqslant m_1(\ell(w)+j-1)$ and $w_j = 1$; a contradiction to the amenability of the tableau.

Suppose we have two amenable tableaux T and T' of shape $D_{\lambda/\nu}$. If there are boxes (x, y) such that $T(x, y) \in \{2', 2\}$ and $T'(x, y) \in \{1', 1\}$ then one of these boxes is either the first or the last box of $T^{(2)}$. But then there is a box (r, s) such that $T(r, s) \in \{1', 1\}$ and $T'(r, s) \in \{2', 2\}$ is the last box or the first box of $T'^{(2)}$, respectively. Without loss of generality we may assume that (x, y) is the first box of $T^{(2)}$. Then $T(x-1, y) = 1$ and $(x-2, y)$ is not part of the diagram. Since $T'(x, y) \in \{1', 1\}$, we have $T'(x-1, y) = 1'$; a contradiction to the fact that there cannot be a box filled with a $1'$ above the uppermost box filled with a 1.

Hence, T and T' differ only by markings on 1s. Let (u, v) be the uppermost rightmost box such that $T'(u, v) = 1'$, say, and $T(u, v) = 1$. Then the boxes $(u+1, v), (u, v-1) \notin T^{(1)} = T'^{(1)}$. Thus, either $(u+1, v) \notin D_{\lambda/\nu}$ or $T(u+1, v) = T'(u+1, v) \in T^{(2)} = T'^{(2)}$. Suppose $T(u+1, v) = T'(u+1, v) \in T^{(2)} = T'^{(2)}$. If we have $(u+1, v) = (x(k), y(k))$ then for $w(T')$ we have $m_2(\ell(w(T')) - k) = m_1(\ell(w(T')) - k)$ and $w_k \in \{2', 2\}$; a contradiction to the amenability of T'. Hence, $(u+1, v) \notin D_{\lambda/\nu}$. By the remark after Definition 2.7, $T^{(1)} = T'^{(1)}$ is fitting. It follows that there is no box (u, v) and, therefore, there are no two amenable tableaux of $D_{\lambda/\nu}$. □

Lemma 4.50. *Let $\lambda = [a, b, c, 1]$ and $\mu = [w, 1]$ where $c \leqslant 2$. Then $Q_{\lambda/\mu}$ is Q-multiplicity-free.*

Proof. Let $n = |D_{\lambda/\mu}|$.

Case 1: $c = 1$.

The only box in the $(a+1)^{\text{th}}$ row is $(a+1, a+1)$. By Lemma 4.41, the filling of the first a rows is unique up to markings. In fact, the filling consists entirely of hooks at the diagonal $\{(s, t) \mid t - s = w\}$. Thus, two different amenable tableaux of the same content differ only by markings. Suppose we have two such tableaux T and T'. Let (y, z) be a box such that $T'(y, z) = k'$, say, and $T(y, z) = k$. Then there must be a box below and to the left of this box with a k. This box is $(a+1, a+1)$ and $y = a$. However, since $T(a, z) = k$, we have $m_{k-1}(n) = m_k(n)$; a contradiction to Lemma 2.10. Thus, there are no two different amenable tableaux of the same content.

Case 2: $c = 2$.

Let T be an amenable tableau of shape $D_{\lambda/\mu}$. By Lemma 4.41, the filling of the first a rows is unique up to markings. In fact, the filling consists entirely of hooks at the diagonal

$\{(s,t) \mid t-s=w\}$. The three boxes below the a^{th} row are $(a+1,a+1)$, $(a+1,a+2)$ and $(a+2,a+2)$.

Case 2.1: $|T(a+1,a+1)| = |T(a+1,a+2)| = k$ for some k.

Then, by Lemma 2.18, we have $|T(a+2,a+2)| > k$. Since $(a,a+1) \in D_{\lambda/\mu}$ we have $k > 1$. If k' or k occur in the first a rows, it follows that $m_k(2n) \geqslant m_{k-1}(2n)$; a contradiction to the amenability of T. Thus, $k = j+1$, where $j = \min\{a, b+3\}$. This is only possible if there are at least three unmarked js, otherwise there is no amenable tableau with these properties. Then $T(a+2,a+2) = k+1 = j+2$ follows and $T(a+1,a+1)$, $T(a+1,a+2)$ and $T(a+2,a+2)$ are unmarked. Additionally, each of the entries in the a^{th} row is unmarked and, therefore, there is no other amenable tableau of the same content.

Case 2.2: $|T(a+1,a+2)| = |T(a+2,a+2)| = k$ for some k.

Since $(a,a+1) \in D_{\lambda/\mu}$ we have $k > 1$. If k' or k occur in the first a rows it follows that $T(a+1,a+1) = k-1$, otherwise $m_k(2n) \geqslant m_{k-1}(2n)$; a contradiction to the amenability of T. Assume there are two different amenable tableaux T and T' of $D_{\lambda/\mu}$ of the same content such that $|T(a+1,a+1)| = |T'(a+1,a+1)| = k-1$, $|T(a+1,a+2)| = |T'(a+1,a+2)| = k$ and $|T(a+2,a+2)| = |T'(a+2,a+2)| = k$. It follows that these tableaux differ only by markings. Then there is some i such that $T'(y,z) = i'$, say, and $T(y,z) = i$. It follows that $y = a$ since the entries in the other rows are determined. It also follows that there is an i in a box which is lower and to the left of (a,z). Thus, we have $i \in \{k-1,k\}$ and, therefore, $k > 2$. If $i = k-1$ then, since $T(a,z) = k-1$, for $w(T)$ we have $m_{k-2}(n) = m_{k-1}(n)$; a contradiction to Lemma 2.10. Hence, we have $i = k$. If $T(a,z-1) = (k-1)'$, then, since $T(a,z) = k$, for $w(T')$ we have $m_{k-1}(n) = m_k(n)$; again a contradiction to Lemma 2.10. If $T(a,z-1) = k-1$, then we have $m_{k-2}(n) = m_{k-1}(n)$; a contradiction to Lemma 2.10 as well. Thus, there are no such two different amenable tableaux of $D_{\lambda/\mu}$.

Case 2.3: $|T(a+1,a+1)| = u$, $|T(a+1,a+2)| = v$ and $|T(a+2,a+2)| = t$ where $u \neq v$, $u \neq t$ and $v \neq t$.

Then we have $u < v < t$. Assume there are two different amenable tableaux T and T' of $D_{\lambda/\mu}$ of the same content in which the boxes $(a+1,a+1)$, $(a+1,a+2)$ and $(a+2,a+2)$ are filled as above. It follows that these tableaux differ only by markings. Then there is some i such that $T'(y,z) = i'$, say, and $T(y,z) = i$. It follows that $y = a$ since the entries in the other rows are determined. It also follows that there is an i in a box which is lower and to the left of the box (a,z). The only possible case is that $i \in \{u,v,t\}$. Arguing as in the cases above, we see that for T we either have $m_{t-1}(n) = m_t(n)$ or $m_{v-1}(n) = m_v(n)$ or $m_{u-1}(n) = m_u(n)$. This contradicts Lemma 2.10.

Hence, there are no such two different amenable tableaux of $D_{\lambda/\mu}$. □

Lemma 4.51. *Let $\lambda = [a,b,c,1]$ and $\mu = [w,1]$ where $w \leqslant 3$ or $w = a+c-1$. Then $Q_{\lambda/\mu}$ is Q-multiplicity-free.*

Proof. Case $w = 1$ follows from Lemma 4.35 and case $w = 2$ follows from Lemma 4.46. For case $w = a+c-1$ the diagram $D^t_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ where $\alpha = [1,c,b,a]$ and $\beta = [b,1]$ and follows from Corollary 4.44. Thus, we only have to prove case $w = 3$.

Since $f^{\lambda}_{\mu\nu} = f^{\lambda}_{\nu\mu}$, we just can look at tableaux of shape $D_{\lambda/\nu}$ and content $\mu = (3, 2, 1)$. By Lemma 2.10, all entries must be unmarked. Assume there are two different amenable tableaux T_1, T_2 of $D_{\lambda/\nu}$ with content μ for some $\nu \in DP$. Thus, we get one tableau from the other by interchanging some entries in certain boxes.

Suppose the 3 is in one of these boxes. Let (a, x) be the upper corner (where $x = a+b+c$) and let (e, e) be the lower corner (where $e = a+c$). Since the 3 is the greatest entry it must be either in (a, x) or in (e, e). Thus, we have $T_1(a, x) = 3$, say, and $T_2(e, e) = 3$. Then, by Lemma 2.11 and since T_1 is amenable, we have $a \geqslant 3$, $T_1(a-1, x) = 2$ and $T_2(a-2, x) = 1$. We have $T_2(a, x) \in \{1, 2\}$. Either way, since all entries are unmarked, we have $T_2(a-2, x) \leqslant T_2(a-1, x) - 1 \leqslant T_2(a, x) - 2$ and, hence, $T_2(a-2, x) \notin \{1, 2, 3\}$. Thus, either $T_1(a, x) = T_2(a, x) = 3$ or $T_1(e, e) = T_2(e, e) = 3$.

Suppose $T_1(a, x) = T_2(a, x) = 3$. Then $T_1(a-1, x) = T_2(a-1, x) = 2$ and $T_1(a-2, x) = T_2(a-1, x) = 1$. Thus, T_1 and T_2 differ only by interchanging one 1 and one 2. Let the boxes containing these entries be (f, t) and (v, g), where $g > t$ and $v < f$. The remaining 1 must be in a box to the right and above (v, g). If $T_1(a-1, x-1) = T_2(a-1, x-1) = 1$ then $T_1(a, x-1) = T_2(a, x-1) = 2$ and both tableaux are the same; a contradiction. Thus, we have $T_1(a, x-1) = T_2(a, x-1) = 1$. The remaining entries must be in two corners below $(a, x-1)$. However, there is only one corner (namely (e, e)), thus, there are no two different amenable tableaux such that $T_1(a, x) = T_2(a, x) = 3$. Therefore, we have $T_1(e, e) = T_2(e, e) = 3$.

Suppose $T_1(a, x) = 1$. Then $T_1(e-1, e) = T_1(e-1, e-1) = 2$ and after inserting the 1s the tableau is determined. Thus, if $T_1(a, x) = 1$, there are no two different amenable tableaux.

Therefore, $T_1(a, x) = T_2(a, x) = 2$. Since T_1 and T_2 are amenable, $T_1(a-1, x) = T_2(a-1, x) = 1$. Thus, T_1 and T_2 differ only by interchanging one 1 and one 2. With the same argument as above we see that $T_1(a, x-1) = T_2(a, x-1) = 1$. Then we have $T_1(e-1, e) = T_2(e-1, e) = 2$ and both tableaux are the same; a contradiction. Thus, there are no two different amenable tableaux of shape $D_{\lambda/\nu}$ and content $\mu = (3, 2, 1)$. □

The Lemmas 4.48, 4.49, 4.50 and 4.51 all together show that 4.33 (v) is Q-multiplicity-free.

Lemma 4.52. *Let $\lambda = [a, b]$ and $\mu = [w, x]$ where $2 \leqslant b \leqslant 4$ or $w \leqslant 2$ or $2 \leqslant x \leqslant 3$ or $a = w + 1$ or $a + b - w - x \leqslant 2$. Then $Q_{\lambda/\mu}$ is Q-multiplicity-free.*

Proof. The diagram $D^{ot}_{\lambda/\mu}$ has shape $D_{\alpha/\beta}$ where $\alpha = [a + b - w - x, w, x - 1, 1]$ and $\beta = [b - 1, 1]$. For each of the given restrictions we have one of the following cases.

Case $2 \leqslant b \leqslant 4$: Then we have $\beta = [w', 1]$ where $w' \leqslant 3$ and Lemma 4.51 proves Q-multiplicity-freeness.

Case $w \leqslant 2$: Then we have $\alpha = [a', b', c', 1]$ where $b' \leqslant 2$ and Lemma 4.49 proves Q-multiplicity-freeness.

Case $2 \leqslant x \leqslant 3$: Then we have $\alpha = [a', b', c', 1]$ where $c' \leqslant 2$ and Lemma 4.50 proves Q-multiplicity-freeness.

Case $a = w + 1$: Then we have $\alpha = [a', b', c', 1]$ and $\beta = [w', 1]$ where we have $a' = a+b-w-x = b-x+1$ and, hence, $w' = b-1 = (b-x+1)+(x-1)-1 = a'+c'-1$ and Lemma 4.51 proves Q-multiplicity-freeness.

Case $a+b-w-x \leqslant 2$: Then we have $\alpha = [a', b', c', 1]$ where $a' \leqslant 2$ and Lemma 4.48 proves Q-multiplicity-freeness. □

We have now proven that all the skew Schur Q-functions occurring in Proposition 4.33 are indeed Q-multiplicity-free, and hence we are now able to state this result as our final classification theorem.

Theorem 4.53. *Let $\lambda, \mu \in DP$ and $a, b, c, d, w, x, y \in \mathbb{N}$ such that $D_{\lambda/\mu}$ is basic. $Q_{\lambda/\mu}$ is Q-multiplicity-free if and only if λ and μ satisfy one of the following conditions:*

(i) λ *is arbitrary and* $\mu \in \{\emptyset, (1)\}$,

(ii) $\lambda = (a+b-1, a+b-2, \ldots, b)$ *where* $b \in \{1, 2\}$ *and* μ *is arbitrary,*

(iii) $\lambda = (a+b-1, a+b-2, \ldots, b)$ *and* $\mu = (w+x+y, w+x+y-1, \ldots, x+y+2, x+y+1, y, y-1, \ldots, 1)$ *where* $w = 1$ *or* $x = 1$ *or* $b \leqslant 3$ *or* $a+b-w-x-y-1 = 1$,

(iv) $\lambda = (a+b+c+d-1, a+b+c+d-2, \ldots, b+c+d+1, b+c+d, c+d-1, c+d-2, \ldots, d)$ *where* $d \neq 1$ *and* $\mu = (w, w-1, \ldots, 1)$ *where* $1 \in \{a, b, c\}$ *or* $w \leqslant 2$,

(v) $\lambda = (a+b+c, a+b+c-1, \ldots, b+c+2, b+c+1, c, c-1, \ldots, 1)$ *and* $\mu = (w, w-1, \ldots, 1)$ *where* $a \leqslant 2$ *or* $b \leqslant 2$ *or* $c \leqslant 2$ *or* $w \leqslant 3$ *or* $w = a+c-1$,

(vi) $\lambda = (a+b-1, a+b-2, \ldots, b)$ *and* $\mu = (w+x-1, w+x-2, \ldots, x)$ *where* $2 \leqslant b \leqslant 4$ *or* $w \leqslant 2$ *or* $x \leqslant 3$ *or* $a = w+1$ *or* $a+b-w-x \leqslant 2$.

Some of these cases overlap.

Proof. Using the shape path notation of Definition 4.19 we have:

- 4.53 (ii) is the case $\lambda = [a, b]$ where $b \in \{1, 2\}$ and μ is arbitrary.
- 4.53 (iii) is the case $\lambda = [a, b]$ and $\mu = [w, x, y, 1]$ where $w = 1$ or $x = 1$ or $b \leqslant 3$ or $a+b-w-x-y-1 = 1$.
- 4.53 (iv) is the case $\lambda = [a, b, c, d]$ such that $d \neq 1$ and $\mu = [w, 1]$ where $1 \in \{a, b, c\}$ or $w \leqslant 2$.
- 4.53 (v) is the case $\lambda = [a, b, c, 1]$ and $\mu = [w, 1]$ where $a \leqslant 2$ or $b \leqslant 2$ or $c \leqslant 2$ or $w \leqslant 3$ or $w = a+c-1$.
- 4.53 (vi) is the case $\lambda = [a, b]$ and $\mu = [w, x]$ where $2 \leqslant b \leqslant 4$ or $w \leqslant 2$ or $x \leqslant 3$ or $a = w+1$ or $a+b-w-x \leqslant 2$.

By Proposition 4.33, only these cases can be Q-multiplicity-free. Lemma 4.35 states that 4.53 (i) is Q-multiplicity-free. Lemma 4.39 states that 4.53 (ii) is Q-multiplicity-free. Lemmas 4.43, 4.45 and 4.46 and Corollary 4.44 state that 4.53 (iv) is Q-multiplicity-free. Lemma 4.47 states that 4.53 (iii) is Q-multiplicity-free. Lemmas 4.48, 4.49, 4.50 and 4.51 state that 4.53 (v) is Q-multiplicity-free. Lemma 4.52 states that 4.53 (vi) for $x \neq 1$ is Q-multiplicity-free. Lemma 4.41 states that for 4.53 (vi) for $x = 1$ we have $Q_{\lambda/\mu} = Q_\alpha$ for some α (see the remark after Lemma 4.41). Hence, 4.53 (vi) for $x = 1$ is Q-multiplicity-free. Thus, all cases in Theorem 4.53 are Q-multiplicity-free. □

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