Latin squares with no transversals

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Abstract

A k-plex in a latin square of order n is a selection of kn entries that includes k representatives from each row and column and k occurrences of each symbol. A 1-plex is also known as a transversal.

It is well known that if n is even then B_n, the addition table for the integers modulo n, possesses no transversals. We show that there are a great many latin squares that are similar to B_n and have no transversal. As a consequence, the number of species of transversal-free latin squares is shown to be at least $n^{n^{3/2}(1/2-o(1))}$ for even $n \to \infty$.

We also produce various constructions for latin squares that have no transversal but do have a k-plex for some odd $k > 1$. We prove a 2002 conjecture of the second author that for all even orders $n > 4$ there is a latin square of order n that contains a 3-plex but no transversal. We also show that for odd k and $m \geqslant 2$, there exists a latin square of order $2km$ with a k-plex but no k'-plex for odd $k' < k$.

Keywords: latin square; transversal; plex; triplex;

1 Introduction

A $k \times n$ latin rectangle is an array containing n symbols such that each symbol occurs once in each row and at most once in each column. A latin square of order n is an $n \times n$ latin rectangle. We index rows and columns by the set $N_n = \{0, 1, \ldots, n-1\}$ and also use N_n for our symbols. A latin square may then be specified as a set of ordered (row, column, symbol) triples called *entries*. Each entry is an element of $N_n \times N_n \times N_n$. This viewpoint allows a natural action of the wreath product $\mathcal{S}_n \wr \mathcal{S}_3$ on the latin squares of

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order n, where $\mathcal{S}_n$ denotes the symmetric group of degree n. Orbits under this action are known as *species* (sometimes also called *main classes*).

A k-plex of a latin square of order n is a selection of kn entries such that exactly k entries are chosen from each row and each column, and each symbol is chosen k times. A 1-plex is also known as a *transversal* and a 3-plex is known as a triplex. A k-plex is said to be an *odd plex* if k is odd. See [7] for a survey on transversals and plexes more generally.

One of our goals is to show that there are very many transversal-free Latin squares for each even order. We do this in the next section. Our second major goal is to prove the following conjecture from [6], which we do in §4.

Conjecture 1. *For all even $n > 4$ there is a latin square of order n that contains a triplex but no transversal.*

It is clear that this conjecture cannot extend to $n = 4$, since the complement of a transversal is always an $(n-1)$-plex. Part of the motivation for studying plexes comes from their use in creating orthogonal partitions in experiment design. Basic existence questions for plexes seem to be very difficult. It has been conjectured in [6] that every latin square of order n has a k-plex for every even $k \leqslant n$. There seems to be more diversity regarding existence of odd plexes, and our result adds to the possibilities that are known to occur.

The following lemma will be crucial to our work. It is one variant of a result known as the Delta lemma which has been employed in several papers including [2, 3]. See [7] for a discussion of other applications.

Lemma 1. *Let L be a latin square of even order n indexed by N_n. Suppose that m is an odd divisor of n. We define a function Δ_m on the entries of L by specifying that $\Delta_m(r,c,s)$ is the integer of least absolute value that satisfies*

$$\Delta_m(r,c,s) \equiv \left\lfloor \frac{s}{m} \right\rfloor - \left\lfloor \frac{r}{m} \right\rfloor - \left\lfloor \frac{c}{m} \right\rfloor \mod \frac{n}{m}.$$

Let k be an odd positive integer. If K is a k-plex of L then,

$$\sum_{(r,c,s)\in K} \Delta_m(r,c,s) \equiv \frac{n}{2m} \mod \frac{n}{m}.$$

Proof. By definition,

$$\begin{aligned}
\sum_{(r,c,s)\in K} \Delta_m(r,c,s) &\equiv k\sum_{s=0}^{n-1} \left\lfloor \frac{s}{m} \right\rfloor - k\sum_{r=0}^{n-1} \left\lfloor \frac{r}{m} \right\rfloor - k\sum_{c=0}^{n-1} \left\lfloor \frac{c}{m} \right\rfloor \\
&\equiv -km \sum_{i=0}^{n/m-1} i = -km\left(\frac{n}{m} - 1\right)\frac{n}{2m} \\
&\equiv \frac{n}{2m} \mod \frac{n}{m},
\end{aligned}$$

since km is odd and n/m is even. $\square$

Most applications simply use $m = 1$, but we will need the more general version in the next section. Also, the requirement that Δ_m has the least absolute value in its residue class will be vital in inequalities throughout the paper.

The number of transversals is invariant within a species. We call a latin square *transversal-free* if it possesses no transversals. Define B_n to be the addition table for the integers modulo n. It is immediate from Lemma 1 (using $m = 1$) that B_n is transversal-free when n is even. In turns out that many latin squares that resemble B_n are also transversal-free. We will use two results on this theme. One is a new result that we prove in the next section. The other is the following classical result due to Maillet [5], which was generalised from transversals to all odd plexes in [6].

Lemma 2. *Let $n = bm$ where b is even and m is odd. Let $L = [L_{ij}]$ be a latin square indexed by N_n. If*

$$\lfloor L_{ij}/m \rfloor \equiv \lfloor i/m \rfloor + \lfloor j/m \rfloor \mod b \tag{1}$$

for all $i, j \in N_n$, then L has no odd plexes.

Lemma 2 is an immediate consequence of Lemma 1. Latin squares that satisfy (1) for all $i, j \in N_n$ are sometimes said to be of *"step-type"*. Part of the original motivation for Conjecture 1 was to find a family of latin squares that are transversal-free but are structurally different to step-type latin squares. That goal was achieved in [2], but without proving the original conjecture. It was shown that there are latin squares that have k-plexes for some odd k but not for any small odd k. By proving Conjecture 1, we demonstrate yet another possible structure.

The other background result that we need is the following, from [4, p.186]:

Lemma 3. *Let R be a $k \times n$ latin rectangle. The number of $n \times n$ latin squares obtained by adding rows to R is at least*

$$\prod_{i=k}^{n-1} n!(1 - i/n)^n = n!^{\,n-k}(n-k)!^{\,n}/n^{n(n-k)}.$$

2 Number of latin squares with no transversals

It was famously conjectured by Ryser (see [7] for the history of this conjecture) that all latin squares of odd order have transversals. Our aim in this section is to show that there are a great many different species of latin squares of even order that have no transversals.

Theorem 1. *Let $n = 2^a m$ for positive integers a and m, where m is odd. There are at least $(m/e^2)^{n^2}$ latin squares of order n that have no odd plexes.*

Proof. By Lemma 2, we can construct an order n latin square with no odd plexes by patching together 2^{2a} latin subsquares of order m. For each subsquare we have at least $m!^{\,2m}/m^{m^2}$ choices, by Lemma 3, and these choices can be made independently. The result now follows from Stirling's approximation, given that $m! > (m/e)^m$ for all $m \geqslant 1$. $\square$

Corollary 2. *For $m \to \infty$ with fixed $a \geqslant 1$, the number of species of transversal-free latin squares of order $n = 2^a m$ is at least $n^{n^2(1-o(1))}$.*

Proof. The number of transversal-free latin squares is at least $(cn)^{n^2}$ for the constant $c = 2^{-a}e^{-2} > 0$. The result now follows, since the number of latin squares in each species is at most $6(n!)^3 = n^{O(n)}$. □

Of course, Theorem 1 does not tell us much for orders that are powers of 2. This is unavoidable because Lemma 2 only applies to one species of such an order, namely the species containing B_n. Our next result will allow us to show that there are many species of transversal-free latin squares for *all* even orders.

Theorem 3. *Let $n = bm$ where b is even and m is odd. Let r be a nonnegative integer and k an odd positive integer. Suppose L is any latin square of order n indexed by N_n, which satisfies* (1) *for all $j \in N_n$ and $0 \leqslant i < n - mr$. If $km^2r(r-1) < n$ then L has no k-plexes.*

Proof. Suppose that K is a k-plex of L. For $0 \leqslant i < n$, let $\{e_{i,a} : 0 \leqslant a < k\}$ be the set of entries in K from the i-th row of L. By assumption, $\Delta_m(e_{i,a}) = 0$ whenever $0 \leqslant i < n - mr$. Now consider some $e_{i,a} = (i, c, s)$ where $i \geqslant n - mr$. Given that s cannot match any of the symbols which are used in the first $n - mr$ rows of column c, we know that $b - r - \lfloor i/m \rfloor \leqslant \Delta_m(e_{i,a}) \leqslant b - 1 - \lfloor i/m \rfloor$. It follows that

$$\left|\sum_{i=0}^{n-1}\sum_{a=0}^{k-1}\Delta_m(e_{i,a})\right| \leqslant \sum_{a=0}^{k-1}\left|\sum_{i=n-mr}^{n-1}\Delta_m(e_{i,a})\right| \leqslant km\sum_{j=0}^{r-1} j = kmr(r-1)/2.$$

The result now follows from Lemma 1. □

In particular, when a step-type latin square of order n is transversal-free, this property is quite robust in the sense that no transversal will be introduced by arbitrarily changing any (roughly) $\sqrt{n}$ consecutive rows. Putting $k = m = 1$ in Theorem 3, we see that:

Corollary 4. *For even n there are no transversals in any latin square which agrees with B_n outside of some set of $\lfloor\sqrt{n}\rfloor$ consecutive rows.*

Corollary 5. *For even $n \to \infty$, there are at least $n^{n^{3/2}(1/2-o(1))}$ species of transversal-free latin squares of order n.*

Proof. Let $s = \lfloor\sqrt{n}\rfloor$. By Lemma 3 and Stirling's approximation, there are at least $(s!)^n(n!)^s/n^{sn} = n^{n^{3/2}(1/2-o(1))}$ ways to complete the first $n - s$ rows of B_n to a latin square. Again, when we divide by $n^{O(n)}$, the maximum number of latin squares in a species, this factor gets absorbed in the error term. □

For $m = 1$ and $r \leqslant 2$, Theorem 3 is not useful since in this case the only way to change r consecutive rows is to permute them, which does not change the species. Hence the smallest case when Theorem 3 is interesting is when $n = 8$ and $r = 3$. There are 264 latin squares that agree with B_8 in the first 5 rows, and these fall in 9 distinct transversal-free species. Representatives of the 8 species other than B_8 may be defined by specifying the non-zero values of the Δ_1 function in the last three rows:

$$\begin{pmatrix} \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -2 & \cdot & -2 & \cdot & -2 & \cdot & -2 \end{pmatrix} \begin{pmatrix} \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & 2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & -1 \\ \cdot & -2 & \cdot & -2 & \cdot & -2 & -1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \cdot & 2 & \cdot & 2 & \cdot & 2 & \cdot & 2 \\ \cdot & \cdot & 1 & -1 & \cdot & \cdot & 1 & -1 \\ \cdot & -2 & -1 & -1 & \cdot & -2 & -1 & -1 \end{pmatrix} \begin{pmatrix} \cdot & 2 & \cdot & 1 & 2 & \cdot & 1 & 2 \\ \cdot & \cdot & \cdot & 1 & -1 & \cdot & 1 & -1 \\ \cdot & -2 & \cdot & -2 & -1 & \cdot & -2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \cdot & 2 & \cdot & 1 & 2 & \cdot & 1 & 2 \\ \cdot & \cdot & \cdot & 1 & -1 & 1 & -1 & \cdot \\ \cdot & -2 & \cdot & -2 & -1 & -1 & \cdot & -2 \end{pmatrix} \begin{pmatrix} \cdot & 2 & \cdot & 1 & 2 & \cdot & 1 & 2 \\ \cdot & \cdot & 1 & -1 & \cdot & \cdot & 1 & -1 \\ \cdot & -2 & -1 & \cdot & -2 & \cdot & -2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \cdot & 2 & \cdot & 1 & 1 & 1 & 1 & 2 \\ \cdot & \cdot & \cdot & 1 & -1 & 1 & -1 & \cdot \\ \cdot & -2 & \cdot & -2 & \cdot & -2 & \cdot & -2 \end{pmatrix} \begin{pmatrix} \cdot & 2 & \cdot & 1 & 1 & 2 & \cdot & 2 \\ \cdot & \cdot & \cdot & 1 & -1 & \cdot & 1 & -1 \\ \cdot & -2 & \cdot & -2 & \cdot & -2 & -1 & -1 \end{pmatrix}$$

It is clear from Theorem 3 that there are a great many species of transversal-free latin squares of even order. The true number is still far from known, but we would expect it to be negligible compared to the number of all latin squares.

3 Latin squares with an odd plex but no transversal

The remainder of the paper is devoted to constructions of latin squares that have no transversal but do have k-plexes for at least one odd value of k. Our strategy will always be to start with B_n (which has no odd plexes by Lemma 1). In B_n we will locate a structure J which is close to being a k-plex. Then we will alter B_n slightly and in the process relocate a few entries from J in order to make it into a k-plex. Since we always begin with B_n it is convenient to define a notation $(x; y)$ to be the triple $(x, y, z) \in N_n \times N_n \times N_n$ for which $z \equiv x + y \mod n$. We also adopt the convention that the result of all calculations for indices will be reduced mod n to an element of N_n.

Our method for changing B_n will be to use the well known theory of latin trades (see, for example, the survey [1]). A *latin trade* in B_n is a subset Q of the entries of B_n that can be removed and replaced by a disjoint set Q' of entries to produce a new latin square. The set Q' is known as the *disjoint mate* for Q. The sets Q, Q' are sets of entries (triples) with the property that $\pi(Q) = \pi(Q')$ for any of the three projections π onto two coordinates. Checking that our latin trades and their mates have this property will be left as a routine exercise to the reader.

For the remainder of this section, k is odd and $n = 2km$ for some integer $m \geqslant 2$. Our aim is to establish the existence of latin squares of order n which contain a k-plex but no smaller odd plexes.

We start by identifying a set of entries inside B_n, which we denote by J. We let $J = J_0 \cup J_1 \cup J_2 \cup J_3$ where:

$$
\begin{aligned}
J_0 &= \big\{(i; 2jm + i - 1) : 1 \leqslant i \leqslant m,\ 0 \leqslant j \leqslant k-1\big\} \\
J_1 &= \big\{(i; 2jm + i) : m \leqslant i \leqslant 2m-1,\ 0 \leqslant j \leqslant k-1\big\} \\
J_2 &= \big\{(2m(2\ell - 1) + i; 2jm + i) : 0 \leqslant i \leqslant 2m-1,\ 0 \leqslant j \leqslant k-1,\ 1 \leqslant \ell \leqslant (k-1)/2\big\} \\
J_3 &= \big\{(4m\ell + i; 2jm + i + 1) : 0 \leqslant i \leqslant 2m-1,\ 0 \leqslant j \leqslant k-1,\ 1 \leqslant \ell \leqslant (k-1)/2\big\}.
\end{aligned}
$$

Below we exhibit J when $k = m = 3$.

1						7						13					
	3						9						15				
		5	6					11	12					17	0		
				8						14						2	
					10						16						4
6						12						0					
	8						14						2				
		10						16						4			
			12						0						6		
				14						2						8	
					16						4						10
	13						1						7				
		15						3						9			
			17						5						11		
				1						7						13	
					3						9						15
17						5						11					

Lemma 4. *Each column of B_n contains precisely k elements of J. Each symbol in N_n appears in precisely k elements of J. Each row of B_n contains precisely k elements of J, except for the first row which contains no elements and row m which contains $2k$ elements.*

Proof. The claim about rows is straightforward to check. Each column appears once in $J_0 \cup J_1$, $(k-1)/2$ times in J_2 (once for each choice of ℓ) and $(k-1)/2$ times in J_3 (again, once for each choice of ℓ). Each even symbol occurs once in J_1 and $k-1$ times in J_2 (twice for each choice of ℓ). Each odd symbol occurs once in J_0 and $k-1$ times in J_3 (twice for each choice of ℓ). $\square$

Our aim is to find a latin trade in B_n which allows us to shift precisely k elements of J from row m to row 0 without making further changes to J. This will allow us to show:

Theorem 6. *Let k be odd and $m \geqslant 2$. Then there exists a latin square of order $n = 2km$ with a k-plex but no k'-plex for odd $k' < k$.*

Proof. Observe that $\{(0; jm), (m; jm) : 0 \leqslant j < 2k\}$ defines a latin trade T in B_n. The disjoint mate T' of T is formed by swapping the two symbols in each column of T. Moreover, $T \cap J = \{(m; m + 2mj) : 0 \leqslant j \leqslant k - 1\}$ since $m \geqslant 2$.

Thus replacing T with T' in B_n has the effect of shifting k entries of J from row m to row 0. From Lemma 4, $L' := (B_n \setminus T) \cup T'$ contains a k-plex.

Suppose that L' contains a k'-plex K for some odd k' such that $k' < k$. For each $(r, c, s) \in T'$, $\Delta_1(r, c, s) = m$ if $r = 0$ and $\Delta_1(r, c, s) = -m$ if $r = m$. Thus

$$\left| \sum_{(r,c,s)\in K} \Delta_1(r, c, s) \right| \leqslant mk' < n/2.$$

Hence by Lemma 1, there is no k'-plex in L'. □

In the extreme case $m = 2$, the previous theorem implies the existence of a latin square of order $4k$ with a k-plex but no smaller odd plexes; such a structure was first shown to exist in [2]. Note that the k-plex constructed in Theorem 6 is necessarily indivisible in the sense that it cannot be partitioned into two or more smaller plexes. In [3], it was shown that for all $n \notin 2, 6$, if k is any proper divisor of n then there exists a latin square of order n that can be partitioned into indivisible k-plexes. That result is in the spirit of Theorem 6, though neither implies the other.

4 Latin squares with a triplex but no transversal

In this section we prove Conjecture 1. Our proof splits into several subcases. Recall that B_n has no odd plexes, by Lemma 1. In each of several cases we will show that it is possible to change a small number of the entries of B_n so that a triplex is created and yet there are still no transverals.

First we construct transversal-free latin squares that have a triplex, for certain small orders that are missed by the general constructions that we will subsequently give. The examples were found by asking a computer to complete a partial triplex in a specific latin square L_n. The construction of L_n is as follows. Let $n = 4m+2$. For $i, j \in \{0, 1, \dots, n-1\}$, define

$$L_n[i, j] \bmod n \equiv \begin{cases} i + j + 1 & \text{if } n - 1 - m \leqslant i \leqslant n - 2 \text{ and } j \in \{m, 3m + 1\}, \\ i + j - 1 & \text{if } n - m \leqslant i < n \text{ and } j \in \{m + 1, 3m + 2\}, \\ i + j + m & \text{if } i = n - 1 - m \text{ and } j \in \{0, m + 1, 2m + 1, 3m + 2\}, \\ i + j - m & \text{if } i = n - 1 \text{ and } j \in \{0, m, 2m + 1, 3m + 1\}, \\ i + j & \text{otherwise.} \end{cases}$$

Theorem 7. *For $n \in \{10, 14, 18, 22, 26, 34, 38, 46, 50, 62\}$, there is a triplex but no transversal in L_n.*

Proof. Let $h = n/2$ and $m = (n - 2)/4$. Suppose that T is a transversal of L_n. By Lemma 1 the sum, S, of the Δ_1 function over T must be $h \mod n$. However T can have at most one entry in row $n-1-m$ and at most one entry in each of columns $m, 3m+1$. It follows that $S \leqslant m+2 < 2m+1 = h$. Similarly, T can have at most one entry in row $n-1$ and at most one entry in each of columns $m+1, 3m+2$, so $S > -h$. It follows that $S \not\equiv h \mod n$, so L_n has no transversal.

Next, we specify a triplex P in L_n as follows. We start by choosing the cells in columns $i, i+h-1, i+h$ of row i for $0 \leqslant i < h$. Next we choose the cells in columns $3(i-h), 3(i-h)+1, 3(i-h)+2$ of row i for $h \leqslant i < h + \lfloor n/6 \rfloor$. For the rows with index $h + \lfloor n/6 \rfloor$ to $n-1$ (in that order) we list the column indices of the cells to choose in E_n, where E_n is as follows:

$$E_{10} = [[6,7,9],[0,5,8],[1,2,3],[3,4,9]],$$

$$E_{14} = [[6,10,11],[0,4,13],[7,8,13],[3,5,9],[1,2,12]],$$

$$E_{18} = [[1,13,14],[0,5,10],[11,15,16],[6,9,12],[4,7,17],[2,3,17]],$$

$$E_{22} = [[17,18,21],[14,15,16],[12,20,21],[0,8,9],[1,7,9],[4,6,10],[2,3,13],[5,11,19]],$$

$$\begin{aligned}E_{26} = [&[19,20,25],[13,16,21],[14,18,22],[9,15,23],[2,11,17],[8,24,25],\\ &[1,10,12],[3,4,5],[0,6,7]],\end{aligned}$$

$$\begin{aligned}E_{34} = [&[2,25,26],[27,28,32],[23,29,33],[20,21,24],[13,17,19],[12,14,15],\\ &[10,15,16],[3,6,11],[1,7,33],[18,30,31],[4,5,22],[0,8,9]],\end{aligned}$$

$$\begin{aligned}E_{38} = [&[18,19,20],[21,23,25],[26,28,29],[0,10,27],[30,31,32],[33,34,35],\\ &[2,36,37],[3,22,37],[1,6,24],[4,7,14],[8,9,11],[15,16,17],[5,12,13]],\end{aligned}$$

$$\begin{aligned}E_{46} = [&[1,34,35],[36,37,38],[31,33,39],[29,40,42],[26,27,45],[19,20,28],\\ &[21,23,25],[16,18,21],[13,14,43],[2,3,4],[17,44,45],[5,6,8],[9,30,41],\\ &[7,10,24],[15,22,32],[0,11,12]],\end{aligned}$$

$$\begin{aligned}E_{50} = [&[2,37,38],[39,40,49],[32,36,44],[33,41,42],[30,31,35],[25,27,43],\\ &[23,24,47],[20,21,29],[16,17,48],[3,22,49],[1,4,5],[7,15,46],[8,10,14],\\ &[6,11,19],[9,26,28],[18,34,45],[0,12,13]],\end{aligned}$$

$$\begin{aligned}E_{62} = [&[30,31,32],[1,46,47],[48,49,50],[39,40,42],[51,52,54],[41,45,55],[43,53,56],\\ &[35,37,59],[33,38,60],[3,5,61],[6,34,61],[7,27,28],[24,25,26],[4,9,21],\\ &[10,12,58],[17,18,19],[13,14,20],[8,11,23],[22,36,57],[2,29,44],[0,15,16]].\end{aligned}$$

It is immediate from our construction that P contains exactly 3 entries in each row. It is routine to check that P also has exactly 3 entries in each column, and 3 copies of each symbol in N_n. □

Next we consider the case when n is divisible by 4.

We start by identifying a subset of B_n which we denote by J. We let $J = J_0 \cup J_1 \cup J_2 \cup J_3 \cup J_4$ where:

$$\begin{aligned}
J_0 &= \{(0;0),(0;1),(0;2),(0;3),(0;4)\}\\
J_1 &= \{(i;3i+2),(i;3i+3),(i;3i+4) : 1 \leqslant i \leqslant n/4-1\}\\
J_2 &= \{(n/4;3n/4+2)\}\\
J_3 &= \{(i;3i),(i;3i+1),(i;3i+2) : n/4+1 \leqslant i \leqslant n/2-1\}\\
J_4 &= \{(i;i-n/2+1),(i;i),(i;i+1) : n/2 \leqslant i \leqslant n-1\}.
\end{aligned}$$

We exhibit J when $n = 12$:

0	1	2	3	4							
					6	7	8				
								10	11	0	
											2
4	5	6									
			8	9	10						
	7					0	1				
		9					2	3			
			11					4	5		
				1					6	7	
					3					8	9
11						5					10

Lemma 5. *Each column of B_n contains precisely 3 elements of J. Each symbol in N_n appears in precisely 3 elements of J. Each row of J contains precisely 3 elements of J, except for the first row which contains 5 elements and row $n/4$ which contains 1 element.*

Proof. First observe that $|J| = |J_0|+|J_1|+|J_2|+|J_3|+|J_4| = 5+3(n/4-1)+1+3(n/4-1)+3(n/2) = 3n$. The statements about rows are easy to check. Next consider the columns. Columns 0 through to 4 appear once each in J_0; columns 5 through to $3n/4+1$ appear once each in J_1; column $3n/4+2$ appears once in J_2; columns $3n/4+3$ through to $n-1$ and 0 through to $n/2-1$ appear in J_3. Including elements of J_4 on the main diagonal, we have each column appearing exactly twice. The set $\{i+1, i-n/2+1 : n/2 \leqslant i < n\} = N_n$ so each column of J has exactly 3 filled cells.

Next, consider symbols. Observe that J_4 contains each odd symbol exactly twice and each even symbol exactly once. Meanwhile, J_1 contains exactly one copy of the symbols from 6 to $n-1$ (and 0) except for those congruent to 1 modulo 4. Also, J_3 contains exactly one copy of the symbols from 4 to $n-2$ except for those congruent to 3 modulo 4. Since $1, 3 \in J_0$, each odd symbol occurs thrice in J. Finally, since $0, 2, 4 \in J_0$ and $2 \in J_2$ each even symbol also occurs thrice in J. □

We next describe how to change B_n to obtain a latin square with a triplex but no transversal. For each $i \in \{0, 1\}$, let $T_i \subset B_n$ be the latin trade of cardinality 8 consisting of all symbols in rows 0 and $n/4$ which are congruent to i modulo $n/4$. The (unique) disjoint mates T_i' are obtained by swapping the two symbols in each column of T. In each case we will show that one of the following latin squares will have the desired properties:

$$\begin{aligned} L_1 &:= (B_n \setminus T_0) \cup T_0', \\ L_2 &:= (B_n \setminus T_1) \cup T_1', \\ L_3 &:= (B_n \setminus (T_0 \cup T_1)) \cup T_0' \cup T_1'. \end{aligned}$$

Lemma 6. *Let n be divisible by 4 and let $n \geqslant 8$. For each $i \in \{1,2,3\}$, the latin square L_i does not contain a transversal. If $n = 8$, L_2 contains a triplex. If $n \in \{12,16\}$, L_1 contains a triplex. If $n \geqslant 20$, L_3 contains a triplex.*

Proof. In each case, only rows 0 and $n/4$ contain entries distinct from those in B_n. Thus $\Delta_1(r,c,s) = 0$ if $r \notin \{0, n/4\}$; $\Delta_1(r,c,s) \in \{0, n/4\}$ if $r = 0$ and $\Delta_1(r,c,s) \in \{0, -n/4\}$ if $r = n/4$. Since a transversal contains exactly one entry in each row, if K is a transversal,

$$\left| \sum_{(r,c,s)\in K} \Delta_1(r,c,s) \right| \leqslant n/4,$$

contradicting Lemma 1. The remaining claims follow from Lemma 5, by observing that in each case we have shifted precisely 2 entries from row 0 of J to row $n/4$ (and have made no other changes to J). □

Corollary 8. *Let n be divisible by 4 and let $n \geqslant 8$. There exists a latin square of order n which contains a triplex but no transversal.*

It remains to consider the case when $n \equiv 2 \mod 4$. This splits into subcases according to the value of $n \mod 3$.

The $n \equiv 6 \mod 12$ case is easiest. Let $m = n/6$. The $m = 1$ case is well known (for an explicit example, see [6]). The case $m = 3$ is done in Theorem 7. For odd $m \geqslant 5$, we may apply Theorem 6.

Next we consider the case when $n \equiv 10 \mod 12$. Let $n = 12m - 2$. By Theorem 7 we may assume that $m \geqslant 5$.

We start by identifying a subset of B_n which we denote by J. We let $J = J_0 \cup J_1 \cup J_2 \cup J_3$ where:

$$\begin{aligned} J_0 &= \big\{(i; 3i-2), (i; 3i-1), (i; 3i) : 1 \leqslant i \leqslant 2m-1\big\} \\ J_1 &= \big\{(2m-1; 6m+1), (2m; 6m-2), (2m; 6m-1)\big\} \\ J_2 &= \big\{(i; 3i+2), (i; 3i+3), (i; 3i+4) : 2m \leqslant i \leqslant n/2-1\big\} \\ J_3 &= \big\{(i; i-n/2), (i; i), (i; i+1) : n/2 \leqslant i \leqslant n-1\big\}. \end{aligned}$$

Observe the following lemma. We omit the proof, which is elementary and similar to that of Lemma 5.

Lemma 7. *Each column of B_n contains precisely 3 elements of J. Each symbol in N_n appears in precisely 3 elements of J. Each row of B_n contains precisely 3 elements of J, except for the first row which contains no elements, row $2m-1$ which contains 4 elements and row $2m$ which contains 5 elements of J.*

As in previous cases we wish to use latin trades in B_n to create a triplex without introducing a transversal. To this end we describe the following latin trades T_0 and T_1 within the first $2m$ (respectively, $2m+1$) rows of B_n.

$$\begin{aligned} T_0 = &\big\{(i;2jm),(i;2jm+1) : 1 \leqslant j \leqslant 4,\ 0 \leqslant i \leqslant 2m-1\big\} \\ &\cup \big\{(0;1),(2m-1;1),(0;10m),(2m-1;10m)\big\}. \\ T_1 = &\big\{(0;2jm-2),(2m;2jm-2) : 1 \leqslant j \leqslant 4\big\} \\ &\cup \big\{(0;10m-2),(2m;12m-4)\big\} \\ &\cup \big\{(2i;12m-4-2i),(2i+2;12m-4-2i) : 0 \leqslant i \leqslant m-1\big\}. \end{aligned}$$

To verify that T_0 and T_1 each give latin trades in B_n we exhibit their respective (unique) disjoint mates T_0' and T_1'.

$$\begin{aligned} T_0' = &\big\{(i,2jm,i+2jm+1),(i,2jm+1,i+2jm) : 1 \leqslant j \leqslant 4,\ 0 < i < 2m-1\big\} \\ &\cup \big\{(0,2jm,2jm+1),(0,2jm+1,2(j+1)m),(2m-1,2jm,2jm), \\ &\quad (2m-1,2jm+1,2(j+1)m-1) : 1 \leqslant j \leqslant 4\big\} \\ &\cup \big\{(0,1,2m),(2m-1,1,1),(0,10m,1),(2m-1,10m,10m)\big\}. \\ T_1' = &\big\{(0,2jm-2,2(j+1)m-2),(2m,2jm-2,2jm-2) : 1 \leqslant j \leqslant 4\big\} \\ &\cup \big\{(0,10m-2,12m-4),(0,12m-4,2m-2)\big\} \\ &\cup \big\{(2m,10m-2,10m-2),(2m,12m-4,0)\big\} \\ &\cup \big\{(2i,12m-4-2i,0),(2i,12m-2-2i,12m-4) : 1 \leqslant i \leqslant m-1\big\}. \end{aligned}$$

Lastly we define $T_2 = \big\{(r,c+5,s+5) : (r,c,s) \in T_1\big\}$ which is clearly a latin trade in B_n with disjoint mate $T_2' = \big\{(r,c+5,s+5) : (r,c,s) \in T_1'\big\}$.

Since $m \geqslant 5$, the latin trades T_0, T_1 and T_2 are pairwise disjoint. Moreover, $(T_1 \cup T_2) \cap J = \big\{(2m;6m-2),(2m;6m+3)\big\}$. Next, J intersects T_0 at $\big\{(2m-1;6m+1)\big\}$ and

$$\big\{(\lceil x/3 \rceil;x) : x \in \{2m,2m+1,4m,4m+1\}\big\}. \tag{2}$$

It follows that $L' = \big(B_n \setminus (T_0 \cup T_1 \cup T_2)\big) \cup (T_0' \cup T_1' \cup T_2')$ contains a triplex. To see this, adjust J by replacing $(2m,6m-2,8m-2)$, $(2m,6m+3,8m+3)$ and $(2m-1,6m+1,8m)$ with $(0,6m-2,8m-2)$, $(0,6m+3,8m+3)$ and $(0,6m+1,8m)$, respectively. Finally, replace (2) with the triples of L' associated with the following cells:

$$\big\{(\lceil 2m/3 \rceil,2m+1),(\lceil (2m+1)/3 \rceil,2m),(\lceil 4m/3 \rceil,4m+1),(\lceil (4m+1)/3 \rceil,4m)\big\}$$

The resultant structure is a triplex in L'.

Suppose, for the sake of contradiction, L' has a transversal K. Recall in the following that K intersects each row, column and symbol exactly once. If $(r,c,s) \in T_0' \cup T_1' \cup T_2'$, $1 \leqslant \Delta_1(0,c,s) \leqslant 2m$, $\Delta_1(2m-1,c,s) \in \big\{-1, -(2m-1)\big\}$ and $\Delta_1(2m,c,s) \in \{-2m, -2m+2\}$. Summing over $(r,c,s) \in K$ with $r \in \{0, 2m-1, 2m\}$:

$$\Big|\sum \Delta_1(r,c,s)\Big| \leqslant 4m - 1.$$

Otherwise the only non-zero values for Δ_1 occur strictly between rows 0 and $2m-1$ of $T_0' \cup T_1' \cup T_2'$, so we consider only when $0 < r < 2m-1$. In this case $\Delta_1(r,c,s) = 1$ if $c \in C = \{2m, 4m, 6m, 8m\}$ and $\Delta_1(r,c,s) = -1$ if $c \in C' = \{2m+1, 4m+1, 6m+1, 8m+1\}$. Summing over $(r,c,s) \in K$ with $r \notin \{0, 2m-1, 2m\}$ and $c \in C \cup C'$:

$$\Big|\sum \Delta_1(r,c,s)\Big| \leqslant 4.$$

Similarly, the most the Δ_1 function can accrue from $T_1' \cup T_2'$ in rows strictly between 0 and $2m-1$ is 4 (in absolute terms). Thus:

$$\left|\sum_{(r,c,s)\in K} \Delta_1(r,c,s)\right| \leqslant 4m + 7 < n/2,$$

a contradiction.

Finally, we consider the case when $n \equiv 2 \mod 12$. Let $n = 12m + 2$. By Theorem 7 we may assume that $m \geqslant 6$.

We start by identifying a subset of B_n which we denote by J. We let $J = J_0 \cup J_1 \cup J_2 \cup J_3$ where:

$$\begin{aligned}
J_0 &= \big\{(i; 3i-2), (i; 3i-1), (i; 3i) : 1 \leqslant i \leqslant 2m\big\} \\
J_1 &= \big\{(2m; 6m+3), (2m; 6m+4), (2m+1; 6m+1)\big\} \\
J_2 &= \big\{(i; 3i+2), (i; 3i+3), (i; 3i+4) : 2m+1 \leqslant i \leqslant n/2 - 1\big\} \\
J_3 &= \big\{(i; i-n/2), (i; i), (i; i+1) : n/2 \leqslant i \leqslant n-1\big\}.
\end{aligned}$$

Observe the following lemma. We omit the proof, which is elementary and similar to that of Lemma 5.

Lemma 8. *Each column of B_n contains precisely 3 elements of J. Each symbol in N_n appears in precisely 3 elements of J. Each row of B_n contains precisely 3 elements of J, except for the first row which contains no elements, row $2m$ which contains 5 elements and row $2m+1$ which contains 4 elements of J.*

As in previous cases we wish to use latin trades in B_n to introduce a triplex but not a transversal. To this end we describe the following latin trades T_0 and T_1 within the first

$2m$ (respectively, $2m+1$) rows of B_n.

$$
\begin{aligned}
T_0 = &\big\{(0;2jm-2),(2m;2jm-2) : 3 \leqslant j \leqslant 6\big\}\\
&\cup \big\{(i;2m-4),(i;2m-3),(i;4m-3),(i;4m-2) : 0 \leqslant i \leqslant 2m\big\},\\
T_1 = &\big\{(0;(2m+1)j-2),(2m+1;(2m+1)j-2) : 0 \leqslant j \leqslant 4\big\}\\
&\cup \big\{(0;10m+1),(1;10m),(1;10m+1),(2;10m),(2;10m+1)\big\}\\
&\cup \big\{(2m+1;10m+1),(1;12m-1),(1;12m),(2;12m-1),(2;12m)\big\}\\
&\cup \big\{(2i+3;10m-2i-2),(2i+3;10m-2i),\\
&\qquad (2i+3;12m-2i-3),(2i+3;12m-2i-1) : 0 \leqslant i \leqslant m-2\big\}.
\end{aligned}
$$

To verify that T_0 and T_1 each give latin trades in B_n we exhibit their respective (unique) disjoint mates T_0' and T_1'.

$$
\begin{aligned}
T_0' = &\big\{(0,2jm-2,2(j+1)m-2),(2m,2jm-2,2jm-2) : 3 \leqslant j \leqslant 6\big\}\\
&\cup \big\{(0,2m-4,2m-3),(0,2m-3,4m-3),(2m,2m-4,2m-4)\big\}\\
&\cup \big\{(2m,2m-3,4m-4),(0,4m-3,4m-2),(0,4m-2,6m-2)\big\}\\
&\cup \big\{(2m,4m-3,4m-3),(2m,4m-2,6m-3)\big\}\\
&\cup \big\{(i,2m-4,2m+i-3),(i,2m-3,2m+i-4),(i,4m-3,4m+i-2),\\
&\qquad (i,4m-2,4m+i-3) : 0 < i < 2m\big\}.\\
T_1' = &\big\{(0,(2m+1)j-2,(2m+1)(j+1)-2),\\
&\qquad (2m+1,(2m+1)j-2,(2m+1)j-2) : 1 \leqslant j \leqslant 3\big\}\\
&\cup \big\{(0,12m,2m-1),(2m+1,12m,0),(0,8m+2,10m+1)\big\}\\
&\cup \big\{(2m+1,8m+2,8m+2),(0,10m+1,12m),(1,10m,10m+2)\big\}\\
&\cup \big\{(1,10m+1,10m+1),(2,10m,10m+3),(2,10m+1,10m+2)\big\}\\
&\cup \big\{(2m+1,10m+1,10m+3),(1,12m-1,12m+1),(1,12m,12m)\big\}\\
&\cup \big\{(2,12m-1,0),(2,12m,12m+1)\big\}\\
&\cup \big\{(2i+3,10m-2i-2,10m+3),(2i+3,10m-2i,10m+1),\\
&\qquad (2i+3,12m-2i-3,0),(2i+3,12m-2i-1,12m) : 0 \leqslant i \leqslant m-2\big\}.
\end{aligned}
$$

We also define $T_2 = \big\{(r,c+6,s+6) : (r,c,s) \in T_0\big\}$ which is clearly a latin trade in B_n with disjoint mate $T_2' = \big\{(r,c+6,s+6) : (r,c,s) \in T_0'\big\}$.

Let $m \geqslant 6$. Observe that the latin trades T_0, T_1 and T_2 are pairwise disjoint. Moreover, $T_0 \cup T_2$ intersects J at $(2m;6m-2),(2m;6m+4)$ and

$$\big\{(\lceil x/3\rceil;x) : x \in \{2m-4,2m-3,2m+2,2m+3,4m-3,4m-2,4m+3,4m+4\}\big\} \tag{3}$$

Also, J intersects T_1 at $\big\{(2m+1;6m+1)\big\}$.

It follows that $L' = \big(B_n \setminus (T_0 \cup T_1 \cup T_2)\big) \cup (T_0' \cup T_1' \cup T_2')$ contains a triplex. To see this, adjust J by replacing $(2m,6m-2,8m-2)$, $(2m,6m+4,8m+4)$ and $(2m+1,6m+1,8m+2)$

with $(0,6m-2,8m-2)$, $(0,6m+4,8m+4)$ and $(0,6m+1,8m+2)$, respectively. Finally, replace (3) with the triples of L' associated with the following cells:

$$\begin{aligned}\big\{&(\lceil(2m-4)/3\rceil,2m-3),(\lceil(2m-3)/3\rceil,2m-4),\\&(\lceil(2m+2)/3\rceil,2m+3),(\lceil(2m+3)/3\rceil,2m+2),\\&(\lceil(4m-3)/3\rceil,4m-2),(\lceil(4m-2)/3\rceil,4m-3),\\&(\lceil(4m+3)/3\rceil,4m+4),(\lceil(4m+4)/3\rceil,4m+3)\big\}.\end{aligned}$$

The resultant structure is a triplex in L'.

Suppose, for the sake of contradiction, L' has a transversal K. If $(r,c,s) \in T_0' \cup T_1' \cup T_2'$, $1 \leqslant \Delta_1(0,c,s) \leqslant 2m+1$, $\Delta_1(2m,c,s) \in \{-1,-2m\}$ and $\Delta_1(2m+1,c,s) \in \{-2m-1,-2m+1\}$. Summing over $(r,c,s) \in K$ with $r \in \{0,2m,2m+1\}$:

$$\left|\sum \Delta_1(r,c,s)\right| \leqslant 4m+1.$$

Otherwise the only non-zero values for Δ_1 occur strictly between rows 0 and $2m$ of $T_0' \cup T_1' \cup T_2'$, so we consider only when $0 < r < 2m$. In this case $\Delta_1(r,c,s) = 1$ if $c \in C = \{2m-4, 2m+2, 4m-3, 4m+3\}$ and $\Delta_1(r,c,s) = -1$ if $c \in C' = \{2m-3, 2m+3, 4m-2, 4m+4\}$. Summing over $(r,c,s) \in K$ with $r \notin \{0,2m,2m+1\}$ and $c \in C \cup C'$:

$$\left|\sum \Delta_1(r,c,s)\right| \leqslant 4.$$

Similarly, the most the Δ_1 function can accrue from T_1' in rows strictly between 0 and $2m$ is 6 (in absolute terms). Thus:

$$\left|\sum_{(r,c,s)\in K} \Delta_1(r,c,s)\right| \leqslant 4m+11 < n/2,$$

a contradiction.

5 Conclusion

It has been known since the 19th century that there are no transversals in step-type Latin squares of even order composed of odd ordered subsquares. This family includes the Cayley table of the cyclic group of any even order. We showed in §2 that the absence of transversals in these squares is a surprisingly robust property. Specifically, the entries in up to $\sqrt{n}$ consecutive rows may be rearranged in any way and there will still be no transversal. A consequence is that there are at least $n^{n^{3/2}(1/2-o(1))}$ species of transversal-free latin squares of each even order n.

Our other main result was to prove Conjecture 1, that for all even $n > 4$ there is a latin square of order n that contains a triplex but no transversal. It would be interesting to know how far this result generalises. We propose:

Conjecture 2. *For each odd k there exists N such that for all even $n \geqslant N$ there exists a latin square of order n that contains a k-plex but no k'-plex for odd $k' < k$.*

In [2], examples were constructed where the smallest odd k-plexes have $k = n/4 - O(1)$, where n is the order of the latin square. This raises the interesting question of whether $n/4 - O(1)$ is as large as possible in a result of this type.

Another unsolved question from [2] is whether there exists any latin square that has an a-plex and a c-plex but no b-plex, for odd integers $a < b < c$.

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References

[1] N. J. Cavenagh, The theory and application of latin bitrades: a survey, *Math. Slovac.*, **58** (2008), 1–28.

[2] J. Egan and I. M. Wanless, Latin squares with no small odd plexes, *J. Combin. Designs* **16** (2008), 477–492.

[3] J. Egan and I. M. Wanless, Indivisible partitions of latin squares, *J. Statist. Plann. Inference* **141** (2011) 402–417.

[4] J. H. van Lint and R. M. Wilson, *A Course in Combinatorics* (2nd ed.), Cambridge University Press, 2001.

[5] E. Maillet, Sur les carrés latins d'Euler, *Assoc. Franc. Caen.* **23** (1894), 244–252.

[6] I. M. Wanless, A generalisation of transversals for latin squares, *Electron. J. Combin.*, **9(1)** (2002), #R12.

[7] I. M. Wanless, "Transversals in latin squares: A survey", in R. Chapman (ed.), *Surveys in Combinatorics 2011*, London Math. Soc. Lecture Note Series **392**, Cambridge University Press, 2011, pp403–437.