Waiter–Client and Client–Waiter colourability and k–SAT games

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Submitted: Jul 7, 2016; Accepted: Jun 18, 2017; Published: Jun 30, 2017
Mathematics Subject Classifications: 05C57, 91A24, 05C15

Abstract

Waiter – Client and Client – Waiter games are two–player, perfect information games, with no chance moves, played on a finite set (board) with special subsets known as the winning sets. Each round of the biased $(1 : q)$ Waiter–Client game begins with Waiter offering $q + 1$ previously unclaimed elements of the board to Client, who claims one and leaves the remaining q elements to be claimed by Waiter immediately afterwards. In a $(1 : q)$ Client–Waiter game, play occurs in the same way except in each round, Waiter offers t elements for any t in the range $1 \leqslant t \leqslant q+1$. If Client fully claims a winning set by the time all board elements have been offered, he wins in the Client–Waiter game and loses in the Waiter–Client game. We give an estimate for the threshold bias (*i.e.* the unique value of q at which the winner of a $(1 : q)$ game changes) of the $(1 : q)$ Waiter–Client and Client–Waiter versions of two different games: the non–2–colourability game, played on the edge set of a complete k–uniform hypergraph, and the k–SAT game. More precisely, we show that the threshold bias for the Waiter–Client and Client–Waiter versions of the non–2–colourability game is $\frac{1}{n}\binom{n}{k}2^{\mathcal{O}_k(k)}$ and $\frac{1}{n}\binom{n}{k}2^{-k(1+o_k(1))}$ respectively. Additionally, we show that the threshold bias for the Waiter–Client and Client–Waiter versions of the k–SAT game is $\frac{1}{n}\binom{n}{k}$ up to a factor that is exponential and polynomial in k respectively. This shows that these games exhibit the *probabilistic intuition*.

Keywords: Positional games; k-SAT; graph theory; colouring

1 Introduction

A positional game is a two–player perfect information game where each player takes turns to claim previously unclaimed (free) elements of a finite set (board) X until all members of X have been claimed, at which point the game ends. The winner is determined by the winning criteria of the specific type of positional game in play. Such criteria are defined by a set $\mathcal{F} \subseteq 2^X$ of so–called *winning sets* which are known to both players before the game begins. A game with board X and set $\mathcal{F}$ of winning sets is often denoted by the pair $(X, \mathcal{F})$. Popular examples of positional games include Tic–Tac–Toe and Hex [28]. For an extensive survey on positional games, the interested reader may refer to the monographs of Beck [11] and Hefetz, Krivelevich, Stojaković and Szabó [30].

In biased $(1 : q)$ Waiter–Client and Client–Waiter games, where q is a positive integer, the two players, Waiter and Client, play in the following way. At the beginning of each round of the $(1 : q)$ Waiter–Client game $(X, \mathcal{F})$, Waiter offers exactly $q + 1$ free elements of X to Client. Client claims one of these, and the remaining q elements are then claimed by Waiter. If, in the last round, only $1 \leqslant s < q + 1$ free elements remain, Waiter claims all of them. Waiter wins the game if he can force Client to fully claim a winning set in $\mathcal{F}$, otherwise Client wins. In the $(1 : q)$ Client–Waiter game $(X, \mathcal{F})$, each round begins with Waiter offering $1 \leqslant t \leqslant q + 1$ free elements of X to Client. Client then claims one of these elements, and the remainder of the offering (if any) is claimed by Waiter. In this game, Client wins if he can fully claim a winning set in $\mathcal{F}$, otherwise Waiter wins.

Since these games are finite, perfect information, two–player games with no chance moves and no possibility of a draw, a classical result from Game Theory guarantees a winning strategy (*i.e.* a strategy that, if followed, ensures a win regardless of how the opponent plays). Also, both games are bias monotone in Waiter's bias q. This means that, if Client has a winning strategy for a $(1 : q)$ Waiter–Client game then Client also has a winning strategy for the same game with bias $(1 : q+1)$. The analogous implication is true when Waiter has a strategy to win a $(1 : q)$ Client–Waiter game. Thus, for each (non–trivial) $(1 : q)$ Waiter–Client or Client–Waiter game $(X, \mathcal{F})$, there exists a unique value of q at which the winner of the game changes. This value is known as the *threshold bias* of the game.

We give bounds on the threshold bias of two specific types of Waiter–Client and Client–Waiter games and show that they exhibit the *probabilistic intuition* (see Section 1.2). The first game of interest is the non–2–colourability game $(E(K_n^{(k)}), \mathcal{NC}_2)$ played on the edge set $E(K_n^{(k)})$ of the complete n–vertex k–uniform hypergraph $K_n^{(k)}$, for some positive integer k. In this, the set $\mathcal{NC}_2$ of winning sets is defined to be

$$\mathcal{NC}_2 = \{F \subseteq E(K_n^{(k)}) : \chi(F) > 2\},$$

where χ denotes the *weak chromatic number* (see Section 1.3). The second game we consider is the k–SAT game $(\mathcal{C}_n^{(k)}, \mathcal{F}_{SAT})$, where k is a positive integer. This is played on the set $\mathcal{C}_n^{(k)}$ of all $\binom{2n}{k}$ possible *k–clauses*, where each k–clause is the disjunction of exactly k literals taken from n fixed boolean variables $x_1, \ldots, x_n$. By *literal*, we mean a boolean variable x_i or its negation $\neg x_i$. The set $\mathcal{F}_{SAT}$ of winning sets is defined to be

$$\mathcal{F}_{SAT} = \{\mathcal{S} \subseteq \mathcal{C}_n^{(k)} : \bigwedge \mathcal{S} \text{ is not satisfiable}\},$$

where $\bigwedge \mathcal{S}$ denotes the conjunction of all k–clauses in $\mathcal{S}$.

1.1 The Results

1.1.1 The non–2–colourability game

We prove that the threshold bias for the $(1:q)$ Waiter–Client and Client–Waiter versions of $(E(K_n^{(k)}),\mathcal{NC}_2)$ is $\frac{1}{n}\binom{n}{k}2^{\mathcal{O}_k(k)}$ and $\frac{1}{n}\binom{n}{k}2^{-k(1+o_k(1))}$ respectively.

Theorem 1. *Let k, q and n be positive integers, with n sufficiently large and $k \geqslant 2$ fixed, and consider the $(1:q)$ Waiter–Client game $(E(K_n^{(k)}),\mathcal{NC}_2)$. If $q \leqslant \binom{\lceil n/2\rceil}{k}\frac{\ln 2}{2((1+\ln 2)n+\ln 2)}$, then Waiter can force Client to build a non–2–colourable hypergraph. Also, Client can keep his hypergraph 2–colourable throughout the game whenever $q \geqslant 2^{k/2}e^{k/2+1}k\binom{n}{k}/n$.*

Theorem 2. *Let k, q and n be positive integers, with n sufficiently large and $k \geqslant 2$ fixed, and consider the $(1:q)$ Client–Waiter game $(E(K_n^{(k)}),\mathcal{NC}_2)$. If $q \leqslant \binom{\lceil n/2\rceil}{k}\frac{\ln 2}{(1+\ln 2)n}$, then Client can build a non–2–colourable hypergraph. However, when $q \geqslant k^3 2^{-k+5}\binom{n}{k}/n$, Waiter can ensure that Client has a 2–colourable hypergraph at the end of the game.*

Thus, we have a multiplicative gap of $(1+o(1))(1+1/\ln 2)2^{3k/2+1}e^{k/2+1}k$ and $(1+o(1))(1+1/\ln 2)2^5k^3$ between the upper and lower bounds of q for the Waiter–Client and Client–Waiter versions respectively.

Remark 3. Our proofs of Theorems 1 and 2 generalise easily to the Waiter–Client and Client–Waiter non–r–colourability game $(E(K_n^{(k)}),\mathcal{NC}_r)$, for any fixed $r,k \geqslant 2$, where

$$\mathcal{NC}_r = \{F \subseteq E(K_n^{(k)}) : \chi(F) > r\}.$$

In particular, their generalisation shows that the threshold bias for the $(1:q)$ Waiter–Client and Client–Waiter versions of $(E(K_n^{(k)}),\mathcal{NC}_r)$ is $\frac{1}{n}\binom{n}{k}r^{\mathcal{O}_k(k)}$ and $\frac{1}{n}\binom{n}{k}r^{-k(1+o_k(1))}$ respectively. Since this generalisation is straightforward, we only include our proofs for the case $r=2$ in this paper.

1.1.2 The k–SAT game

We prove that the threshold bias for the $(1:q)$ Waiter–Client and Client–Waiter versions of $(\mathcal{C}_n^{(k)},\mathcal{F}_{SAT})$ is $\frac{1}{n}\binom{n}{k}$ up to a factor that is exponential and polynomial in k respectively.

Theorem 4. *Let k, q and n be positive integers, with n sufficiently large and $k \geqslant 2$ fixed, and consider the $(1:q)$ Waiter–Client game $(\mathcal{C}_n^{(k)},\mathcal{F}_{SAT})$. When $q \leqslant \binom{n}{k}/(2(n+1))$, Waiter can ensure that the conjunction of all k–clauses claimed by Client by the end of the game is not satisfiable. However, when $q \geqslant 2^{3k/2}e^{k/2+1}k\binom{n}{k}/n$, Client can ensure that the conjunction of all k–clauses he claims remains satisfiable throughout the game.*

Theorem 5. *Let k, q and n be positive integers, with n sufficiently large and $k \geqslant 2$ fixed, and consider the $(1:q)$ Client–Waiter game $(\mathcal{C}_n^{(k)},\mathcal{F}_{SAT})$. When $q < \binom{n}{k}/n$, Client can ensure that the conjunction of all k–clauses he claims by the end of the game is not satisfiable. However, when $q \geqslant 2^9k^3\binom{n}{k}/n$, Waiter can ensure that the conjunction of all k–clauses claimed by Client is satisfiable throughout the game.*

Thus, we have a multiplicative gap of $(1+o(1))2^{3k/2+1}e^{k/2+1}k$ and 2^9k^3 between the upper and lower bounds of q for the Waiter–Client and Client–Waiter versions respectively.

1.2 Motivation and Related Work

Waiter–Client and Client–Waiter games were first introduced by Beck (see *e.g.* [10]) under the names Picker–Chooser and Chooser–Picker. Since their introduction, much work has gone into finding their threshold bias, particularly when the winning sets are defined by various graph properties (see *e.g.* [31, 13]), and recent research (see *e.g.* [10, 20, 12, 33]) has revealed interesting connections with the highly popular Maker–Breaker games. The $(1:q)$ Waiter–Client and Client–Waiter non–r–colourability games $(E(K_n^{(k)}), \mathcal{NC}_r)$ have been considered before in the case $k=2$. Hefetz, Krivelevich and Tan [31] found that the threshold bias of these games has order $\Theta_{n,k}(n/(k\log k))$, and it was by generalising the techniques used in their paper that we obtained our results stated in Section 1.1. To our knowledge, the k–SAT game has not yet been considered in the literature. However, the closely related Achlioptas process for k–SAT has been studied (see *e.g.* [35, 34, 21]).

1.2.1 The Probabilistic Intuition.

The main motivation behind our results was to investigate a heuristic known as the *probabilistic intuition*. First employed by Chvátal and Erdős [15], it suggests that the player with the highest chance of winning a game $(X, \mathcal{F})$ when both players play randomly is the player with the winning strategy. Thus, given a threshold $m_{\mathcal{F}}$ for the number of board elements Client needs to contain some $A \in \mathcal{F}$ *whp* (with probability tending to 1 as $|X| \to \infty$) when Waiter and Client play randomly, we say the $(1:q)$ game exhibits the probabilistic intuition if the threshold bias $q_{\mathcal{F}}$ has the same order of magnitude as $|X|/m_{\mathcal{F}}$. Surprisingly, this occurs for many Waiter–Client and Client–Waiter games played on graphs (see *e.g.* [31, 13, 14, 19]).

Since extensive work has gone into understanding r–colourable hypergraphs and satisfiable k–CNF boolean formulae (*i.e.* the conjunction of k–clauses) in the random setting, these games are perfect candidates for developing our understanding of this phenomenon. Denoting the random n–vertex k–uniform hypergraph with cn edges by $\mathcal{H}_k(n, cn)$, the threshold $c_{r,k}$ (although only conjectured to exist) for c regarding the r–colourability of $\mathcal{H}_k(n, cn)$ currently stands at $r^{k-1}\ln r + o_k(k)$ when $r=2$ (see [7, 1, 3, 18]) and when r is sufficiently large (see [5, 23, 9]). Therefore, the probabilistic intuition predicts that the threshold bias for the $(1:q)$ non–r–colourability game $(E(K_n^{(k)}), \mathcal{NC}_r)$ is $\frac{1}{n}\binom{n}{k}r^{-k(1+o_k(1))}$ for such r. This matches the threshold bias (up to the error term in the exponent) given by Theorems 1 and 2 when $r=2$, and in Remark 3 when r is large. Additionally, Coja–Oghlan and Panagiotou [17] found that the threshold number of clauses regarding satisfiability of the conjunction of random k–clauses in $\mathcal{C}_n^{(k)}$ is $(2^k\ln 2 - o_k(k))n$ (see [25, 16, 27, 2, 26, 4, 6, 22] for related work), thereby showing that the probabilistic intuition predicts a threshold bias of $\frac{1}{n}\binom{n}{k}(\ln 2 - o_k(1))^{-1}$ for the $(1:q)$ k–SAT game $(\mathcal{C}_n^{(k)}, \mathcal{F}_{SAT})$. This has the same order of magnitude as the threshold bias given in Theorem 4 and 5. In conclusion, each game we study in this paper exhibits the probabilistic intuition.

1.3 Preliminaries

Whenever necessary, we assume that the number n of vertices or boolean variables is sufficiently large. We use the following notation in our proofs.

Let $\mathcal{H}$ be any k–uniform hypergraph with vertex set $V(\mathcal{H})$ and edge set $E(\mathcal{H}) \subseteq 2^{V(\mathcal{H})}$, where each edge consists of exactly k vertices. For a subset $S \subseteq V(\mathcal{H})$, let $E_{\mathcal{H}}(S)$ denote the set of edges of $\mathcal{H}$ that contain exactly one vertex in S and let $d_{\mathcal{H}}(S) = |E_{\mathcal{H}}(S)|$. When $S = \{v\}$ for some $v \in V(\mathcal{H})$, we abuse notation slightly and write $d_{\mathcal{H}}(v)$ instead of $d_{\mathcal{H}}(\{v\})$. Let $\mathcal{H}[S]$ denote the hypergraph with vertex set S and edge set $\{e \in E(\mathcal{H}) : e \subseteq S\}$. The *maximum degree* of $\mathcal{H}$ is defined by $\Delta(\mathcal{H}) = \max\{d_{\mathcal{H}}(v) : v \in V(\mathcal{H})\}$ and the *minimum degree* of $\mathcal{H}$ is $\delta(\mathcal{H}) = \min\{d_{\mathcal{H}}(v) : v \in V(\mathcal{H})\}$. We say that S is *independent* in $\mathcal{H}$ if $\{e \in E(\mathcal{H}) : e \subseteq S\} = \emptyset$. The *independence number* of $\mathcal{H}$, denoted by $\alpha(\mathcal{H})$, is the maximum size of an independent set of vertices in $\mathcal{H}$. A subhypergraph $\mathcal{H}' \subseteq \mathcal{H}$ (*i.e.* a hypergraph $\mathcal{H}'$ with $V(\mathcal{H}') \subseteq V(\mathcal{H})$ and $E(\mathcal{H}') \subseteq E(\mathcal{H})$) is a *clique* in $\mathcal{H}$ if every set of k vertices in $V(\mathcal{H}')$ is an edge of $\mathcal{H}'$. We sometimes refer to a clique with t vertices as a t–clique. The *clique number* of $\mathcal{H}$, denoted by $\omega(\mathcal{H})$, is the largest t such that $\mathcal{H}$ contains a t–clique. The *weak chromatic number* of $\mathcal{H}$, denoted by $\chi(\mathcal{H})$, is the smallest integer r for which $V(\mathcal{H})$ can be partitioned into r independent sets. For a set $F \subseteq E(\mathcal{H})$, we abuse notation slightly by using $\chi(F)$ to denote the chromatic number of the hypergraph with vertex set $V(\mathcal{H})$ and edge set F. Given some partition $\mathcal{P} = \{V_1, \dots, V_{\mathcal{P}}\}$ of $V(\mathcal{H})$ and an edge $e \in E(\mathcal{H})$, we define $\mathcal{P}(e) = \{V_i \in \mathcal{P} : e \cap V_i \neq \emptyset\}$. We define a *linear forest* in $\mathcal{H}$ with respect to partition $\mathcal{P}$ to be a sequence $(e_1, \dots, e_m)$ of edges in $E(\mathcal{H})$ such that $\mathcal{P}(e_i) \cap \mathcal{P}(e_j) \neq \emptyset$ only if $j \in \{i-1, i, i+1\}$. Two distinct edges $e, e' \in E(\mathcal{H})$ with vertices in parts $V_{i_1}, \dots, V_{i_k}$ of $\mathcal{P}$ are *complementary* if $e \cap e' = \emptyset$.

Let us denote the complete n–vertex k–uniform hypergraph by $K_n^{(k)}$ (*i.e.* $K_n^{(k)}$ is an n–clique). At any given moment in a Waiter–Client or Client–Waiter game, played on $E(K_n^{(k)})$, let E_C denote the set of edges currently owned by Client. We denote the hypergraph with vertex set $V(K_n^{(k)})$ and edge set E_C by $\mathcal{H}_C$. Moreover, let $\mathcal{H}_F$ be the hypergraph consisting of all edges of $K_n^{(k)}$ that are free at a given moment.

1.3.1 Useful Tools

We will use the following two lemmas which follow as immediate corollaries of the Lovász Local Lemma ([24], see also *e.g.* Chapter 5 in [8], [29],[30]).

Lemma 6 (Corollary 1 in [24])**.** *Let $\mathcal{H}$ be a k–uniform hypergraph with maximum degree $\Delta(\mathcal{H}) \leqslant 2^k/(8k)$. Then $\mathcal{H}$ is 2–colourable.*

Lemma 7 (see Theorem 1 in [29])**.** *Let $k \geqslant 2$ be an integer. Any k–CNF boolean formula in which no variable appears in more than $2^{k-4}/k$ k–clauses is satisfiable.*

We will also use the following game theoretic tools. The first two apply to the *transversal game* $(X, \mathcal{F}^*)$. For a finite set X and $\mathcal{F} \subseteq 2^X$, the *transversal family* of $\mathcal{F}$ is $\mathcal{F}^* := \{A \subseteq X : A \cap B \neq \emptyset$ for every $B \in \mathcal{F}\}$.

Theorem 8 ([32])**.** *Let q be a positive integer, let X be a finite set and let $\mathcal{F}$ be a family of subsets of X. If*

$$\sum_{A\in\mathcal{F}}\left(\frac{q}{q+1}\right)^{|A|}<1,$$

then Client has a winning strategy for the $(1:q)$ Client–Waiter game $(X,\mathcal{F}^*)$.

Theorem 9 ([12])**.** *Let q be a positive integer, let X be a finite set and let $\mathcal{F}$ be a family of subsets of X. If*

$$\sum_{A\in\mathcal{F}}2^{-|A|/(2q-1)}<\frac{1}{2},$$

then Waiter has a winning strategy for the $(1:q)$ Waiter–Client game $(X,\mathcal{F}^*)$.

Theorem 10 (implicit in [11], see Theorem 2.1 in [31] for an explicit proof)**.** *Let q be a positive integer, let X be a finite set, let $\mathcal{F}$ be a family of (not necessarily distinct) subsets of X and let $\Phi(\mathcal{F})=\sum_{A\in\mathcal{F}}(q+1)^{-|A|}$. Then, when playing the $(1:q)$ Waiter–Client game $(X,\mathcal{F})$, Client has a strategy to avoid fully claiming more than $\Phi(\mathcal{F})$ sets in $\mathcal{F}$.*

The rest of this paper is organised as follows: in Section 2.1 we prove Theorems 1 and 2. In Section 2.2 we prove Theorems 4 and 5. Finally, in Section 3 we present some open problems.

2 Proofs

We begin with some core lemmas that will be useful in both the non–2–colourability game and the k–SAT game. In these, let $\mathcal{H}^{\mathcal{P}}$ denote some k–uniform hypergraph with a partition $\mathcal{P}=\{V_1,\ldots,V_n\}$ of its vertex set such that each part has size $\ell\in\{1,2\}$ and contains at most one vertex from any edge in $E(\mathcal{H}^{\mathcal{P}})$. In each round i of a game played by Waiter and Client on $E(\mathcal{H}^{\mathcal{P}})$, we denote the edge claimed by Client by e_i and the parts of $\mathcal{P}$ in which e_i has a vertex by $V_{i_1},\ldots,V_{i_k}$ (ordered arbitrarily).

Lemma 11. *Let S denote some set of $\lceil n/2\rceil$–cliques in $K_n^{(k)}$. In a $(1:q)$ game on $E(K_n^{(k)})$, a strategy to ensure that $\mathcal{H}_C$ contains an edge in every member of S at the end of the game exists for Waiter in the Waiter–Client version if $q\leqslant\binom{\lceil n/2\rceil}{k}\frac{\ln 2}{2\ln(2|S|)}$ and for Client in the Client–Waiter version if $q<\binom{\lceil n/2\rceil}{k}\ln 2/\ln(|S|)$.*

Proof. Let $\mathcal{F}=\{E(\mathcal{H}):\mathcal{H}\in S\}$ and let us first suppose that $q\leqslant\binom{\lceil n/2\rceil}{k}\frac{\ln 2}{2\ln(2|S|)}$. Observe that

$$\sum_{A\in\mathcal{F}}2^{-|A|/(2q-1)}<|S|2^{-\binom{\lceil n/2\rceil}{k}/(2q)}\leqslant\frac{1}{2},$$

where the final inequality follows from our choice of q. Thus, by Theorem 9, Waiter can force Client to claim an edge in every member of S as stated.

Now suppose that $q < \binom{\lceil n/2\rceil}{k} \ln 2/\ln(|S|)$ and observe that

$$\sum_{A\in\mathcal{F}} \left(\frac{q}{q+1}\right)^{|A|} \leqslant \sum_{A\in\mathcal{F}} 2^{-|A|/q} \leqslant |S| 2^{-\binom{\lceil n/2\rceil}{k}/q} < 1,$$

where the final inequality follows from our choice of q. Thus, by Theorem 8, Client can claim an edge in every member of S by the end of the game as stated. □

Corollary 12. *In a $(1:q)$ game on $E(K_n^{(k)})$, a strategy to ensure that $\chi(\mathcal{H}_C) > 2$ at the end of the game exists for Waiter in the Waiter–Client version if $q \leqslant \binom{\lceil n/2\rceil}{k}\frac{\ln 2}{2((1+\ln 2)n+\ln 2)}$ and for Client in the Client–Waiter version if $q < \binom{\lceil n/2\rceil}{k}\frac{\ln 2}{(1+\ln 2)n}$.*

Proof. By taking S to be the set of all $\lceil n/2\rceil$–cliques in $K_n^{(k)}$, Lemma 11 provides strategies for Waiter and Client to ensure that $\alpha(\mathcal{H}_C) < \lceil n/2\rceil$ at the end of the game. Since $\chi(\mathcal{H}_C)\alpha(\mathcal{H}_C) \geqslant n$, the result follows. □

Lemma 13. *Consider a $(1:q)$ Waiter–Client game on $E(\mathcal{H}^{\mathcal{P}})$ and let n be sufficiently large. If $q \geqslant (2\ell^2)^{k/2}e^{k/2+1}k\binom{n}{k}/n$, then Client can ensure that, for every $S \subseteq \mathcal{P}$, there exists some $A \in S$ such that $d_{\mathcal{H}_C[\cup S]}(A) \leqslant 1$ at the end of the game.*

Proof. Let $\mathcal{F} = \{F : \exists S \subseteq \mathcal{P} \text{ s.t. } S \neq \emptyset, F \subseteq E(\mathcal{H}^{\mathcal{P}}[\cup S]) \text{ and } |F| = \lceil 2|S|/k\rceil\}$. Observe that

$$\begin{aligned}
\Phi(\mathcal{F}) &= \sum_{A\in\mathcal{F}} (q+1)^{-|A|} \leqslant \sum_{t=k+\ell-1}^{n} \binom{n}{t}\binom{\binom{\ell t}{k}}{\lceil 2t/k\rceil} q^{-\lceil 2t/k\rceil} \leqslant \sum_{t=k+\ell-1}^{n} \left(\frac{en}{t}\right)^t \left(\frac{e\binom{\ell t}{k}}{\lceil 2t/k\rceil q}\right)^{\lceil 2t/k\rceil} \\
&\leqslant \sum_{t=k+\ell-1}^{n} \left(\frac{en}{t}\right)^t \left(\frac{ek\binom{\ell t}{k}}{2tq}\right)^{\lceil 2t/k\rceil} \leqslant \sum_{t=k+\ell-1}^{n} \left[\frac{en}{t}\left(\frac{ek\binom{\ell t}{k}}{2tq}\right)^{2/k}\right]^t \\
&\leqslant \sum_{t=1}^{n} \left[\frac{en}{t}\left(\frac{e\ell^k t^{k-1}}{2q(k-1)!}\right)^{2/k}\right]^t \leqslant \sum_{t=1}^{n} \left[\frac{en}{t}\left(\frac{1}{(2e)^{k/2}}\left(\frac{t}{n}\right)^{k-1}\right)^{2/k}\right]^t \\
&= \sum_{t=1}^{n} \left[\frac{1}{2}\left(\frac{t}{n}\right)^{\frac{2}{k}(k-1)-1}\right]^t < \sum_{t=1}^{\infty} \left[\frac{1}{2}\right]^t = 1,
\end{aligned}$$

where the fourth and sixth inequalities follow from our choice of q and since n is sufficiently large. Thus, by Theorem 10, Client can avoid claiming any member of $\mathcal{F}$. In particular, this means that, for every $S \subseteq \mathcal{P}$, Client must have strictly less than $2|S|/k$ edges in $\mathcal{H}^{\mathcal{P}}[\cup S]$ at the end of the game. Hence, in every $S \subseteq \mathcal{P}$ there exists some $A \in S$ for which $d_{\mathcal{H}_C[\cup S]}(A) \leqslant 1$, as stated. □

Lemma 14. *Consider a $(1:q)$ Client–Waiter game on $E(\mathcal{H}^{\mathcal{P}})$ where $q \geqslant 4k-2$. Waiter has a strategy to ensure that*

$$d_{\mathcal{H}_C}(V_j) \leqslant \frac{2kd_{\mathcal{H}^{\mathcal{P}}}(V_j)}{q} + 2$$

for every $j \in [n]$ at the end of the game.

Proof. In the first round, Waiter offers $q+1$ arbitrary free edges. In each round i, let us denote the edge claimed by Client by e_i and the parts of $\mathcal{P}$ in which e_i has a vertex $V_{i_1}, \ldots, V_{i_k}$ ordered arbitrarily. In round $i+1$, Waiter responds to Client's claim of e_i in the following way. With $S_{i_j} = \{e \in E(\mathcal{H}_F) : V_{i_j} \cap e \neq \emptyset\}$ for each $j \in [k]$, let $F_{i_1} \subseteq S_{i_1}$ with size $|F_1| = \min\{d_{\mathcal{H}_F}(V_{i_1}), \lfloor (q+1)/k \rfloor\}$ and, for each $2 \leqslant j \leqslant k$, let $F_{i_j} \subseteq S_{i_j} \setminus \cup\{F_{i_\ell} : 1 \leqslant \ell < j\}$ with size $|F_j| = \min\{|S_{i_j} \setminus \cup\{F_{i_\ell} : 1 \leqslant \ell < j\}|, \lfloor (q+1)/k \rfloor\}$. Immediately after Client has claimed e_i, Waiter offers all edges in $\cup\{F_{i_j} : j \in [k]\}$. Recall that, in any round of a Client–Waiter game, Waiter may offer less than $q+1$ edges if he desires. If no free edge contains a vertex from $\cup\{V_{i_1}, \ldots, V_{i_k}\}$, Waiter performs his response on an edge that Client claimed earlier on in the game. If this is not possible, Waiter simply offers $\min\{q+1, |E(\mathcal{H}_F)|\}$ arbitrary free edges. It is clear that, by responding to each of Client's moves in this way, Waiter offers every edge of $\mathcal{H}^{\mathcal{P}}$ in the game.

Consider an arbitrary part V_j from $\mathcal{P}$. Every time Client claims an edge containing a vertex from V_j, Waiter offers at least $\lfloor (q+1)/k \rfloor$ free edges containing a vertex from V_j in the next round, except for perhaps the last time he offers edges touching V_j when there may be less than $\lfloor (q+1)/k \rfloor$ such edges available. Every time Waiter offers edges touching V_j, Client may claim at most one of these. Hence, at the end of the game,

$$d_{\mathcal{H}_C}(V_j) \leqslant \left\lceil \frac{d_{\mathcal{H}^{\mathcal{P}}}(V_j) - 1}{\lfloor (q+1)/k \rfloor} \right\rceil + 1 < \frac{2k d_{\mathcal{H}^{\mathcal{P}}}(V_j)}{q} + 2,$$

for every $j \in [n]$, by our choice of q, as stated. □

Lemma 15. *Consider a $(1:q)$ Client–Waiter game on $E(\mathcal{H}^{\mathcal{P}})$. If $q \geqslant k^2 \ell^k \binom{n}{k}/n$, then Waiter can ensure that $\mathcal{H}_C$ is a linear forest with respect to partition $\mathcal{P}$, that does not contain a pair of complementary edges, at the end of the game.*

Proof. In each round $i \geqslant 1$, Waiter identifies an inclusion maximal set $A_i \subseteq \mathcal{P}$ such that the number of free edges with at least one vertex in $\cup A_i$ is at most $q+1$. Waiter offers all such free edges in round i. When $i \geqslant 2$, $\cup A_i$ must also contain e_{i-1}. Note that for each edge e_i there are at most $k\ell^k \binom{n-1}{k-1}$ other edges in $\mathcal{H}^{\mathcal{P}}$ with at least one vertex in $\cup\{V_{i_1}, \ldots, V_{i_k}\}$. Therefore, by our choice of q, Waiter always offers at least one edge in each round when following the described strategy.

Observe that this strategy ensures that, at the end of round $i+1$ for every $i \geqslant 1$, $d_{\mathcal{H}_F}(V_{i_j}) = 0$ for every $j \in [k]$ and Client has claimed at most one edge with at least one vertex in $\cup\{V_{i_1}, \ldots, V_{i_k}\}$. Also, any edge of $\mathcal{H}^{\mathcal{P}}$ complementary to edge e_i was offered with e_i in round i and therefore cannot be claimed by Client in any subsequent round. Hence, at the end of the game, $\mathcal{H}_C$ forms a linear forest with respect to partition $\mathcal{P}$ and does not contain a pair of complementary edges. □

2.1 The non–2–colourability game

2.1.1 The Waiter–Client non–2–colourability game

Proof of Theorem 1. Fix $k \geqslant 2$. Waiter's strategy follows immediately from Corollary 12.

Client's Strategy: Suppose that $q \geqslant 2^{k/2}e^{k/2+1}k\binom{n}{k}/n$. Then, by choosing partition $\mathcal{P} = \{\{v\} : v \in V(K_n^{(k)})\}$ and $\ell = 1$ in Lemma 13, Client can ensure that, for every $S \subseteq V(\mathcal{H}_C)$, there exists a vertex $v \in S$ with $d_{\mathcal{H}_C[S]}(v) \leqslant 1$ at the end of the game. Thus, $\chi(\mathcal{H}_C) \leqslant 1 + \max_{\mathcal{H}' \subseteq \mathcal{H}_C} \delta(\mathcal{H}') \leqslant 2$. □

2.1.2 The Client–Waiter non–2–colourability game

Proof of Theorem 2. Fix $k \geqslant 2$. Client's strategy follows immediately from Corollary 12.
Waiter's Strategy: Suppose that $q \geqslant k^3 2^{-k+5}\binom{n}{k}/n$ and let us first consider the case $k \geqslant 8$. By Lemma 6, it suffices for Waiter to ensure that $\Delta(\mathcal{H}_C) \leqslant 2^k/(8k)$ at the end of the game. Indeed, with partition $\mathcal{P} = \{\{v\} : v \in V(K_n^{(k)})\}$, Lemma 14 tells us that Waiter can ensure that

$$d_{\mathcal{H}_C}(u) \leqslant \frac{2k\binom{n-1}{k-1}}{q} + 2 \leqslant \frac{2^k}{8k},$$

for each $u \in V(K_n^{(k)})$, where the final inequality follows from our choice of k and q.

Now consider the case $2 \leqslant k \leqslant 7$. Using the same partition $\mathcal{P}$ as above, setting $\ell = 1$ and noting that $q \geqslant k^3 2^{-k+5}\binom{n}{k}/n \geqslant k^2 \ell^k \binom{n}{k}/n$ for our choice of q and k, Lemma 15 gives Waiter a strategy to ensure that $\mathcal{H}_C$ is a linear forest with respect to $\mathcal{P}$ at the end of the game. In particular, $\delta(\mathcal{H}') \leqslant 1$ for each $\mathcal{H}' \subseteq \mathcal{H}_C$ and thus, $\chi(\mathcal{H}_C) \leqslant 1 + \max_{\mathcal{H}' \subseteq \mathcal{H}} \delta(\mathcal{H}') \leqslant 2$. □

2.2 The *k*–SAT game

For the sake of clarity, in both of the following proofs we analyse a game that is analogous to the k–SAT game; namely $(E(K_{2n}^{(k)}), \mathcal{F}'_{SAT})$. In this, each vertex of $V(K_{2n}^{(k)})$ is labeled with a unique literal in $\cup_{i \in [n]}\{x_i, \neg x_i\}$ over the set $\{x_i : i \in [n]\}$ of boolean variables and $\mathcal{P}_{SAT} = \{B_i : i \in [n]\}$ denotes a fixed partition of $V(K_{2n}^{(k)})$, where $B_i = \{x_i, \neg x_i\}$ for each $i \in [n]$. An edge containing a pair of vertices that lie within the same part B_i will be referred to as a *satisfied edge*. The winning sets are defined by $\mathcal{F}'_{SAT} = \{F \subseteq E(K_{2n}^{(k)}) : \wedge\{\vee e : e \in F\}$ is *not* satisfiable$\}$. We will use the following corollary of Lemma 11 in our proofs.

Corollary 16. *In a $(1 : q)$ game on $E(K_{2n}^{(k)})$, a strategy to ensure that $E(\mathcal{H}_C) \notin \mathcal{F}'_{SAT}$ at the end of the game exists for Waiter in the Waiter–Client version if $q \leqslant \binom{n}{k}/(2(n+1))$ and for Client in the Client–Waiter version if $q < \binom{n}{k}/n$.*

Proof. By taking $S = \{\mathcal{H} \subseteq K_{2n}^{(k)} : \mathcal{H}$ is an n–clique without a satisfied edge$\}$, Lemma 11 provides strategies for Waiter and Client to ensure that $\mathcal{H}_C$ contains an edge in every n–clique of S by the end of the game. Since every $\{0, 1\}$–assignment to the boolean variables $x_1, \ldots, x_n$ defines a 2–colouring of $V(K_{2n}^{(k)})$ where each colour class contains a member of S, Client always has an edge that is monochromatic in colour 0 under every possible assignment. Hence, the k–CNF boolean formula corresponding to Client's edges at the end of the game cannot be satisfiable. □

2.2.1 The Waiter–Client k–SAT game

Proof of Theorem 4. Fix $k \geqslant 2$. Waiter's strategy follows immediately from Corollary 16.

Client's Strategy: Let $q \geqslant 2^{3k/2}e^{k/2+1}k\binom{n}{k}/n$. Since a k–clause corresponding to a satisfied edge of $K_{2n}^{(k)}$ is satisfiable under every $\{0,1\}$–assignment to the boolean variables x_i, its conjunction with any k–CNF boolean formula ϕ does not affect the satisfiability of ϕ. Hence, whenever Client is offered a satisfied edge, he takes it. In rounds where no such edge is offered, Client follows the strategy dictated by Lemma 13 using partition $\mathcal{P}_{SAT}$, $\ell = 2$, and hypergraph $\mathcal{H} \subseteq K_{2n}^{(k)}$ whose edge set consists of all unsatisfied edges of $K_{2n}^{(k)}$. Doing so ensures that, for every $S \subseteq \mathcal{P}_{SAT}$ there exists some $B \in S$ in which at most one unsatisfied edge of Client's has a vertex. Consequently, there exists an ordering $B_{i_1}, \ldots, B_{i_n}$ of the elements of $\mathcal{P}_{SAT}$ such that, for each $j \in [n]$, Client has at most one unsatisfied edge contained in $\cup\{B_{i_t} : t \leqslant j\}$ with a vertex $v_{i_j} \in B_{i_j}$. Assigning the value, 0 or 1, to the variable x_{i_j} such that the literal labeling v_{i_j} has value 1 for every $j \in [n]$, and assigning arbitrary values to any remaining variables, provides a satisfying assignment for the formula corresponding to $\mathcal{H}_C$ at the end of the game. $\square$

2.2.2 The Client–Waiter k–SAT game

Proof of Theorem 5. Fix $k \geqslant 2$. Client's strategy follows immediately from Corollary 16.

Waiter's Strategy: Let $q \geqslant 2^9k^3\binom{n}{k}/n$. Waiter's strategy consists of two stages. In Stage 1, Waiter only offers satisfied edges until no more are left, at which point Stage 2 begins. We denote Client's hypergraph built during Stage 1 and Stage 2 by $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Stage 2 depends on the following two cases.

We first consider the case $k \geqslant 10$. By Lemma 7, $E(\mathcal{H}_2) \in \mathcal{F}'_{SAT}$ if Waiter can ensure that every $B_i \in \mathcal{P}_{SAT}$ has non–empty intersection with at most $2^{k-4}/k$ edges in $\mathcal{H}_2$ at the end of the game. Indeed, with partition $\mathcal{P}_{SAT}$, Lemma 14 tells us that Waiter can force

$$d_{\mathcal{H}_2}(B_i) \leqslant \frac{2^{k+1}k\binom{n-1}{k-1}}{q} + 2 \leqslant \frac{2^{k-4}}{k},$$

for each $i \in [n]$, where the final inequality follows from our choice of k and q. Thus, in Stage 2, Waiter follows the strategy given by Lemma 14.

Now consider the case $2 \leqslant k \leqslant 9$. By using partition $\mathcal{P}_{SAT}$, setting $\ell = 2$ and noting that $q \geqslant 2^9k^3\binom{n}{k}/n \geqslant k^2\ell^k\binom{n}{k}/n$ for our choice of q and k, Lemma 15 gives Waiter a strategy to ensure that $\mathcal{H}_2$ is a linear forest with respect to $\mathcal{P}_{SAT}$, that does not contain a pair of complementary edges, at the end of the game. Hence, there exists an ordering $e_1, \ldots, e_m$ of the edges in $E(\mathcal{H}_2)$ such that $\mathcal{P}_{SAT}(e_i) \cap \mathcal{P}_{SAT}(e_j) \neq \emptyset$ only when $j \in \{i-1, i, i+1\}$. By assigning values from $\{0,1\}$ to the boolean variables such that the literals labelling vertices in $e_i \cap \cup(\mathcal{P}_{SAT}(e_i) \cap \mathcal{P}_{SAT}(e_{i+1}))$ when $\mathcal{P}_{SAT}(e_i) \cap \mathcal{P}_{SAT}(e_{i+1}) \neq \emptyset$ and those labelling vertices in $e_j \setminus \cup(\mathcal{P}_{SAT}(e_{j-1}) \cap \mathcal{P}_{SAT}(e_j))$ when $\mathcal{P}_{SAT}(e_j) \cap \mathcal{P}_{SAT}(e_{j+1}) = \emptyset$ or $j = m$ have value 1, we obtain a satisfying assignment for the formula corresponding to $\mathcal{H}_2$.

Since any k–clause corresponding to a satisfied edge of $K_{2n}^{(k)}$ is satisfiable under every $\{0,1\}$–assignment to the boolean variables x_i, the conjunction of the formula corresponding to $\mathcal{H}_1$ with that corresponding to $\mathcal{H}_2$ is also satisfiable in both of our considered cases. Hence, Waiter can ensure that $E(\mathcal{H}_C) = E(\mathcal{H}_1) \cup E(\mathcal{H}_2) \notin \mathcal{F}_{SAT}$ at the end of the game. $\square$

3 Concluding remarks and open problems

There is room to improve the bounds on the threshold bias for all of the games we consider in this paper, especially in the Waiter–Client versions where the multiplicative gap is exponential in k. In particular, we believe that the threshold bias of these games is asymptotically equivalent to that predicted by the probabilistic intuition in the following sense.

Conjecture 17. Let the threshold bias for the $(1:q)$ Waiter–Client and Client–Waiter non–2–colourability games $(E(K_n^{(k)}), \mathcal{NC}_2)$ be denoted by $b_{\mathcal{NC}_2}^{WC}$ and $b_{\mathcal{NC}_2}^{CW}$ respectively and let

$$f(x) := \lim_{k\to\infty} \left\{ \lim_{n\to\infty} \frac{1}{n}\binom{n}{k}\frac{1}{x\cdot 2^{k-1}\ln 2} \right\}.$$

Then $f(b_{\mathcal{NC}_2}^{WC}) = f(b_{\mathcal{NC}_2}^{CW}) = 1$.

Conjecture 18. Let the threshold bias for the $(1:q)$ Waiter–Client and Client–Waiter k–SAT games $(\mathcal{C}_n^{(k)}, \mathcal{F}_{SAT})$ be denoted by $b_{\mathcal{F}_{SAT}}^{WC}$ and $b_{\mathcal{F}_{SAT}}^{CW}$ respectively and let

$$g(x) := \lim_{k\to\infty} \left\{ \lim_{n\to\infty} \frac{1}{n}\binom{n}{k}\frac{1}{x\ln 2} \right\}.$$

Then $g(b_{\mathcal{F}_{SAT}}^{WC}) = g(b_{\mathcal{F}_{SAT}}^{CW}) = 1$.

Such similarity between the threshold bias and the probabilistic intuition has been observed before in other games such as the Waiter–Client K_t–minor game, played on the edge set $E(K_n)$ of the complete graph K_n (see [31]).

Acknowledgments

The author would like to thank Will Perkins, Dan Hefetz and the anonymous referees for their very helpful comments and suggestions.

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