Transversals and Independence in Linear Hypergraphs with Maximum Degree Two

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Abstract

For $k \geqslant 2$, let H be a k-uniform hypergraph on n vertices and m edges. Let S be a set of vertices in a hypergraph H. The set S is a transversal if S intersects every edge of H, while the set S is strongly independent if no two vertices in S belong to a common edge. The transversal number, $\tau(H)$, of H is the minimum cardinality of a transversal in H, and the strong independence number of H, $\alpha(H)$, is the maximum cardinality of a strongly independent set in H. The hypergraph H is linear if every two distinct edges of H intersect in at most one vertex. Let $\mathcal{H}_k$ be the class of all connected, linear, k-uniform hypergraphs with maximum degree 2. It is known [European J. Combin. 36 (2014), 231–236] that if $H \in \mathcal{H}_k$, then $(k+1)\tau(H) \leqslant n+m$, and there are only two hypergraphs that achieve equality in the bound. In this paper, we prove a much more powerful result, and establish tight upper bounds on $\tau(H)$ and tight lower bounds on $\alpha(H)$ that are achieved for infinite families of hypergraphs. More precisely, if $k \geqslant 3$ is odd and $H \in \mathcal{H}_k$ has n vertices and m edges, then we prove that $k(k^2-3)\tau(H) \leqslant (k-2)(k+1)n + (k-1)^2 m + k - 1$ and $k(k^2-3)\alpha(H) \geqslant (k^2+k-4)n - (k-1)^2 m - (k-1)$. Similar bounds are proven in the case when $k \geqslant 2$ is even.

Keywords: Transversal; Hypergraph; Linear hypergraph; Strong independence
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1 Introduction

In this paper, we study transversals and independence in hypergraphs. Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A *hypergraph* $H = (V, E)$ is a finite set $V = V(H)$ of elements, called *vertices*, together with a finite multiset $E = E(H)$ of subsets of V, called *hyperedges* or simply *edges*. The *order* of H is $n(H) = |V|$ and the *size* of H is $m(H) = |E|$. The hypergraph H is said to be k*-uniform* if every edge of H is of size k. Every (simple) graph is a 2-uniform hypergraph. Thus graphs are special hypergraphs. The *degree* of a vertex v in H, denoted by $d_H(v)$, is the number of edges of H which contain v. A vertex of degree r in H is called a *degree-*r *vertex*. The *rank* of H is the maximum size of an edge in H. The hypergraph H is r-regular if $d_H(v) = r$ for all $v \in V(H)$. The minimum and maximum degrees among the vertices of H is denoted by $\delta(H)$ and $\Delta(H)$, respectively. We use the standard notation $[k] = \{1, 2, \ldots, k\}$.

Two vertices x and y of H are *adjacent* if there is an edge e of H such that $\{x, y\} \subseteq V(e)$. Two vertices x and y of H are *connected* if there is a sequence $x = v_0, v_1, v_2 \ldots, v_k = y$ of vertices of H in which v_{i-1} is adjacent to v_i for $i \in [k]$. A *connected hypergraph* is a hypergraph in which every pair of vertices is connected. A maximal connected subhypergraph of H is a *component* of H. Thus, no edge in H contains vertices from different components.

For a subset $X \subseteq V(H)$ of vertices in H, let $H[X]$ denote the hypergraph induced by the vertices in X, in the sense that $V(H[X]) = X$ and $E(H[X]) = \{e \cap X \mid e \in E(H)$ and $|e \cap X| \geqslant 1\}$; that is, $E(H[X])$ is obtained from $E(H)$ by shrinking edges $e \in E(H)$ that intersect X to the edges $e \cap X$. For a subset $X \subset V(H)$ of vertices in H, we define $H - X$ to be the hypergraph obtained from H by deleting the vertices in X and all edges incident with X, and deleting all isolated vertices, if any, from the resulting hypergraph.

A subset T of vertices in a hypergraph H is a *transversal* (also called *vertex cover* or *hitting set* in many papers) if T intersects every edge of H. Equivalently, a set of vertices S is transversal in H if and only if $V(H) \setminus S$ is a weakly independent set in H. That is, no edge lies completely within $V(H) \setminus S$. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H. Transversals in hypergraphs are well studied in the literature (see, for example, [5, 6, 14, 18, 26, 30]).

A set S of vertices in a hypergraph H is *strongly independent* if no two vertices in S belong to a common edge. The *strong independence number* of H, which we denote by $\alpha(H)$, is the maximum cardinality of a strongly independent set in H. The independence number is one of the most fundamental and well-studied graph and hypergraph parameters (see, for example, [1, 2, 4, 9, 11, 10, 12, 13, 15, 16, 17, 21, 22, 23, 25, 27]).

A hypergraph H is called an *intersecting hypergraph* if every two distinct edges of H have a non-empty intersection, while H is called a *linear hypergraph* if every two distinct edges of H intersect in at most one vertex. Intersecting and linear hypergraphs are well studied in the literature (see, for example, [8, 20]).

Two edges in a graph G are *independent* if they are not adjacent in G. A set of pairwise independent edges of G is called a *matching* in G, while a matching of maximum cardinality is a *maximum matching*. The number of edges in a maximum matching of G is the *matching number* of G which we denote by $\alpha'(G)$. Matchings in graphs are extensively studied in the literature (see, for example, the classical book on matchings by Lovász and Plummer [24], and the excellent survey articles by Plummer [28] and Pulleyblank [29]).

Given a graph G, we define a hypergraph H_G as follows. Let the edges of G become vertices in H_G and the vertices of G become hyperedges in H_G, containing all edges that are incident with that vertex in the graph. Thus, $V(H_G) = E(G)$ and $E(H_G)$ contains a hyperedge for every vertex $v \in V(G)$ which consists of all elements of $V(H_G)$ that correspond with edges incident with v in G. Therefore, $n(H_G) = m(G)$ and $m(H_G) = n(G)$. We call H_G the *dual hypergraph* of G.

2 Known Matching Results

We shall need the following results by the authors [19] which establish a tight lower bound on the matching number of a graph in terms of its maximum degree, order, and size.

Theorem 1. ([19]) *If $k \geqslant 2$ is an even integer and G is a connected graph of order n, size m and maximum degree $\Delta(G) \leqslant k$, then*

$$\alpha'(G) \geqslant \frac{n}{k(k+1)} + \frac{m}{k+1} - \frac{1}{k(k+1)},$$

unless the following holds.

(a) *G is k-regular and $n = k+1$, in which case $\alpha'(G) = \frac{n-1}{2} = \frac{n}{k(k+1)} + \frac{m}{k+1} - \frac{1}{k}$.*

(b) *G is k-regular and $n = k+3$, in which case $\alpha'(G) = \frac{n-1}{2} = \frac{n}{k(k+1)} + \frac{m}{k+1} - \frac{3}{k(k+1)}$.*

Let $k \geqslant 4$ be even and let $r \geqslant 1$ be arbitrary and let $\ell = r(k-1)+1$. Let $X_1, X_2, \ldots, X_\ell$ be a number of vertex disjoint graphs such that each X_i where $i \in [\ell]$ is either a single vertex or it is a K_{k+1} where an arbitrary edge has been deleted. Let $Y = \{y_1, y_2, \ldots, y_r\}$ and build the graph $G_{k,r}$ as follows. Let $G_{k,r}$ be obtained from the disjoint union of the graphs $X_1, X_2, \ldots, X_\ell$ by adding to it the vertices in Y and furthermore, for every $i \in [r]$, adding an edge from y_i to a vertex in each graph $X_{(i-1)(k-1)+1}$, $X_{(i-1)(k-1)+2}$, $X_{(i-1)(k-1)+3}, \ldots, X_{(i-1)(k-1)+k}$ in such a way that no vertex degree becomes more than k. Let $\mathcal{G}_{k,r}$ be the family of all such graph $G_{k,r}$. When $k = 4$ and $r = 2$, an example of a graph G in the family $\mathcal{G}_{k,r}$ is illustrated in Figure 1, where G has order $n = 21$, size $m = 35$ and matching number $\alpha'(G) = 8$.

Proposition 2. ([19]) *For $k \geqslant 4$ an even integer and $r \geqslant 1$ arbitrary, if $G \in \mathcal{G}_{k,r}$ has order n and size m, then*

$$\alpha'(G) = \frac{n}{k(k+1)} + \frac{m}{k+1} - \frac{1}{k(k+1)}.$$

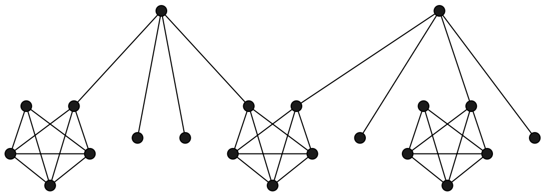


Figure 1: A graph G in the family $\mathcal{G}_{4,2}$

Theorem 3. ([19]) *If $k \geqslant 3$ is an odd integer and G is a connected graph of order n, size m, and with maximum degree $\Delta(G) \leqslant k$, then*

$$\alpha'(G) \geqslant \left(\frac{k-1}{k(k^2-3)}\right) n + \left(\frac{k^2-k-2}{k(k^2-3)}\right) m - \frac{k-1}{k(k^2-3)}.$$

For $k \geqslant 3$ odd, let H_{k+2} be the graph of (odd) order $k+2$ whose complement $\overline{H_{k+2}}$ is isomorphic to $P_3 \cup (\frac{k-1}{2})P_2$. We note that every vertex in H_{k+2} has degree k, except for exactly one vertex, which has degree $k-1$. We call the vertex of degree $k-1$ in H_{k+2} the *link vertex* of H_{k+2}.

For $k \geqslant 3$ odd and $r \geqslant 1$ arbitrary, let $T_{k,r}$ be a tree with maximum degree at most k and with partite sets V_1 and V_2, where $|V_2| = r$. Let $H_{k,r}$ be obtained from $T_{k,r}$ as follows: For every vertex x in V_2 with $d_{T_{k,r}}(x) < k$, add $k - d_{T_{k,r}}(x)$ copies of the subgraph H_{k+2} to $T_{k,r}$ and in each added copy of H_{k+2}, join the link vertex of H_{k+2} to x. We note that every vertex in the resulting graph $H_{k,r}$ has degree k, except possibly for vertices in the set V_1 whose degrees belong to the set $\{1, 2, \ldots, k\}$. Let $\mathcal{F}_{k,r}$ be the family of all such graphs $H_{k,r}$.

When $k = 3$ and $r = 4$, an example of a graph G in the family $\mathcal{F}_{k,r}$ is illustrated in Figure 2, where G has order $n = 29$, size $m = 40$ and matching number $\alpha'(G) = 12$.

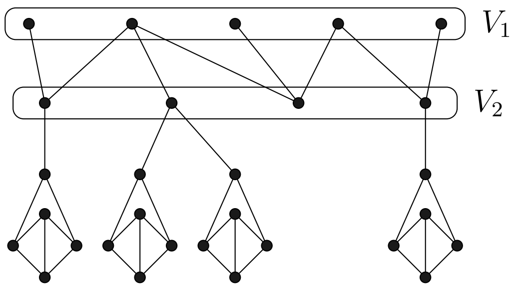


Figure 2: A graph G in the family $\mathcal{F}_{3,4}$

Proposition 4. ([19]) *For $k \geqslant 3$ an odd integer and $r \geqslant 1$ arbitrary, if $G \in \mathcal{F}_{k,r}$ has order n and size m, then*

$$\alpha'(G) = \left(\frac{k-1}{k(k^2-3)}\right) n + \left(\frac{k^2-k-2}{k(k^2-3)}\right) m - \frac{k-1}{k(k^2-3)}.$$

3 Three Families of Hypergraphs

In this section, we define three families of hypergraphs, $\mathcal{H}_k$, $\mathcal{H}'_k$ and $\mathcal{H}''_k$. For a hypergraph H with maximum degree at most 2 we let $V_1(H)$ and $V_2(H)$ denote the set of vertices in H of degree 1 and 2, respectively. Further, we let $n_i(H) = |V_i(H)|$ for $i \in [2]$.

3.1 The Family $\mathcal{H}_k$

Definition 5. Let $\mathcal{H}_k$ be the class of all connected, linear, k-uniform hypergraphs with maximum degree 2.

For a hypergraph $H \in \mathcal{H}_k$ we define a graph G_H as follows. Let the vertices of G_H be the edges of H and let the edges of G_H correspond to the $n_2(H)$ vertices of degree 2 in H: if a vertex of H is contained in the edges e and f of H, then the corresponding edge of the multigraph G_H joins vertices e and f of G_H. Thus, $V(G_H) = E(H)$ and for every $v \in V_2(H)$, contained in the two edges e and f, add an edge between e and f in G_H. By the linearity of H, the multigraph G_H is indeed a graph, called the *dual graph* of H. Since H is k-uniform and $\Delta(H) = 2$, the maximum degree, $\Delta(G_H)$, in G_H is at most k. Since H is connected, so too is G_H. By construction, $n(G_H) = m(H)$ and $m(G_H) = n_2(H)$. We note that if $H \in \mathcal{H}_k$ is 2-regular, then the dual graph, G_H, of H is k-regular.

3.2 The Family $\mathcal{H}'_k$

In order to define the family $\mathcal{H}'_k$, we first define a hypergraph, which we call L_k.

The Hypergraph L_k. For $k \geqslant 2$, let L_k be the 2-regular, k-uniform hypergraph of size $k+1$ and order $k(k+1)/2$ defined inductively as follows. We define $L_2 = K_3$ and we define L_3 to be the hypergraph with $V(L_3) = \{v_1, v_2, \ldots, v_6\}$ and let $E(L_3) = \{e_1, e_2, e_3, e_4\}$, where $e_1 = \{v_1, v_2, v_3\}$, $e_2 = \{v_1, v_4, v_5\}$, $e_3 = \{v_2, v_4, v_6\}$ and $e_4 = \{v_3, v_5, v_6\}$. For $k \geqslant 2$, suppose the hypergraph L_k has been constructed and that $E(L_k) = \{e_1, e_2, \ldots, e_{k+1}\}$. Let L_{k+2} be the hypergraph of order $n(L_k) + 2k + 3$ with $V(L_{k+2}) = V(L_k) \cup \{v\} \cup \{u_1, u_2, \ldots, u_{k+1}\} \cup \{w_1, w_2, \ldots, w_{k+1}\}$ and with edge set $E(L_{k+2}) = \{f_1, f_2, \ldots, f_{k+3}\}$, where $f_i = e_i \cup \{u_i, w_i\}$ for $i \in [k+1]$ and where $f_{k+2} = \{v, u_1, \ldots, u_{k+1}\}$ and $f_{k+3} = \{v, w_1, \ldots, w_{k+1}\}$. The hypergraphs L_2, L_4 and L_6 are illustrated in Figure 3(a), 3(b), and 3(c), respectively.

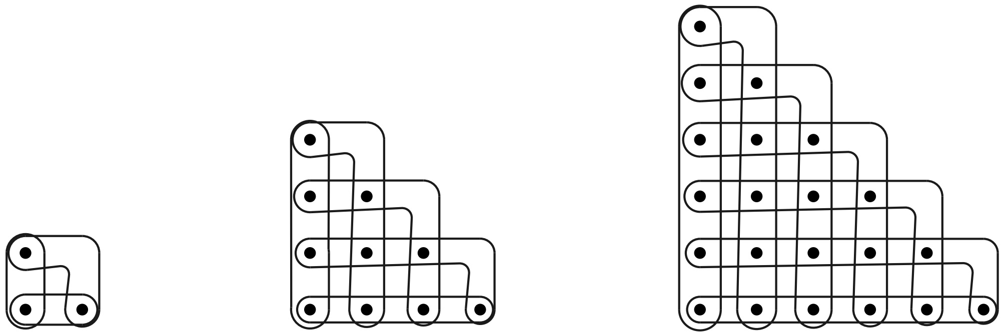
We shall need the following result from [7].

Theorem 6. ([7]) *For $k \geqslant 2$, the hypergraph L_k is the unique k-uniform, 2-regular, linear, intersecting hypergraph.*

Definition 7. Let $\mathcal{H}'_k = \mathcal{H}_k \setminus \{L_k\}$.

3.3 The Family $\mathcal{H}''_k$

For a hypergraph $H \in \mathcal{H}_k$, let $\alpha_2(H)$ be the maximum cardinality of a strongly independent set in H consisting only of degree-2 vertices in H. Every strongly independent set in H corresponds to a matching in the dual graph G_H of H. Conversely, every matching



(a) L_2 (b) L_4 (c) L_6

Figure 3: The hypergraphs L_2, L_4 and L_6.

M in the dual graph G_H of H corresponds to a strongly independent set $V_M \subseteq V_2(H)$ in H. This immediately implies the following observation.

Observation 8. *If $H \in \mathcal{H}_k$ and G_H is the dual graph of H, then $\alpha'(G_H) = \alpha_2(H)$.*

The following result is well-known (see, for example, [7]). However, since it is central to our discussions, we give a short proof for completeness.

Proposition 9. *If $H \in \mathcal{H}_k$ and G_H is the dual graph of H, then $\alpha'(G_H) = |E(H)| - \tau(H)$.*

Proof. Let $H \in \mathcal{H}_k$ and let G_H be the dual graph of H. If M is a maximum matching, then the corresponding set $V_M \subseteq V_2(H)$ is a maximum strong independent set in $V_2(H)$ by Observation 8. Therefore, V_M covers $2|V_M|$ distinct edges in H. Using an additional $|E(H)| - 2|V_M|$ vertices in H, one from each of the edges not covered by V_H, we can extend the set V_M to a transversal in H. Therefore, $\tau(H) \leqslant |V_M| + (|E(H)| - 2|V_M|) = |E(H)| - \alpha'(G_H)$, or, equivalently, $\alpha'(G_H) \leqslant |E(H)| - \tau(H)$.

Conversely, let T be a minimum transversal in H, and so, $\tau(H) = |T|$. If a vertex $x \in T$ covers only one hyperedge in H that is not covered by $T \setminus \{x\}$, then delete this vertex from T and the edge it covers from H. We continue this process removing r vertices from T, resulting in a set T', and r associated edges from H, resulting in a hypergraph H', until every vertex in T' covers two distinct edges in H' that are not covered by any other vertex of T'. Therefore, T' corresponds to a matching in G_H, and $|E(H)| = |E(H')| + r = 2|T'| + r = 2|T'| + (|T| - |T'|) = |T'| + |T|$. Thus, $\alpha'(G_H) \geqslant |T'| = |E(H)| - |T| = |E(H)| - \tau(H)$. As observed earlier, $\alpha'(G_H) \leqslant |E(H)| - \tau(H)$. Consequently, $\alpha'(G_H) = |E(H)| - \tau(H)$. □

The Family $\mathcal{M}_k$. Let $\mathcal{M}_k$ be the class of all connected, linear, k-uniform, 2-regular hypergraphs H with $k+3$ edges. We note that $\mathcal{M}_k$ is a subclass of $\mathcal{H}_k$. The dual graph, G_H, of a hypergraph $H \in \mathcal{M}_k$ is a k-regular graph of order $k+3$. We note that the complement $\overline{G_H}$ of G_H is a 2-regular graph on $k+3$ vertices. Thus, G_H can be constructed from K_{k+3} by removing the edges of a cycle factor of K_{k+3}. Using this approach, we observe that the number of non-isomporphic hypergraphs in $\mathcal{M}_k$ is equal to the number of non-isomorphic cycle factors in K_{k+3}. For example, $|\mathcal{M}_4| = 2$ (the cycle factors in K_7 are either a Hamilton cycle or the union of a 3-cycle and a 4-cycle) and $|\mathcal{M}_6| = 4$ (consider cycle factors with cycle lengths (9), $(6,3)$, $(5,4)$ and $(3,3,3)$). We state this formally as follows.

Observation 10. *The following holds.*

(a) *If $H \in \mathcal{M}_k$, then the dual graph of H is a k-regular graph of order $k+3$.*

(b) *If G is a k-regular graph of order $k+3$, then the dual hypergraph of G belongs to $\mathcal{M}_k$ and has order $k(k+3)/2$.*

Definition 11. Let $\mathcal{H}''_k = \mathcal{H}'_k \setminus \mathcal{M}_k = \mathcal{H}_k \setminus (\mathcal{M}_k \cup \{L_k\})$.

4 Main Results

In what follows, we adopt the following notation. If $H \in \mathcal{H}_k$, we let H have order n and size m, and so $n = n(H)$ and $m = m(H)$. Further, we let $n_i = n_i(H)$ for $i \in [2]$, and so n_1 and n_2 denote the number of vertices of degree 1 and 2, respectively, in H. We note that $km = n_1 + 2n_2$. We denote the number of components of a hypergraph H by $c(H)$.

4.1 Transversal Number

Our first result establishes an upper bound on the transversal number of a connected, linear, k-uniform hypergraph with maximum degree 2 for $k \geqslant 2$ even.

Theorem 12. *For all even $k \geqslant 2$ the following holds.*

(a) *If $H \in \mathcal{H}_k$, then $\tau(H) \leqslant \frac{kn+(k-1)m+k+1}{k(k+1)}$.*

(b) *If $H \in \mathcal{H}'_k$, then $\tau(H) \leqslant \frac{kn+(k-1)m+3}{k(k+1)}$.*

(c) *If $H \in \mathcal{H}''_k$, then $\tau(H) \leqslant \frac{kn+(k-1)m+1}{k(k+1)}$.*

Proof. Let $k \geqslant 2$ be even and let $H \in \mathcal{H}_k$. Let G_H be the dual graph of H. If $H = L_k$, then, by Theorem 6, we note that $m = k+1$ and G_H is a k-regular graph of order $k+1$. If $H \in \mathcal{M}_k$, then $m = k+3$ and, by Observation 10, the graph G_H is a k-regular graph of order $k+3$. If $H \in \mathcal{H}''_k$, then G_H has maximum degree $\Delta(G) \leqslant k$. Further, if G_H is k-regular (and still $H \in \mathcal{H}''_k$), then $n(G_H) \notin \{k+1, k+3\}$. In all cases, we note that G_H is a connected graph of order $n(G_H) = m$ and size $m(G_H) = n_2$. Let

$$\theta = \begin{cases} 1 & \text{if } H \in \mathcal{H}''_k \\ 3 & \text{if } H \in \mathcal{M}_k \\ k+1 & \text{if } H = L_k. \end{cases}$$

By Theorem 1 and our definition of θ, the following holds.

$$\alpha'(G_H) \geqslant \frac{m}{k(k+1)} + \frac{n_2}{k+1} - \frac{\theta}{k(k+1)}.$$

By Proposition 9, we note that the following therefore holds.

$$\begin{array}{rcl}\tau(H) & = & m-\alpha'(G_H)\\ & \leqslant & m-\left(\dfrac{m}{k(k+1)}+\dfrac{n_2}{k+1}-\dfrac{\theta}{k(k+1)}\right)\\ & = & \left(1-\dfrac{1}{k(k+1)}\right)\left(\dfrac{n_1+2n_2}{k}\right)-\dfrac{n_2}{k+1}+\dfrac{\theta}{k(k+1)}\\ & = & \left(\dfrac{k(k+1)-1}{k^2(k+1)}\right)n_1+\left(\dfrac{2(k(k+1)-1)-k^2}{k^2(k+1)}\right)n_2+\dfrac{\theta}{k(k+1)}.\end{array}$$

Simplifying and multiplying through with $k^2(k+1)$ we obtain the following.

$$\begin{array}{rcl}k^2(k+1)\tau(H) & \leqslant & (k^2+k-1)n_1+(k^2+2k-2)n_2+k\theta\\ & = & k^2(n_1+n_2)+(k-1)(n_1+2n_2)+k\theta\\ & = & k^2n+(k-1)(km)+k\theta.\end{array}$$

This implies the desired result. □

We discuss next the hypergraphs $H\in\mathcal{H}_k$ for $k\geqslant 4$ even that achieve the upper bound for the transversal number in the statement of Theorem 12. If $H=L_k$, then $m(H)=k+1$ and $n(H)=k(k+1)/2$, and the dual graph of H is the graph K_{k+1}. Therefore, by Proposition 9,

$$\tau(H)=m(H)-\alpha'(K_{k+1})=(k+1)-\frac{k}{2}=\frac{k+2}{2}=\frac{kn+(k-1)m+k+1}{k(k+1)},$$

and equality holds in the statement of Theorem 12(a). If $H\in\mathcal{M}_k$, then $m(H)=k+3$ and $n(H)=k(k+3)/2$. By Observation 10, the dual graph, G_H, of H is a k-regular graph of order $k+3$. Therefore, by Proposition 9,

$$\tau(H)=m(H)-\alpha'(G_H)=(k+3)-\frac{k+2}{2}=\frac{k+4}{2}=\frac{kn+(k-1)m+3}{k(k+1)},$$

and equality holds in the statement of Theorem 12(b). We show next that there is an infinite family of hypergraphs $H\in\mathcal{H}''_k$ that satisfy

$$\tau(H)=\frac{kn+(k-1)m+1}{k(k+1)}.$$

For $k\geqslant 4$ an even integer and $r\geqslant 1$, let G be an arbitrary graph in the family $\mathcal{G}_{k,r}$. We show that associated with the graph G, there exists a hypergraph $H\in\mathcal{H}''_k$ for which equality holds in the statement of Theorem 12(c), constructed as follows. Let H_G be the dual hypergraph of G, and so the edges of G become vertices in H_G and the vertices of G become hyperedges in H_G, containing all edges that are incident with that vertex in the graph. We note that $n(H_G)=m(G)$ and $m(H_G)=n(G)$.

Since $\Delta(G) = k$, we note that the rank of H_G is k. We note further that the edges of size 1 in H_G, if any, correspond to the pendant edges in G (that are incident with a vertex of degree 1). The edges of size 2 in H_G, if any, correspond to vertices of degree 2 in G (that have both neighbors in Y). All other edges in H_G have size $k-1$ or k.

We now expand all edges of H_G of size less than k to edges of size k by adding new vertices of degree 1 to each such edge. For example, if e_v is an edge of size 1 in H_G containing the vertex v, then we add $k-1$ new vertices and expand the edge e_v to an edge of size k that contains these new vertices and the vertex v. Let H_G^k denote the resulting hypergraph, and let $\mathcal{H}_{k,r}^{\rm even}$ be the family of all such hypergraphs H_G^k. For example, given the graph $G \in \mathcal{G}_{4,2}$ shown in Figure 1 we obtain the associated hypergraph $H \in \mathcal{H}_{4,2}^{\rm even}$ shown in Figure 4.

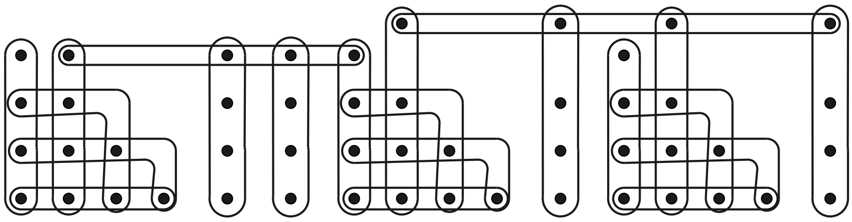


Figure 4: The hypergraph $H \in \mathcal{H}_{4,2}^{\rm even}$ associated with the graph $G \in \mathcal{G}_{4,2}$ shown in Figure 1.

Proposition 13. *For $k \geqslant 4$ an even integer and $r \geqslant 1$ arbitrary, if $H \in \mathcal{H}_{k,r}^{\rm even}$ has order n and size m, then*

$$\tau(H) = \frac{kn + (k-1)m + 1}{k(k+1)}.$$

Proof. We consider the graph $G \in \mathcal{G}_{k,r}$ used to construct the hypergraph $H \in \mathcal{H}_{k,r}^{\rm even}$, and so $H = H_G^k$. Assume that when building the graph G, we have ℓ_1 single vertices and ℓ_2 copies of K_{k+1}'s minus an edge in $X_1, X_2, \ldots, X_\ell$. We note that $\ell_1 + \ell_2 = \ell = r(k-1)+1$ and $n(G) = r + \ell_1 + \ell_2(k+1)$. Further,

$$\alpha'(G) = r + \left(\frac{k}{2}\right)\ell_2 = \frac{\ell_1 + \ell_2 - 1}{k-1} + \left(\frac{k}{2}\right)\ell_2 = \frac{2\ell_1 + (k^2 - k + 2)\ell_2 - 2}{2(k-1)}.$$

The order of H_G^k is

$$n(H_G^k) = k\ell_1 + \left(\frac{k^2+k+2}{2}\right)\ell_2.$$

Further, $m(H_G^k) = m(H_G) = n(G) = r + \ell_1 + \ell_2(k+1)$, implying that the size of H_G^k is

$$m(H_G^k) = \left(\frac{k}{k-1}\right)\ell_1 + \left(\frac{k^2}{k-1}\right)\ell_2 - \frac{1}{k-1}.$$

We remark that the graph $G \in \mathcal{G}_{k,r}$ used to construct the hypergraph $H_G^k \in \mathcal{H}_{k,r}^{\rm even}$ is in fact the dual graph (see Section 3.1) of H_G^k. Therefore, letting $H = H_G^k$, $n = n(H_G^k)$ and $m = m(H_G^k)$, and applying Proposition 9 to H and its dual graph G, we have

$$\begin{aligned}
\tau(H) &= m - \alpha'(G) \\
&= \left(\left(\frac{k}{k-1} \right) \ell_1 + \left(\frac{k^2}{k-1} \right) \ell_2 - \frac{1}{k-1} \right) \\
&\quad - \left(\left(\frac{1}{k-1} \right) \ell_1 + \left(\frac{k^2-k+2}{2(k-1)} \right) \ell_2 - \frac{1}{k-1} \right) \\
&= \ell_1 + \left(\frac{k^2+k-2}{2(k-1)} \right) \ell_2 \\
&= \ell_1 + \left(\frac{k+2}{2} \right) \ell_2
\end{aligned}$$

and

$$\begin{aligned}
\frac{kn + (k-1)m + 1}{k(k+1)} &= \left(\frac{k}{k(k+1)} \right) \left(k\ell_1 + \left(\frac{k^2+k+2}{2} \right) \ell_2 \right) \\
&\quad + \left(\frac{k-1}{k(k+1)} \right) \left(\left(\frac{k}{k-1} \right) \ell_1 + \left(\frac{k^2}{k-1} \right) \ell_2 - \frac{1}{k-1} \right) \\
&\quad + \frac{1}{k(k+1)} \\
&= \ell_1 + \left(\frac{k+2}{2} \right) \ell_2.
\end{aligned}$$

Equality therefore holds in the statement of Theorem 12(c). □

Next we consider the case when $k \geqslant 3$ is odd.

Theorem 14. *For $k \geqslant 3$ an odd integer, if $H \in \mathcal{H}_k$, then*

$$\tau(H) \leqslant \frac{(k-2)(k+1)n + (k-1)^2 m + k - 1}{k(k^2-3)}.$$

Proof. Let $k \geqslant 3$ be odd and let $H \in \mathcal{H}_k$. Let G_H be the dual graph of H and note that G_H has maximum degree $\Delta(G) \leqslant k$. Further, we note that G_H is a connected graph of order $n(G_H) = m$ and size $m(G_H) = n_2$. By Theorem 3, the following holds.

$$\alpha'(G_H) \geqslant \left(\frac{k-1}{k(k^2-3)} \right) m + \left(\frac{k^2-k-2}{k(k^2-3)} \right) n_2 - \frac{k-1}{k(k^2-3)}.$$

By Proposition 9, we note that the following therefore holds.

$$
\begin{aligned}
\tau(H) &= m-\alpha'(G_H) \\
&\leqslant m-\left(\left(\frac{k-1}{k(k^2-3)}\right)m+\left(\frac{k^2-k-2}{k(k^2-3)}\right)n_2-\frac{k-1}{k(k^2-3)}\right) \\
&= \left(1-\frac{k-1}{k(k^2-3)}\right)\left(\frac{n_1+2n_2}{k}\right)-\left(\frac{k^2-k-2}{k(k^2-3)}\right)n_2+\frac{k-1}{k(k^2-3)} \\
&= \left(\frac{k^3-4k+1}{k^2(k^2-3)}\right)n_1+\left(\frac{2(k^3-4k+1)-k(k^2-k-2)}{k^2(k^2-3)}\right)n_2+\frac{k-1}{k(k^2-3)}.
\end{aligned}
$$

Simplifying and multiplying through with $k^2(k^2-3)$ we obtain the following.

$$
\begin{aligned}
k^2(k^2-3)\tau(H) &\leqslant (k^3-4k+1)n_1+(k^3+k^2-6k+2)n_2+k(k-1) \\
&= (k^3-2k)(n_1+n_2)-(2k-1)(n_1+2n_2)+k^2\cdot n_2+k(k-1) \\
&= (k^3-2k)n-(2k-1)(km)+k^2\cdot n_2+k(k-1) \\
&= (k^3-2k)n-(2k-1)(km)+k^2(km-n)+k(k-1) \\
&= k(k-2)(k+1)n+k(k-1)^2m+k(k-1).
\end{aligned}
$$

This implies the desired result. □

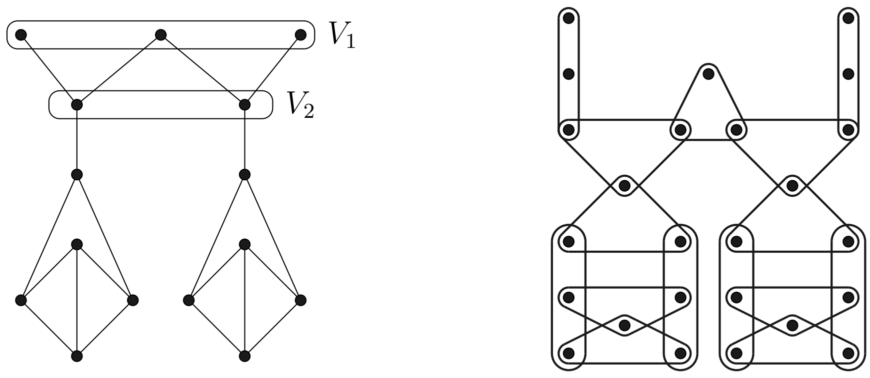
We discuss next the hypergraphs $H \in \mathcal{H}_k$ for $k \geqslant 3$ odd that achieve the upper bound for the transversal number in the statement of Theorem 14. For $k \geqslant 3$ an even integer and $r \geqslant 1$, let G be an arbitrary graph in the family $\mathcal{F}_{k,r}$. Analogously as with the case when k is even, we let H_G be the dual hypergraph of G, and we let H_G^k be the hypergraph obtained from H_G by expanding all edges of H_G of size less than k to edges of size k by adding new vertices of degree 1 to each such edge. Let H_G^k denote the resulting hypergraph, and let $\mathcal{H}_{k,r}^{\rm odd}$ be the family of all such hypergraphs H_G^k. For example, given the graph $G \in \mathcal{F}_{3,2}$ shown in Figure 5(a) we obtain the associated hypergraph $H \in \mathcal{H}_{3,2}^{\rm odd}$ shown in Figure 5(b).

Proposition 15. *For $k \geqslant 3$ an odd integer and $r \geqslant 1$ arbitrary, if $H \in \mathcal{H}_{k,r}^{\rm odd}$ has order n and size m, then*

$$
\tau(H)=\frac{(k-2)(k+1)n+(k-1)^2m+k-1}{k(k^2-3)}.
$$

Proof. We consider the graph $G \in \mathcal{F}_{k,r}$ used to construct the hypergraph $H \in \mathcal{H}_{k,r}^{\rm odd}$, and so $H = H_G^k$. Assume that ℓ copies of the graph H_{k+2} were added when constructing the graph G. Thus, as observed in [19],

$$
\ell = (k-1)|V_2| - |V_1| + 1.
$$



(a) $G \in \mathcal{F}_{3,2}$ (b) $H \in \mathcal{H}^{\rm odd}_{3,2}$

Figure 5: The hypergraph $H \in \mathcal{H}^{\rm odd}_{3,2}$ associated with the graph $G \in \mathcal{F}_{3,2}$.

Further, the order, size and matching number of G are as follows.

$$\begin{array}{rcl} n(G) & = & (k^2+k-1)|V_2| - (k+1)|V_1| + (k+2) \\ 2m(G) & = & (k^3+k^2-k+1)|V_2| - (k^2+2k-1)|V_1| + (k^2+2k-1) \\ 2\alpha'(G) & = & (k^2+1)|V_2| - (k+1)|V_1| + (k+1). \end{array}$$

For $i \in [k]$, let $n_{1,i}$ be the number of vertices in V_1 that have degree i in G. Thus, if $[V_1, V_2]$ denotes the set of edges between V_1 and V_2 in G, then

$$\sum_{i=1}^{k} n_{1,i} = |V_1| \quad \text{and} \quad \sum_{i=1}^{k} i \cdot n_{1,i} = |[V_1, V_2]| = k|V_2| - \ell = |V_1| + |V_2| - 1.$$

Recall that $H = H_G^k$, $n = n(H_G^k)$ and $m = m(H_G^k)$. The order of H is

$$\begin{array}{rcl} n & = & m(G) + \displaystyle\sum_{i=1}^{k} (k-i) \cdot n_{1,i} \\ & = & m(G) + k \left(\displaystyle\sum_{i=1}^{k} n_{1,i} \right) - \left(\displaystyle\sum_{i=1}^{k} i \cdot n_{1,i} \right) \\ & = & m(G) + (k-1)|V_1| - |V_2| + 1 \\ & = & \left(\dfrac{k^3+k^2-k-1}{2} \right) |V_2| - \left(\dfrac{k^2+1}{2} \right) |V_1| + \left(\dfrac{k^2+2k+1}{2} \right). \end{array}$$

Further, H has size $m = m(H_G^k) = m(H_G) = n(G)$, and so

$$m = (k^2+k-1)|V_2| - (k+1)|V_1| + (k+2).$$

We remark that the graph $G \in \mathcal{F}_{k,r}$ used to construct the hypergraph $H_G^k \in \mathcal{H}_{k,r}^{\mathrm{odd}}$ is in fact the dual graph (see Section 3.1) of H_G^k. Therefore, applying Proposition 9 to H and its dual graph G, we have

$$\begin{aligned}\tau(H) &= m - \alpha'(G) \\ &= \big((k^2+k-1)|V_2| - (k+1)|V_1| + (k+2)\big) \\ &\quad -\frac{1}{2}\big((k^2+1)|V_2| - (k+1)|V_1| + (k+1)\big) \\ &= \left(\frac{k^2+2k-3}{2}\right)|V_2| - \left(\frac{k+1}{2}\right)|V_1| + \frac{k+3}{2}\end{aligned}$$

and

$$\begin{aligned}&\frac{(k-2)(k+1)n + (k-1)^2 m + k - 1}{k(k^2-3)} \\ &= \left(\frac{(k-2)(k+1)}{k(k^2-3)}\right)\left(\left(\frac{k^3+k^2-k-1}{2}\right)|V_2| - \left(\frac{k^2+1}{2}\right)|V_1| + \left(\frac{k^2+2k+1}{2}\right)\right) \\ &\quad + \left(\frac{(k-1)^2}{k(k^2-3)}\right)\big((k^2+k-1)|V_2| - (k+1)|V_1| + (k+2)\big) \\ &\quad + \frac{k-1}{k(k^2-3)} \\ &= \left(\frac{k^2+2k-3}{2}\right)|V_2| - \left(\frac{k+1}{2}\right)|V_1| + \frac{k+3}{2}.\end{aligned}$$

Equality therefore holds in the statement of Theorem 14. □

4.2 Strong Independence Number

In this section we establish a lower bound on the strong independence number of a connected, linear, k-uniform hypergraph H with maximum degree 2 for $k \geqslant 2$. For this purpose, we first establish a lower bound on a maximum strong independent set consisting only of degree-2 vertices in H.

Theorem 16. *For all even $k \geqslant 2$ the following holds.*

(a) *If $H \in \mathcal{H}_k$, then $\alpha_2(H) \geqslant \frac{n_1+(k^2+2)n_2-k(k+1)}{k^2(k+1)}$.*

(b) *If $H \in \mathcal{H}'_k$, then $\alpha_2(H) \geqslant \frac{n_1+(k^2+2)n_2-3k}{k^2(k+1)}$.*

(c) *If $H \in \mathcal{H}''_k$, then $\alpha_2(H) \geqslant \frac{n_1+(k^2+2)n_2-k}{k^2(k+1)}$.*

Proof. Let $k \geqslant 2$ be even and let $H \in \mathcal{H}_k$, and let G_H be the dual graph of H. We adopt the notation in the proof of Theorem 12. Analogously as in the proof of Theorem 12,

$$\alpha'(G_H) \geqslant \frac{m}{k(k+1)} + \frac{n_2}{k+1} - \frac{\theta}{k(k+1)}.$$

By Observation 8, we note that the following therefore holds.

$$\begin{aligned}\alpha_2(H) &= \alpha'(G_H)\\ &\geqslant \frac{m}{k(k+1)} + \frac{n_2}{k+1} - \frac{\theta}{k(k+1)}\\ &= \left(\frac{1}{k(k+1)}\right)\left(\frac{n_1+2n_2}{k}\right) + \frac{n_2}{k+1} - \frac{\theta}{k(k+1)}.\end{aligned}$$

Multiplying through with $k^2(k+1)$ we obtain the following.

$$k^2(k+1)\alpha_2(H) \geqslant n_1 + (k^2+2)n_2 - k\theta.$$

This implies the desired result. □

We proceed further with the following simple lemma.[1]

Lemma 17. ([3]) *If H is a k-uniform hypergraph of order n and size m with $\delta(H) \geqslant 1$ and with c components, then $(k-1)m + c \geqslant n$.*

Proof. Replace each hyperedge $e \in E(H)$ by a star of $k-1$ edges on the vertex set of e to produce a graph G. If H has c components, then so too does G. Since G has $(k-1)m$ edges, n vertices and c components, we have that $(k-1)m + c \geqslant n$. □

As a special case of Lemma 17, we note that if H is a connected k-uniform hypergraph of order n and size m, then $(k-1)m + 1 \geqslant n$.

Theorem 18. *For all even $k \geqslant 2$ the following holds.*

(a) *If $H \in \mathcal{H}_k$, then $\alpha(H) \geqslant \frac{(k+2)n-(k-1)m-(k+1)}{k(k+1)}$.*

(b) *If $H \in \mathcal{H}'_k$, then $\alpha(H) \geqslant \frac{(k+2)n-(k-1)m-3}{k(k+1)}$.*

(c) *If $H \in \mathcal{H}''_k$, then $\alpha(H) \geqslant \frac{(k+2)n-(k-1)m-1}{k(k+1)}$.*

Proof. Let $k \geqslant 2$ be even and let $H \in \mathcal{H}''_k$ be arbitrary. Let $V_1(H)$ denote the set of all vertices of degree 1 in H, and let S be the set of all edges of H that contain at least one vertex in $V_1(H)$. Let R be the vertices in H which belong to two edges of S, and let $r = |R|$. Let $X = V(H) \setminus (V_1(H) \cup R)$ and consider the hypergraph $H[X]$ induced by the vertices in X. Let S' be the set of edges in $H[X]$ of size less than k. We note that each edge in S' was obtained by shrinking an edge in S by removing from it vertices in $V_1(H) \cup R$. We note that $H[X]$ contains at most $r+1$ components; that is, $c(H[X]) \leqslant r+1$.

[1] We have not been able to find the original source of this lemma, but as remarked in [3], "it definitely seems to have been known already at least in the early 1960's." For completeness, we provide the short proof given in [3].

Let H' be obtained from $H[X]$ by removing all edges in $H[X]$ of size less than k. Equivalently, H' is obtained from H be removing all edges in S and all resulting isolated vertices. We note that H' has order

$$n(H') = n(H) - n_1(H) - r$$

and may possibly be the empty hypergraph. For every $i = \{0\} \cup [k-1]$, let T_i denote the subset of edges of S which contain vertices from exactly i different components in H' and let $t_i = |T_i|$. We note that for $i \in [k-1] \setminus \{1\}$, the removal of all edges in T_i from $H[X]$ gives rise to at most $(i-1)t_i$ additional components. Thus,

$$c(H') \leqslant c(H[X]) + \sum_{i=2}^{k-1}(i-1)t_i.$$

As observed earlier, $c(H[X]) \leqslant r+1$, implying that

$$\sum_{i=2}^{k-1}(i-1)t_i \geqslant c(H') - r - 1.$$

Every edge in T_i contains at most $k-i$ vertices of degree 1 in H, and at least i vertices from different components of H', in addition to possibly some vertices of R. Thus,

$$\begin{aligned} n_1(H) &\leqslant k|S| - \left(\sum_{i=1}^{k-1} i \cdot t_i\right) - 2r \\ &= k|S| - \left(\sum_{i=0}^{k-1} t_i\right) - \left(\sum_{i=2}^{k-1}(i-1)t_i\right) - 2r + t_0 \\ &\leqslant k|S| - |S| - (c(H') - r - 1) - 2r + t_0 \\ &= (k-1)|S| - c(H') - r + t_0 + 1. \end{aligned}$$

We now obtain a strong independent set in H by taking a maximum strong independent set of degree-2 vertices in H' and adding to this set a vertex of degree one from each edge in S. Therefore the following holds by Theorem 16, as no component belongs to $\{L_k\} \cup \mathcal{M}_k$ (recall that $H \in \mathcal{H}''_k$).

$$\alpha(H) \geqslant |S| + \alpha_2(H') \geqslant |S| + \frac{n_1(H') + (k^2+2)n_2(H') - k \cdot c(H')}{k^2(k+1)}.$$

As $n_1(H') + n_2(H') = n(H') = n(H) - n_1(H) - r$, we note that

$$n_2(H') = n_2(H) - n_1(H') - r.$$

Furthermore,

$$n_1(H') = k|S| - n_1(H) - 2r,$$

as the $|S|$ edges in S each have k vertices and every vertex with degree 1 in H' belongs to an edge in S and does not have degree 1 in H and does not belong to R, and every vertex in R counts two in $k|S| - n_1(S)$ but does not belong to H'. The following now holds by the above observations.

$$\begin{aligned}
& k^2(k+1)\alpha(H) \\
\geqslant\ & k^2(k+1)|S| + n_1(H') + (k^2+2)n_2(H') - k \cdot c(H') \\
=\ & k^2(k+1)|S| + n_1(H') + (k^2+2)(n_2(H) - r - n_1(H')) - k \cdot c(H') \\
=\ & k^2(k+1)|S| + n_1(H')(1-(k^2+2)) + (k^2+2)n_2(H) - (k^2+2)r - k \cdot c(H') \\
=\ & k^2(k+1)|S| + (k|S| - n_1(H) - 2r)(-k^2-1) + (k^2+2)n_2(H) - (k^2+2)r - k \cdot c(H') \\
=\ & (k^3+k^2-k^3-k)|S| + n_1(H)(k^2+1) + (k^2+2)n_2(H) + k^2 r - k \cdot c(H') \\
=\ & (k(k-1)|S| - k \cdot c(H') - kr + kt_0 + k) \\
& \quad -kt_0 + n_1(H)(k^2+1) + (k^2+2)n_2(H) + (k^2+k)r - k \\
\geqslant\ & (k \cdot n_1(H)) - kt_0 + n_1(H)(k^2+1) + (k^2+2)n_2(H) + (k^2+k)r - k \\
=\ & (k^2+k+1)n_1(H) + (k^2+2)n_2(H) + (k^2+k)r - kt_0 - k \\
=\ & (k^2+2k)(n_1(H)+n_2(H)) - (k-1)(n_1(H)+2n_2(H)) + (k^2+k)r - kt_0 - k \\
=\ & (k^2+2k)n(H) - (k-1)(k \cdot m(H)) + (k^2+k)r - kt_0 - k.
\end{aligned}$$

Note that every edge in T_0 must contain a vertex from R. In particular, if $r = 0$, then $t_0 = 0$. In this case, dividing though by k the above simplifies to the following.

$$k(k+1)\alpha(H) \geqslant (k+2)n(H) - (k-1)m(H) - 1.$$

Suppose that $r \geqslant 1$. We note that every edge in T_0 contains at most $k-1$ vertices from R, and so $t_0 \leqslant (k-1)r$. Dividing though by k above we get the following.

$$\begin{aligned}
k(k+1)\alpha(H) &\geqslant (k+2)n(H) - (k-1)m(H) + (k+1)r - t_0 - 1 \\
&\geqslant (k+2)n(H) - (k-1)m(H) + (k+1)r - (k-1)r - 1 \\
&= (k+2)n(H) - (k-1)m(H) + 2r - 1 \\
&\geqslant (k+2)n(H) - (k-1)m(H) - 1.
\end{aligned}$$

This implies the theorem in the case when $H \in \mathcal{H}''_k$.

Suppose next that $H \in \mathcal{H}'_k$. If $H \notin \mathcal{M}_k$, then as shown above we have $k(k+1)\alpha(H) \geqslant (k+2)n(H) - (k-1)m(H) - 1$. Suppose, therefore, that $H \in \mathcal{M}_k$. We note that, by Theorem 16,

$$\alpha_2(H) \geqslant \frac{n_1(H) + (k^2+2)n_2(H) - 3k}{k^2(k+1)}.$$

As H is 2-regular, we have $\alpha(H) = \alpha_2(H)$ and $n_1(H) = 0$, and therefore $n(H) = n_2(H) = k(k+3)/2$ and $k \cdot m(H) = 2n_2(H) = k(k+3)$. Therefore,

$$\alpha(H) \geqslant \frac{(k^2+2)n_2(H) - 3k}{k^2(k+1)}$$

$$= \frac{k(k+2)n_2(H) - 2(k-1)n_2(H) - 3k}{k^2(k+1)}$$
$$= \frac{k(k+2)n(H) - (k-1)(k \cdot m(H)) - 3k}{k^2(k+1)}$$
$$= \frac{(k+2)n(H)| - (k-1)m(H) - 3}{k(k+1)}.$$

This implies the theorem in the case when $H \in \mathcal{H}'_k$.

Suppose finally that $H \in \mathcal{H}_k$. From the above, it remains for us to consider the case when $H = L_k$. In this case Theorem 16 implies that

$$\alpha_2(H) \geqslant \frac{n_1(H) + (k^2+2)n_2(H) - k(k+1)}{k^2(k+1)}.$$

As H is 2-regular, we have $\alpha(H) = \alpha_2(H)$ and $n_1(H) = 0$, and therefore $n(H) = n_2(H) = k(k+1)/2$ and $k \cdot m(H) = 2n_2(H) = k(k+1)$. Analogous to the discussion in the previous argument,

$$\alpha(H) \geqslant \frac{(k+2)n(H) - (k-1)m(H) - (k+1)}{k(k+1)},$$

This implies the theorem in the case when $H \in \mathcal{H}_k$. □

We discuss next the hypergraphs $H \in \mathcal{H}_k$ for $k \geqslant 2$ even that achieve the lower bound for the strong independence number in the statement of Theorem 18. If $H = L_k$, then, by Observation 8 and Theorem 1(a), equality holds in the statement of Theorem 18(a). If $H \in \mathcal{M}_k$, then, by Observation 10 and Theorem 1(b), equality holds in the statement of Theorem 18(b).

We show next that there is an infinite family of hypergraphs $H \in \mathcal{H}''_k$ for which equality holds in the statement of Theorem 18(c). For $k \geqslant 4$ an even integer and $r \geqslant 1$, let G be an arbitrary graph in the family $\mathcal{G}_{k,r}$, and let H^k_G be the associated hypergraph in the family $\mathcal{H}^{\rm even}_{k,r}$. For each vertex v of degree 1 in H^k_G, we add $k-1$ new vertices and an edge (of size k) containing v and these new vertices. Let R^k_G denote the resulting hypergraph, and let $\mathcal{R}_{k,r}$ be the family of all such hypergraphs R^k_G.

Proposition 19. *For $k \geqslant 4$ an even integer and $r \geqslant 1$ arbitrary, if $H \in \mathcal{R}^{\rm even}_{k,r}$ has order n and size m, then*

$$\alpha(H) = \frac{(k+2)n - (k-1)m - 1}{k(k+1)}.$$

Proof. Let $G \in \mathcal{G}_{k,r}$ be the graph and $H^k_G \in \mathcal{H}^{\rm even}_{k,r}$ the associated hypergraph used to construct the hypergraph $H \in \mathcal{R}^{\rm even}_{k,r}$, and so $H = R^k_G$.

We show firstly that $\alpha(R^k_G) = n_1(H^k_G) + \alpha'(G)$. Let S be a maximum independent set in $H = R^k_G$ that contains the maximum number of vertices of degree 1 in H. For each vertex v of degree 1 in H^k_G, let e_v be the associated edge containing v that was added to H_G^k when constructing H. We note that every vertex in e_v different from v has degree 1 in H. Let v' be an arbitrary vertex in e_v different from v. If $v \in S$ or if S contains no vertex from e_v, then the set $(S \setminus \{v\}) \cup \{v'\}$ is a maximum independent set containing more vertices of degree 1 than does S, a contradiction. Hence, the set S contains $n_1(H_G^k)$ vertices of degree 1, one from each edge added to H_G^k when constructing H. The remaining vertices of S belong to $V(H_G)$ and have degree 2 in H_G^k, and so $\alpha(H) \leqslant n_1(H_G^k) + \alpha_2(H_G^k)$. Conversely, every maximum independent set of degree-2 vertices in H_G^k can be extended to an independent set in H by adding to it $n_1(H_G^k)$ vertices of degree 1, one vertex from each edge added to H_G^k when constructing H, implying that $\alpha(H) \geqslant n_1(H_G^k) + \alpha_2(H_G^k)$. Consequently, $\alpha(H) = n_1(H_G^k) + \alpha_2(H_G^k)$. We note that G is the dual graph of the hypergraph $H_G^k \in \mathcal{H}_k$, and so, by Observation 8, $\alpha_2(H_G^k) = \alpha'(G)$. Therefore, $\alpha(R_G^k) = n_1(H_G^k) + \alpha'(G)$.

Let G be constructed from ℓ_1 single vertices and ℓ_2 copies of $K_{k+1} - e$. Further, let $\ell_{1,1}$ and $\ell_{1,2}$ be the number of single vertices of degree 1 and degree 2 in G, and let $\ell_{2,1}$ and $\ell_{2,2}$ be the number of copies of $K_{k+1} - e$ joined to one or two vertices in Y, respectively. We note that $(\ell_{1,1}+\ell_{1,2})+(\ell_{2,1}+\ell_{2,2}) = \ell_1+\ell_2 = \ell = r(k-1)+1$ and $n(G) = r+\ell_1+\ell_2(k+1)$. Recall that $n = n(H)$ and $m = m(H)$. We note that

$$\begin{aligned} n &= n(H_G^k) + (k-1)n_1(H_G^k) \\ m &= m(H_G^k) + n_1(H_G^k) \\ n_1(H_G^k) &= (k-1)\ell_{1,1} + (k-2)\ell_{1,2} + \ell_{2,1} \\ r &= \ell_{1,2} + \ell_{2,2} + 1 \end{aligned}$$

Recall (see the proof of Proposition 13) that

$$\begin{aligned} n(H_G^k) &= k\ell_1 + \tfrac{1}{2}(k^2+k+2)\ell_2 \\ m(H_G^k) &= \tfrac{1}{k-1}\left(k\ell_1 + k^2\ell_2 - 1\right) \\ \alpha'(G) &= r + \tfrac{1}{2}k\ell_2. \end{aligned}$$

We note that

$$\begin{aligned} &\tfrac{1}{k}(\ell_{1,1} + 2\ell_{1,2} + \ell_{2,1} + \ell_{2,2}) \\ &= \tfrac{1}{k}(\ell + \ell_{1,2} + 2\ell_{2,2}) \\ &= \tfrac{1}{k}(\ell + r - 1) \\ &= \tfrac{1}{k}((r(k-1)+1) + r - 1) \\ &= r. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{(k+2)n - (k-1)m - 1}{k(k+1)} =& \left(\frac{k+2}{k(k+1)}\right)\left(n(H_G^k) + (k-1)n_1(H_G^k)\right) \\ &- \left(\frac{k-1}{k(k+1)}\right)\left(m(H_G^k) + n_1(H_G^k)\right) \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{k(k+1)}\\
=&\left(\frac{k+2}{k(k+1)}\right)\left(k\ell_1+\left(\frac{k^2+k+2}{2}\right)\ell_2+(k-1)n_1(H_G^k)\right)\\
&-\left(\frac{k-1}{k(k+1)}\right)\left(\left(\frac{k}{k-1}\right)\ell_1+\left(\frac{k^2}{k-1}\right)\ell_2-\frac{1}{k-1}+n_1(H_G^k)\right)\\
&-\frac{1}{k(k+1)}\\
=&\left(\frac{k(k+2)-k}{k(k+1)}\right)\ell_1\\
&+\left(\frac{k^3+k^2+4k+4}{2k(k+1)}\right)\ell_2\\
&+\left(\frac{k^2-1}{k(k+1)}\right)n_1(H_G^k)\\
=&\ell_1+\left(\frac{k^2+4}{2k}\right)\ell_2+\left(\frac{k-1}{k}\right)n_1(H_G^k)\\
=&(\ell_{1,1}+\ell_{1,2})+\left(\frac{k^2+4}{2k}\right)(\ell_{2,1}+\ell_{2,2})\\
&+\left(\frac{k-1}{k}\right)((k-1)\ell_{1,1}+(k-2)\ell_{1,2}+\ell_{2,1})\\
=&\left(\frac{k^2-k+1}{k}\right)\ell_{1,1}+\left(\frac{k^2-2k+2}{2}\right)\ell_{1,2}\\
&+\left(\frac{k^2+2k+2}{2k}\right)\ell_{2,1}+\left(\frac{k^2+4}{2k}\right)\ell_{2,1}\\
=&(k-1)\ell_{1,1}+(k-2)\ell_{1,2}+\ell_{2,1}\\
&+\frac{1}{k}\left(\ell_{1,1}+2\ell_{1,2}+\ell_{2,1}+2\ell_{2,2}\right)+\frac{1}{2}k(\ell_{2,1}+\ell_{2,2})\\
=&n_1(H_G^k)+r+\frac{1}{2}k\ell\\
=&n_1(H_G^k)+\alpha'(G)\\
=&\alpha(H).
\end{aligned}
$$

□

Next we consider the case when $k \geqslant 3$ is odd.

Theorem 20. *For $k \geqslant 3$ odd, if $H \in \mathcal{H}_k$, then*

$$\alpha_2(H) \geqslant \frac{(k-1)n_1+(k^3-k^2-2)n_2-k(k-1)}{k^2(k^2-3)}.$$

Proof. Let $k \geqslant 3$ be odd and let $H \in \mathcal{H}_k$. Let G_H be the dual graph of H and note that G_H has maximum degree $\Delta(G) \leqslant k$. Further, we note that G_H is a connected graph of order $n(G_H) = m$ and size $m(G_H) = n_2$. By Theorem 3, the following holds.

$$\alpha'(G_H) \geqslant \left(\frac{k-1}{k(k^2-3)}\right)m + \left(\frac{k^2-k-2}{k(k^2-3)}\right)n_2 - \frac{k-1}{k(k^2-3)}.$$

By Observation 8, and noting that $km = n_1 + 2n_2$, the following therefore holds.

$$\alpha_2(H) = \alpha'(G_H) \geqslant \left(\frac{k-1}{k(k^2-3)}\right)\left(\frac{n_1+2n_2}{k}\right) + \left(\frac{k^2-k-2}{k(k^2-3)}\right) n_2 - \frac{k-1}{k(k^2-3)}.$$

Multiplying through with $k^2(k^2-3)$, and simplifying, we obtain the following.

$$k^2(k^2-3)\alpha_2(H) \geqslant (k-1)n_1 + (k^3-k^2-2)n_2 - k(k-1).$$

This implies the desired result. □

Theorem 21. *For $k \geqslant 3$ odd, if $H \in \mathcal{H}_k$, then*

$$\alpha(H) \geqslant \frac{(k^2+k-4)n(H) - (k-1)^2 m(H) - (k-1)}{k(k^2-3)}.$$

Proof. Let $k \geqslant 3$ be odd and let $H \in \mathcal{H}_k$. We follow the same notation as introduced in the proof of Theorem 18. Proceeding exactly as in the proof of Theorem 18, we have

$$\begin{array}{rcl} n_1(H) & \leqslant & (k-1)|S| - c(H') - r + t_0 + 1 \\ n_1(H') & = & k|S| - n_1(H) - 2r \\ n_2(H') & = & n_2(H) - n_1(H') - r \end{array}$$

The following holds by Theorem 20.

$$\alpha(H) \geqslant |S| + \alpha_2(H') \geqslant |S| + \frac{(k-1)n_1(H') + (k^3-k^2-2)n_2(H') - k(k-1)c(H')}{k^2(k^2-3)}.$$

Therefore,

$$\begin{aligned} k^2(k^2-3)\alpha(H) \geqslant & k^2(k^2-3)|S| + (k-1)n_1(H') + (k^3-k^2-2)n_2(H') - k(k-1)c(H') \\ = & k^2(k^2-3)|S| + (k-1)n_1(H') + (k^3-k^2-2)(n_2(H) - n_1(H') - r) \\ & - k(k-1)c(H') \\ = & k^2(k^2-3)|S| + (-k^3+k^2+k+1)n_1(H') \\ & + (k^3-k^2-2)n_2(H) - (k^3-k^2-2)r - k(k-1)c(H') \\ = & k^2(k^2-3)|S| + (-k^3+k^2+k+1)(k|S| - n_1(H) - 2r) \\ & + (k^3-k^2-2)n_2(H) - (k^3-k^2-2)r - k(k-1)c(H') \\ = & (k^4-3k^2-k^4+k^3+k^2+k)|S| + (k^3-k^2-k-1)n_1(H) \\ & + (k^3-k^2-2)n_2(H) + (k^3-k^2-2k)r - k(k-1)c(H') \\ = & k(k-1)^2|S| + (k^3-k^2-k-1)n_1(H) \\ & + (k^3-k^2-2)n_2(H) + (k^3-k^2-2k)r - k(k-1)c(H') \\ = & k(k-1)((k-1)|S| - c(H') - r + t_0 + 1) \\ & - k(k-1)t_0 - k(k-1) + (k^3-k^2-k-1)n_1(H) \end{aligned}$$

$$
\begin{aligned}
&+(k^3-k^2-2)n_2(H)+(k^3-3k)r\\
\geqslant &k(k-1)n_1(H)-k(k-1)t_0-k(k-1)+(k^3-k^2-k-1)n_1(H)\\
&+(k^3-k^2-2)n_2(H)+k(k^2-3)r\\
= &(k^3-2k-1)n_1(H)+(k^3-k^2-2)n_2(H)\\
&+k(k^2-3)r-k(k-1)t_0-k(k-1)\\
= &k(k^2+k-4)(n_1(H)+n_2(H))-(k-1)^2(n_1(H)+2n_2(H))\\
&+k(k^2-3)r-k(k-1)t_0-k(k-1)\\
= &k(k^2+k-4)n(H)-k(k-1)^2m(H)+k(k^2-3)r-k(k-1)t_0-k(k-1).
\end{aligned}
$$

Dividing through by k, the above simplifies to

$$k(k^2-3)\alpha(H) \geqslant (k^2+k-4)n(H)-(k-1)^2m(H)+ (k^2-3)r- (k-1)t_0-(k-1).$$

As observed in the proof of Theorem 18, if $r=0$, then $t_0=0$, while if $r \geqslant 1$, then $t_0 \leqslant (k-1)r$. If $r=0$, then the above simplifies to the following.

$$k(k^2-3)\alpha(H) \geqslant (k^2+k-4)n(H)-(k-1)^2m(H)-(k-1).$$

If $r \geqslant 1$, then the above simplifies to the following.

$$
\begin{aligned}
k(k^2-3)\alpha(H) \geqslant &(k^2+k-4)n(H)-(k-1)^2m(H)\\
&+(k^2-3)r-(k-1)^2r-(k-1)\\
= &(k^2+k-4)n(H)-(k-1)^2m(H)+2(k-2)r-(k-1)\\
\geqslant &k(k^2+k-4)n(H)-(k-1)^2m(H)+k-3\\
\geqslant &k(k^2+k-4)n(H)-(k-1)^2m(H)\\
> &k(k^2+k-4)n(H)-(k-1)^2m(H)-(k-1).
\end{aligned}
$$

This completes the proof of Theorem 21. □

We show next that there is an infinite family of hypergraphs $H \in \mathcal{H}_k$ for which equality holds in the statement of Theorem 21. For $k \geqslant 3$ an odd integer and $r \geqslant 1$, let G be an arbitrary graph in the family $\mathcal{F}_{k,r}$, and let H_G^k be the associated hypergraph in the family $\mathcal{H}_{k,r}^{\rm odd}$. For each vertex v of degree 1 in H_G^k, we add $k-1$ new vertices and an edge (of size k) containing v and these new vertices. Let R_G^k denote the resulting hypergraph, and let $\mathcal{R}_{k,r}^{\rm odd}$ be the family of all such hypergraphs R_G^k.

Proposition 22. *For $k \geqslant 3$ an odd integer and $r \geqslant 1$ arbitrary, if $H \in \mathcal{R}_{k,r}^{\rm odd}$ has order n and size m, then*

$$\alpha(H) = \frac{(k^2+k-4)n(H)-(k-1)^2m(H)-(k-1)}{k(k^2-3)}.$$

Proof. Let $G \in \mathcal{F}_{k,r}$ be the graph and $H_G^k \in \mathcal{H}_{k,r}^{\rm odd}$ the associated hypergraph used to construct the hypergraph $H \in \mathcal{R}_{k,r}^{\rm odd}$, and so $H = R_G^k$. Analogous to the proof of Proposition 19, we have that

$$\alpha(H) = n_1(H_G^k) + \alpha'(G).$$

For $i \in [k]$, let $n_{1,i}$ be the number of vertices in V_1 that have degree i in G. As shown in the proof of Proposition 13,

$$\sum_{i=1}^{k} n_{1,i} = |V_1| \quad \text{and} \quad \sum_{i=1}^{k} i \cdot n_{1,i} = |V_1| + |V_2| - 1,$$

implying that

$$n_1(H_G^k) = \sum_{i=1}^{k}(k-i)n_{1,i} = k\sum_{i=1}^{k} n_{1,i} - \sum_{i=1}^{k} i \cdot n_{1,i} = (k-1)|V_1| - |V_2| + 1.$$

Recall that $n = n(H)$ and $m = m(H)$. We note that

$$\begin{aligned} n &= n(H_G^k) + (k-1)n_1(H_G^k) \\ m &= m(H_G^k) + n_1(H_G^k) \end{aligned}$$

Recall (see the proof of Proposition 13) that

$$\begin{aligned} n(H_G^k) &= \left(\frac{k^3+k^2-k-1}{2}\right)|V_2| - \left(\frac{k^2+1}{2}\right)|V_1| + \left(\frac{k^2+2k+1}{2}\right) \\ m(H_G^k) &= (k^2+k-1)|V_2| - (k+1)|V_1| + (k+2) \\ \alpha'(G) &= \tfrac{1}{2}\big((k^2+1)|V_2| - (k+1)|V_1| + (k+1)\big) \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{(k^2+k-4)n - (k-1)^2 m - (k-1)}{k(k^2-3)} \\ =&\left(\frac{k^2+k-4}{k(k^2-3)}\right)\Big(n(H_G^k) + (k-1)n_1(H_G^k)\Big) \\ &-\left(\frac{(k-1)^2}{k(k^2-3)}\right)\Big(m(H_G^k) + n_1(H_G^k)\Big) \\ &-\frac{k-1}{k(k^2-3)} \\ =&\left(\frac{k^2+k-4}{k(k^2-3)}\right)\left(\left(\frac{k^3+k^2-k-1}{2}\right)|V_2| - \left(\frac{k^2+1}{2}\right)|V_1| + \left(\frac{k^2+2k+1}{2}\right)\right. \\ &\left.+(k-1)n_1(H_G^k)\right) - \left(\frac{(k-1)^2}{k(k^2-3)}\right)\Big((k^2+k-1)|V_2| - (k+1)|V_1| \\ &+(k+2) + n_1(H_G^k)\Big) - \frac{k-1}{k(k^2-3)} \\ =&\left(\frac{k^5-2k^3-2k^2-3k+6}{2k(k^2-3)}\right)|V_2| - \left(\frac{k^4-k^3-k^2+3k-6}{2k(k^2-3)}\right)|V_1| \\ &+\left(\frac{k^3-k^2-3k+3}{k(k^2-3)}\right)n_1(H_G^k) + \frac{k^4+k^3-k^2-3k-6}{2k(k^2-3)} \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{k^3+k-2}{2k}\right)|V_2| - \left(\frac{k^2-k+2}{2k}\right)|V_1| + \left(\frac{k-1}{k}\right) n_1(H_G^k) + \frac{k^2+k+2}{2k} \\
&= \left(\frac{k^3+k-2}{2k}\right)|V_2| - \left(\frac{k^2-k+2}{2k}\right)|V_1| + n_1(H_G^k) - \frac{1}{k}\left((k-1)|V_1| - |V_2| + 1\right) \\
&\quad + \frac{k^2+k+2}{2k} \\
&= n_1(H_G^k) + \left(\frac{k^2+1}{2}\right)|V_2| - \left(\frac{k+1}{2}\right)|V_1| + \frac{k+1}{2} \\
&= n_1(H_G^k) + \alpha'(G) \\
&= \alpha(H).
\end{aligned}
$$

This completes the proof of Proposition 22. □

5 Summary

For small values of $k \geqslant 3$, the results in this paper are summarized in Table 1 and Table 2 below.

k even	H has order n and size m.		
	$H \in \mathcal{H}_k$	$H \in \mathcal{H}'_k$	$H \in \mathcal{H}''_k$
$k = 4$	$20\tau(H) \leqslant 4n + 3m + 5$ $20\alpha(H) \geqslant 6n - 3m - 5$	$20\tau(H) \leqslant 4n + 3m + 3$ $20\alpha(H) \geqslant 6n - 3m - 3$	$20\tau(H) \leqslant 4n + 3m + 1$ $20\alpha(H) \geqslant 6n - 3m - 1$
$k = 6$	$42\tau(H) \leqslant 6n + 5m + 7$ $42\alpha(H) \geqslant 8n - 5m - 7$	$42\tau(H) \leqslant 6n + 5m + 3$ $42\alpha(H) \geqslant 8n - 5m - 3$	$42\tau(H) \leqslant 6n + 5m + 1$ $42\alpha(H) \geqslant 8n - 5m - 1$
$k = 8$	$72\tau(H) \leqslant 8n + 7m + 9$ $72\alpha(H) \geqslant 10n - 7m - 9$	$72\tau(H) \leqslant 8n + 7m + 3$ $72\alpha(H) \geqslant 10n - 7m - 3$	$72\tau(H) \leqslant 8n + 7m + 1$ $72\alpha(H) \geqslant 10n - 7m - 1$

Table 1. Results for small values of even $k \geqslant 4$

k odd	$H \in \mathcal{H}_k$ has order n and size m.
$k = 3$	$9\tau(H) \leqslant 2n + 2m + 1$ $9\alpha(H) \geqslant 4n - 2m - 1$
$k = 5$	$55\tau(H) \leqslant 9n + 8m + 2$ $55\alpha(H) \geqslant 13n - 8m - 2$
$k = 7$	$161\tau(H) \leqslant 20n + 18m + 3$ $161\alpha(H) \geqslant 26n - 18m - 3$

Table 2. Results for small values of odd $k \geqslant 3$

We have further shown that in each of the inequality statements involving the transversal number or the independence number, there is an infinite family of hypergraphs $H \in \mathcal{H}_k$ for which equality holds, implying that all the bounds are tight.

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