# Some properties of the Fibonacci sequence on an infinite alphabet*

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**Abstract**

The infinite Fibonacci sequence $\mathbf{F}$, which is an extension of the classic Fibonacci sequence to the infinite alphabet $\mathbb{N}$, is the fixed point of the morphism $\phi$: $(2i) \mapsto (2i)(2i+1)$ and $(2i+1) \mapsto (2i+2)$ for all $i \in \mathbb{N}$. In this paper, we study the growth order and digit sum of $\mathbf{F}$, and give several decompositions of $\mathbf{F}$ using singular words.

**Keywords:** Infinite Fibonacci sequence; singular words; Fibonacci number; digit sum

## 1 Introduction

Let $\mathbf{f} = \{f_n\}_{n\geqslant 0}$ be the Fibonacci sequence that is the fixed point of the Fibonacci morphism $\sigma$ defined by $0 \mapsto 01,\ 1 \mapsto 0$. The Fibonacci sequence $\mathbf{f}$ occurs in the study of combinatorics on words, number theory, dynamical system, quasi-crystals, etc.; see [1, 2, 13, 18] and references therein. It is well known that $\mathbf{f}$ is a characteristic Sturmian word of

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slope $1/\phi^2$ where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. In [9, 10, 11], several generalizations of the Fibonacci sequence were given, and they were proved to be characteristic Sturmian words of irrational slopes, whose simple continued fraction expansions are periodic sequences with period 1 and 2.

The previous extensions of $\mathbf{f}$ are all on the alphabet of 2 letters. There are also extensions on alphabets containing at least 3 letters. A famous one is the Tribonacci sequence, which is the fixed point of Rauzy substitution $0 \mapsto 01, 1 \mapsto 02, 2 \mapsto 0$. Rauzy [8] studied the dynamical and geometrical aspects of the Tribonacci sequences; see also [18]. For an extension to an alphabet of $m$ letters, see [15]. In this paper, we study the following extension of the Fibonacci sequence to an infinite alphabet.

Consider the morphism $\phi$ (over $\mathbb{N}$) which is given by

$$\phi : (2i) \mapsto (2i)(2i+1) \text{ and } (2i+1) \mapsto (2i+2), \quad \text{for all } i \geqslant 0.$$

The *infinite Fibonacci sequence* $\mathbf{F} = \{F_i\}_{i\geqslant 0}$ is the fixed point of $\phi$ starting by 0. The first several terms of $\mathbf{F}$ and $\mathbf{f}$ are

$$\mathbf{F} = 0\,1\,2\,2\,3\,2\,3\,4\,2\,3\,4\,4\,5\,2\,3\,4\,4\,5\,4\,5\,6\cdots$$
$$\mathbf{f} = 0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,0\,1\,0\,1\,0\,0\,1\,0\,1\,0\cdots$$

It is easy to see that $\mathbf{F} = \mathbf{f} \pmod 2$, since $\phi$ is reduced to $\sigma$ while modulo 2. In this sense, these two sequence are similar. Hence the infinite Fibonacci sequence $\mathbf{F}$ may inherit some combinatorial properties of the Fibonacci sequence $\mathbf{f}$. For example, since '11' does not occur in $\mathbf{f}$, two adjacent elements in $\mathbf{F}$ can not be both odd numbers. However, $\mathbf{F}$ also has some properties that $\mathbf{f}$ does not have. For example, one can find arbitrarily long palindromes in $\mathbf{f}$ ([27, Property 2]); while in $\mathbf{F}$, there are no palindromes of length larger than 3 (see Proposition 27). In general, the sequence $\mathbf{F} \pmod k$ is a substitution sequence up to a coding (see Section 6).

In [27], Wen and Wen studied the Fibonacci sequence $\mathbf{f}$ by analyzing singular words of $\mathbf{f}$. They proved that the adjacent singular words of the same order are positively separated, and gave a decomposition of $\mathbf{f}$ using singular words. Levé and Séébold [5] studied the singular factorization of the Fibonacci sequence and its conjugates. Melançon studied the Lyndon factorization of the Fibonacci sequence in [6, 7]. Tan and Wen [4] studied the singular words of Sturmian sequences and Tribonacci sequence. In [26], Huang and Wen studied the occurrence of arbitrary factors of $\mathbf{f}$ and gave a quite general decomposition of $\mathbf{f}$. The study of singular words have applications in the study of combinatorics on words [3, 16, 22, 23, 24] and Padé approximation [14, 21], etc.

There are many works devoted to the study of the substitution sequences and the automatic sequences over infinite alphabets. For the research of infinite-state automata see for example [12, 20] and a survey [25]. For the study of substitution sequences over an infinite alphabets, see for example [19].

In this paper, we first investigate the growth order of $F_n$, and show that for all $n \geqslant 1$,

$$0 \leqslant F_n \leqslant c \log n$$

where $c$ is a constant (see Remark 6). Then we turn to study the digit sum problem of $\mathbf{F}$. We give an exact form of the digit sum of $\phi^n(0)$ $(n \geqslant 1)$ in Theorem 13, and give a formula for the digit sum of the first $n$ digits of $\mathbf{F}$ in Theorem 15. Inspired by the work of Wen and Wen [27], we also study the decomposition (factorisation) of $\mathbf{F}$. In fact, we give the following three decompositions. The first one is a decomposition using its (slightly modified) prefixes (see Theorem 16). The second one is a decomposition using singular words of letter 2 whose lengths are Fibonacci numbers (see Theorem 17). The third one is a decomposition using singular words of a fixed order (see Theorem 19). The main difference here is that the singular words (defined in Section 5) in $\mathbf{F}$ are defined to be classes of words, and while modulo 2, the singular words in $\mathbf{F}$ are only part of the singular words in $\mathbf{f}$ which is defined in [27].

This paper is organized as follows. In Section 2, we state some basic notation and definitions. In Section 3 and Section 4, we study the growth order of the terms of the infinite Fibonacci sequence $\mathbf{F}$ and its digit sums. In Section 5, we give three decompositions of $\mathbf{F}$. In the last section, several other properties of $\mathbf{F}$ are given.

## 2 Preliminary

Let $\mathcal{A} = \{a_0, a_1, \cdots, a_n, \cdots\}$ be an (infinite) alphabet. $\mathcal{A}^k$ denotes the set of all words of length $k$ on $\mathcal{A}$, and $\mathcal{A}^*$ denotes the set of all finite words on $\mathcal{A}$. The length of a finite word $w \in \mathcal{A}^*$ is denoted by $|w|$. For any $V, W \in \mathcal{A}^*$, we write $V \prec W$ when the finite word $V$ is a *factor* of the word $W$, that is, when there exist words $U, U' \in \mathcal{A}^*$, such that $W = UVU'$. We say that $V$ is a *prefix* (resp. *suffix*) of a word $W$, and we write $V \lhd W$ (resp. $V \rhd W$) if there exists a word $U' \in \mathcal{A}^*$, such that $W = VU'$ (resp. $W = U'V$).

Let $\mathbf{u} = u_0u_1u_2\cdots u_n$ be a finite word (or $\mathbf{u} = u_0u_1u_2\cdots u_n\cdots$ be a sequence). For any $i \leqslant j \leqslant n$, define $\mathbf{u}[i,j] := u_iu_{i+1}\cdots u_{j-1}u_j$.

Denote by $W^{-1}$ the *inverse* of $W$, that is, $W^{-1} = w_p^{-1}\cdots w_2^{-1}w_1^{-1}$ where $W = w_1w_2\cdots w_p$. If $V$ is a suffix of $W$, we can write $WV^{-1} = U$, with $W = UV$. This makes sense in $\mathcal{A}^*$, since the reduced word associated with $WV^{-1}$ belongs to $\mathcal{A}^*$. This abuse of language will be very useful in what follows.

For a sequence $\mathbf{u} = (u_n)_{n\geqslant 0}$ and an integer $k \geqslant 0$, we denote $\mathbf{u} \oplus k := (u_n + k)_{n\geqslant 0}$.

## 3 Growth order of the infinite Fibonacci sequence

In this part, we shall study the growth order of the terms of the infinite Fibonacci sequence. For this purpose, we first give some basic rules of the digits in $\mathbf{F}$, which reveal the growth order of the sequence $\mathbf{F}$.

**Theorem 1.** *For any $k \geqslant 1$, we have*

$$\phi^k(2i + j) = \phi^k(j) \oplus (2i)$$

*for any $i, j \geqslant 0$.*

*Proof.* When $k = 1$, the result follows from the definition of $\phi$. Now suppose the result holds for all $k \leqslant n$, we shall prove it for $n+1$. Suppose $j$ is even. For any $i \geqslant 0$,

$$\begin{aligned}\phi^{n+1}(2i+j) &= \phi^n(2i+j)\phi^n(2i+j+1)\\ &= [\phi^n(j) \oplus (2i)][\phi^n(j+1) \oplus (2i)]\\ &= [\phi^n(j)\phi^n(j+1)] \oplus (2i)\\ &= \phi^{n+1}(j) \oplus (2i).\end{aligned}$$

When $j$ is odd, we have for any $i \geqslant 0$,

$$\begin{aligned}\phi^{n+1}(2i+j) &= \phi^n(2i+j+1)\\ &= \phi^n(j+1) \oplus (2i)\\ &= \phi^{n+1}(j) \oplus (2i).\end{aligned}$$

This completes the proof. □

Letting $j = 0$ in Theorem 1, we have the following result.

**Corollary 2.** *For all $k \geqslant 1$ and $i \geqslant 0$, $|\phi^k(2i)| = |\phi^k(0)|$.*

Recall that for any $n \geqslant 0$, the Fibonacci number $L_n$ counts the number of 1's in $\sigma^n(0)$ (or sums the digits of $\sigma^n(0)$), and $L_{n+2}$ represents the length of $\sigma^n(0)$. It is worth to remark that for $n \geqslant 1$,

$$L_{n+1} = L_n + L_{n-1},$$

where $L_0 = 0$ and $L_1 = 1$.

**Lemma 3.** *For all $k \geqslant 0$, $|\phi^k(0)| = L_{k+2}$.*

*Proof.* Note that $\phi \pmod 2$ is the Fibonacci morphism $\sigma$. So $|\phi^k(0)| = |\sigma^k(0)| = L_{k+2}$ for all $k \geqslant 0$. □

**Proposition 4.** *For any $n \geqslant 2$, $i \geqslant 0$,*

$$(n+i-1)(n+i) \rhd \phi^n(i).$$

*Moreover, $n+i$ is the largest digit in $\phi^n(i)$.*

*Proof.* We first prove this result for $i = 0$. Note that $\phi^n(0)$ is a prefix of $\mathbf{F}$, and according to Lemma 3, $|\phi^n(0)| = L_{n+2}$. So $F_{L_{n+2}-1}$ and $F_{L_{n+2}-2}$ are the last and the penultimate digits of $\phi^n(0)$. Now we will show by induction that for all $n \geqslant 2$,

$$\begin{cases}\text{The largest digit of } \phi^n(0) \text{ is } n;\\ F_{L_{n+2}-1} = n \text{ and } F_{L_{n+2}-2} = n-1.\end{cases} \tag{1}$$

It is easy to check that (1) holds for $n = 2$. Assume that (1) holds for all $m \leqslant n$. We shall check it for $m = n+1$. Note that

$$\phi^m(0) = \phi^n(0)\phi^{n-1}(2) = \phi^n(0)(\phi^{n-1}(0) \oplus 2) \tag{2}$$

and by the induction hypothesis the largest digits of $\phi^n(0)$ and $\phi^{n-1}(0) \oplus 2$ are $n$ and $n+1$ respectively. So the largest digit of $\phi^{n+1}(0)$ is $n+1$.

Since $F_{L_{m+2}-2}F_{L_{m+2}-1} \rhd \phi^m(0)$, by (2), we have

$$F_{L_{m+2}-2}F_{L_{m+2}-1} \rhd (\phi^{n-1}(0) \oplus 2)$$

which implies

$$F_{L_{m+2}-2} = F_{L_{n+1}-2} + 2 = n \text{ and } F_{L_{m+2}-1} = F_{L_{n+1}-1} + 2 = n+1.$$

Therefore, (1) holds.

Suppose $\alpha$ and $\beta$ are the last two digits of $\phi^n(i)$. When $i = 2k+1$, we have

$$\phi^n(i) = \phi^n(2k+1) = \phi^{n-1}(2k+2) = \phi^{n-1}(0) \oplus (2k+2),$$

where the last equality follows from Theorem 1. So

$$\begin{aligned} \alpha =& F_{L_{n+1}-2} + 2k + 2 = n + 2k = n + i - 1, \\ \beta =& F_{L_{n+1}-1} + 2k + 2 = n + 2k + 1 = n + i, \end{aligned}$$

and the largest digit of $\phi^n(i)$ is $n - 1 + 2k + 2 = n + i$. The case $i = 2k$ is similar. □

The above result implies that the digits in **F** can be arbitrary large. However, there are also infinitely many small digits, say '2' and '3', in **F**.

**Proposition 5.** *For any $n \geqslant 4$, $F_{L_n} = 2$ and $F_{L_n+1} = 3$.*

*Proof.* By the definition of **F**, for any $k \geqslant 2$, $\phi^{k+1}(0) = \phi^k(0)\phi^{k-1}(2)$ is a prefix of **F**. Since $|\phi^k(0)| = L_{k+2}$, we know that $F_{L_{k+2}}F_{L_{k+2}+1}$ is a prefix of $\phi^{k-1}(2)$. It follows from the definition of $\phi$ that $F_{L_{k+2}}F_{L_{k+2}+1} = 23$ for any $k \geqslant 2$. □

*Remark* 6 (The growth order of $\{F_n\}_{n\geqslant 0}$). Using Proposition 5, we have

$$\lim_{n\to\infty} \frac{F_{L_n}}{\log L_n} = \lim_{n\to\infty} \frac{2}{\log L_n} = 0.$$

On the other hand, Proposition 4 gives

$$\lim_{n\to\infty} \frac{F_{L_n-1}}{\log(L_n - 1)} = \lim_{n\to\infty} \frac{n-2}{\log(\frac{\gamma^n-\psi^n}{\gamma-\psi} - 1)} = \frac{1}{\log\gamma},$$

where $\gamma = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. This implies the limit of $F_n/n$ does not exist. However, we have

$$\liminf_{n\to\infty} \frac{F_n}{\log n} = 0 \text{ and } \limsup_{n\to\infty} \frac{F_n}{\log n} = \frac{1}{\log\gamma}.$$

The first one is trivial. For the limsup we know from Proposition 4 that

$$\text{if } L_k \leqslant n < L_{k+1}, \text{ then } F_n \leqslant F_{L_{k+1}-1}.$$

Let $k = k(n)$ such that $L_k \leqslant n < L_{k+1}$. Then

$$\log n \geqslant \log L_k = k \log \gamma + O(1).$$

Note that $F_n \leqslant F_{L_{k+1}-1} = k - 1$. We have

$$\limsup_{n\to\infty} \frac{F_n}{\log n} \leqslant \limsup_{n\to\infty} \frac{F_{L_{k+1}-1}}{\log n} = \limsup_{n\to\infty} \frac{k-1}{\log n} = \frac{1}{\log \gamma}.$$

Hence, we have $0 \leqslant F_n \leqslant c \log n$, for any $n \geqslant 1$, where $c$ is a constant.

Next, we will show that the block of length $L_{n+1}$ in $\mathbf{F}$ from position $L_{n+2} + 1$ to position $L_{n+3}$ is exact $\phi^{n-1}(2)$.

**Proposition 7.** *For any $n \geqslant 1$, $F_{L_{n+2}} F_{L_{n+2}+1} \cdots F_{L_{n+3}-1} = \phi^{n-1}(2)$.*

*Proof.* Since $\phi^{n+1}(0) = \phi^n(0)\phi^{n-1}(2)$, and $|\phi^n(0)| = L_{n+2}$, we have

$$\phi^{n-1}(2) = F_{L_{n+2}} F_{L_{n+2}+1} \cdots F_{L_{n+3}-1}. \qquad \square$$

**Corollary 8.** *For any $n \geqslant 4$, $0 \leqslant i \leqslant L_n - 1$, we have*

$$F_{i+L_n} - F_i = \begin{cases} 2, & \text{if } 0 \leqslant i \leqslant L_{n-1} - 1, \\ 0, & \text{if } L_{n-1} \leqslant i \leqslant L_n - 1. \end{cases}$$

*Moreover, for $n = 1$ and $n = 2$, $F_1 - F_0 = 1$, for $n = 3$, $F_2 - F_0 = 2$, $F_3 - F_1 = 1$.*

*Proof.* For $0 \leqslant i \leqslant L_{n-1} - 1$, we have known

$$F_0 F_1 \cdots F_{L_{n-1}-1} = \phi^{n-3}(0).$$

By Proposition 7, we have

$$F_{L_n} F_{L_n+1} \cdots F_{L_{n+1}-1} = \phi^{n-3}(2).$$

Since $\phi^{n-3}(2) = \phi^{n-3}(0) \oplus 2$, it follows that $F_{i+L_n} - F_i = 2$.

For $L_{n-1} \leqslant i \leqslant L_n - 1$, from Proposition 7, we have

$$F_{L_{n-1}} F_{L_{n-1}+1} \cdots F_{L_n-1} = \phi^{n-4}(2).$$

By Proposition 7, we have

$$F_{L_{n+1}} F_{L_{n+1}+1} \cdots F_{L_{n+1}+L_{n-1}-1} \lhd \phi^{n-2}(2),$$

and $\phi^{n-4}(2)$ is also the prefix of $\phi^{n-2}(2)$ with the length $L_{n-2}$, which is also the length of $F_{L_{n+1}} F_{L_{n+1}+1} \cdots F_{L_{n+1}+L_{n-1}-1}$. Then

$$F_{L_n+L_{n-1}} F_{L_n+L_{n-1}+1} \cdots F_{L_n+L_n-1} = \phi^{n-4}(2) = F_{L_{n-1}} F_{L_{n-1}+1} \cdots F_{L_n-1}.$$

Thus, $F_{i+L_n} - F_i = 0$. $\square$

From previous discussions, we know that

- $n$ is the largest digit of $\phi^n(0)$. 0 is the smallest digit of $\phi^n(0)$.
- $n$ occurs only once in $\phi^n(0)$. Moreover, $n \rhd \phi^n(0)$.
- 0 occurs only once in $\phi^n(0)$.

It is natural to ask the location of '$n-1$' in $\phi^n(0)$. For this purpose, we need the following decomposition of $\phi^n(0)$.

**Lemma 9.** *For any $n \geqslant 0$, let $k = \lfloor \frac{n-1}{2} \rfloor$. Then*

$$\phi^n(0) = \left( \prod_{i=0}^{k} \phi^{n-1-2i}(2i) \right) \cdot n.$$

*Proof.* Assume that $n-1 = 2k+j$, where $j = 0$ or 1. Then

$$\begin{aligned} \phi^n(0) &= \phi^{n-1}(01) = \phi^{n-1}(0)\phi^{n-1}(1) \\ &= \phi^{n-1}(0)\phi^{n-3}(2)\phi^{n-3}(3) \\ &= \phi^{n-1}(0)\phi^{n-3}(2)\phi^{n-5}(4)\phi^{n-5}(5) \\ &= \cdots \\ &= \left( \prod_{i=0}^{k} \phi^{n-1-2i}(2i) \right) \phi^j(n-j), \end{aligned}$$

where $\phi^j(n-j) = \phi^0(n) = n$ when $j = 0$, and $\phi^j(n-j) = \phi(2k+1) = 2k+2 = n$ when $j = 1$. □

**Proposition 10.** *Let $n-1 = 2k+j$, where $j = 0$ or 1. Then the letter '$n-1$' occurs $(k+1)$ times in $\phi^n(0)$. Moreover, the $l$-th '$n-1$' in $\phi^n(0)$ is the $P(n,l)$-th digit of $\phi^n(0)$, where for $l = 1, \cdots, k+1$,*

$$P(n,l) = \sum_{i=0}^{l-1} L_{n+1-2i}.$$

*Proof.* By Lemma 9, we know that

$$\phi^n(0) = \left( \prod_{i=0}^{k} \phi^{n-1-2i}(2i) \right) \cdot n.$$

By Proposition 4, we have '$n-1$' occurs only once in $\phi^{n-1-2i}(2i)$ (at the end). Note that $|\phi^{n-1-2i}(2i)| = L_{n+1-2i}$. We know that the $l$-th '$n-1$' in $\phi^n(0)$ is

$$\left|\phi^{n-1}(0)\phi^{n-1-2}(2)\cdots\phi^{n-1-2(l-1)}(2(l-1))\right| \text{-th digit}$$

of $\phi^n(0)$. Moreover,

$$P(n,l) = \left|\phi^{n-1}(0)\phi^{n-1-2}(2)\cdots\phi^{n-1-2(l-1)}(2(l-1))\right|$$

which completes the proof. □

*Remark* 11. In the same process, one can also locate '$n-2$' in $\phi^n(0)$ and then '$n-3$' and so on. However, the formulae would not be nice to read.

## 4 Digit sum

This section is devoted to study the digit sum of the Fibonacci sequence and the generalized Fibonacci sequences. For any integer sequence $\mathbf{c} = c_0c_1\cdots$, the *digit sum* of first $n$-terms is denoted by

$$S_{\mathbf{c}}(n) := \sum_{i=0}^{n-1} c_i.$$

Let $(n)_F := a_r a_{r-1}\cdots a_0$ be the unique Fibonacci representation of a non-negative integer $n$ (see [13, Section 3.8]). That is

$$n = \sum_{i=0}^{r} a_i L_{i+2} \tag{3}$$

where $a_i = 0$ or $1$, $a_r \neq 0$ and $a_i a_{i+1} = 0$ for $0 \leqslant i \leqslant r$.

For any $n \geqslant 0$ with Fibonacci representation $(n)_F = a_r a_{r-1}\cdots a_0$, we have

$$S_{\mathbf{f}}(n) = \sum_{i=0}^{r} a_i L_i. \tag{4}$$

In fact, consider the word $w := \sigma^{i_0}(0)\sigma^{i_1}(0)\cdots\sigma^{i_t}(0)$, where the indexes $\{i_k\}_{k=0}^{t}$ satisfy $i_0 \leqslant r$ and for all $k$, $a_{i_k} \neq 0$ and $i_{k+1} < i_k$. Moreover,

$$|w| = \sum_{k=0}^{t} |\sigma^{i_k}(0)| = \sum_{k=0}^{t} L_{i_k+2} = \sum_{i=0}^{r} a_i L_{i+2} = n.$$

Since $w \lhd \sigma(w)$, $w$ is the prefix of $\mathbf{f}$. Therefore (4) holds.

### 4.1 Digit sum of F

For the digit sum of the infinite Fibonacci sequence $\mathbf{F}$, we need a number, say $\widetilde{F}(i,n)$, which sums the digits of $\phi^n(i)$. In the case of $i = 0$, $\widetilde{F}_n := \widetilde{F}(0,n)$. It is easy to see that for all $i \geqslant 1$,

$$\widetilde{F}(2i,n) = \widetilde{F}(2i-1,n+1) = \widetilde{F}_n + 2i \cdot L_{n+2}. \tag{5}$$

**Lemma 12.** *For any $n \geqslant 2$,*

$$\widetilde{F}_n = \widetilde{F}_{n-1} + \widetilde{F}_{n-2} + 2L_n.$$

*In the matrix form, the above equation is*

$$\begin{pmatrix} \widetilde{F}_{n+1} \\ \widetilde{F}_n \\ L_{n+1} \\ L_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \widetilde{F}_n \\ \widetilde{F}_{n-1} \\ L_n \\ L_{n-1} \end{pmatrix}. \tag{6}$$

*Proof.* Since $\phi^n(0) = \phi^{n-1}(01) = \phi^{n-1}(0)\phi^{n-2}(2)$, $\widetilde{F}_n = \widetilde{F}_{n-1} + \widetilde{F}(2, n-2)$. Then, by (5), $\widetilde{F}_n = \widetilde{F}_{n-1} + \widetilde{F}_{n-2} + 2L_n$. □

**Theorem 13** (Digit sum of $\boldsymbol{\phi^n(0)}$)**.** *Let* $\gamma = \frac{1+\sqrt{5}}{2}$ *and* $\psi = \frac{1-\sqrt{5}}{2}$*. For any* $n \geqslant 1$*,*

$$\widetilde{F}_n = \frac{n(\gamma^{n+1} + \psi^{n+1})(\psi^2 + \gamma)}{(\gamma - \psi)^2} + \frac{\psi^n - \gamma^n}{(\gamma - \psi)^3}.$$

*Moreover,*

$$\lim_{n\to\infty} \frac{\widetilde{F}_n}{nL_n} = \gamma.$$

*Proof.* Denote the $4 \times 4$ matrix in (6) by $A$ and $\vec{F}_{n+1} := (\widetilde{F}_{n+1}, \widetilde{F}_n, L_{n+1}, L_n)^t$. Then by Lemma 12, we have

$$\vec{F}_{n+1} = A\vec{F}_n = \cdots = A^n \vec{F}_1.$$

Thus, to evaluate $\widetilde{F}_n$, we only need to compute $A^n$. This can be done by using the Jordan form of

$$A = PJP^{-1}$$

where

$$P = \begin{pmatrix} \frac{2(2-\sqrt{5})}{5} & -\frac{2\sqrt{5}}{25} & \frac{2(2+\sqrt{5})}{5} & \frac{2\sqrt{5}}{25} \\ \frac{3-\sqrt{5}}{5} & \frac{6\sqrt{5}}{25} & \frac{\sqrt{5}+3}{5} & -\frac{6\sqrt{5}}{25} \\ 0 & \frac{5-\sqrt{5}}{10} & 0 & \frac{5+\sqrt{5}}{10} \\ 0 & -\frac{\sqrt{5}}{5} & 0 & \frac{\sqrt{5}}{5} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\psi^4+\psi}{(\gamma-\psi)^2} & -\frac{\gamma^2+\psi}{(\gamma-\psi)^3} & \frac{\gamma^4+\gamma}{(\gamma-\psi)^2} & \frac{\gamma^2+\psi}{(\gamma-\psi)^3} \\ \frac{\psi^4-\psi}{(\gamma-\psi)^2} & \frac{\gamma^4+\psi^3+\psi}{(\gamma-\psi)^3} & \frac{\gamma^4-\gamma}{(\gamma-\psi)^2} & -\frac{\gamma^4+\psi^3+\psi}{(\gamma-\psi)^3} \\ 0 & -\frac{\psi}{\gamma-\psi} & 0 & \frac{\gamma}{\gamma-\psi} \\ 0 & -\frac{1}{\gamma-\psi} & 0 & \frac{1}{\gamma-\psi} \end{pmatrix}$$

and

$$J = \begin{pmatrix} \frac{1-\sqrt{5}}{2} & 1 & 0 & 0 \\ 0 & \frac{1-\sqrt{5}}{2} & 0 & 0 \\ 0 & 0 & \frac{1+\sqrt{5}}{2} & 1 \\ 0 & 0 & 0 & \frac{1+\sqrt{5}}{2} \end{pmatrix}.$$

Then

$$J^n = \begin{pmatrix} \psi^n & n\psi^{n-1} & 0 & 0 \\ 0 & \psi^n & 0 & 0 \\ 0 & 0 & \gamma^n & n\gamma^{n-1} \\ 0 & 0 & 0 & \gamma^n \end{pmatrix}$$

for all $n \geqslant 1$. The inverse of $P$ is

$$P^{-1} = \begin{pmatrix} -\frac{5+3\sqrt{5}}{4} & \frac{5+2\sqrt{5}}{2} & 0 & \frac{7+3\sqrt{5}}{2} \\ 0 & 0 & 1 & -\frac{1+\sqrt{5}}{2} \\ -\frac{5-3\sqrt{5}}{4} & \frac{5-2\sqrt{5}}{2} & 0 & \frac{7-3\sqrt{5}}{2} \\ 0 & 0 & 1 & \frac{\sqrt{5}-1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\gamma^3+\gamma}{\gamma^2+\psi} & \frac{\gamma^4+\gamma^2}{\gamma^2+\psi} & 0 & \gamma^4 \\ 0 & 0 & \gamma+\psi & -\gamma \\ -\frac{\psi^3+\psi}{\gamma^2+\psi} & \frac{\psi^4+\psi^2}{\gamma^2+\psi} & 0 & \psi^4 \\ 0 & 0 & \gamma+\psi & -\psi \end{pmatrix}.$$

Since $\vec{F}_1 = (1,0,1,0)^t$ and $\vec{F}_{n+1} = A^n\vec{F}_1 = PJ^nP^{-1}\vec{F}_1$, we can give the closed form of $L_n$ and $\widetilde{F}_n$:

$$L_n = \frac{\gamma^n - \psi^n}{\gamma - \psi} \text{ and } \widetilde{F}_n = \frac{n(\gamma^{n+1} + \psi^{n+1})(\psi^2 + \gamma)}{(\gamma - \psi)^2} + \frac{\psi^n - \gamma^n}{(\gamma - \psi)^3}.$$

Therefore

$$\lim_{n\to\infty} \frac{\widetilde{F}_{n+1}}{\widetilde{F}_n} = \gamma \text{ and } \lim_{n\to\infty} \frac{\widetilde{F}_n}{nL_n} = \gamma. \qquad \square$$

*Remark* 14. The integer sequence $\{\tilde{F}_n\}_{n\geqslant 0}$ also satisfies the recurrence relation

$$\tilde{F}_n = 2\tilde{F}_{n-1} + \tilde{F}_{n-2} - 2\tilde{F}_{n-3} - \tilde{F}_{n-4}$$

with the initial values $0, 1, 3, 8$. Using this fact, one can show that its generating function is rational. In fact,

$$\sum_{n\geqslant 0} \tilde{F}_n x^n = \frac{x^3 + x^2 + x}{(1 - x - x^2)^2}.$$

**Theorem 15** (Digit sum of $\mathbf{F}$)**.** *Let $n \in \mathbb{N}$ and $(n)_F = a_r a_{r-1} \cdots a_0$. Collect all the index $i$ such that $a_i \neq 0$ and arrange them in descending order. Those indexes are denoted by $i_0 = r, i_1, \cdots, i_t$. Suppose $w$ is a prefix of $\mathbf{F}$ of length $n$, then*

$$w = \phi^{i_0}(0)\phi^{i_1}(2)\cdots\phi^{i_k}(2k)\cdots\phi^{i_t}(2t).$$

*Furthermore,*

$$S_{\mathbf{F}}(n) = \sum_{k=0}^{t} \widetilde{F}(2k, i_k).$$

*Proof.* Let $w' := \phi^{i_0}(0)\phi^{i_1}(2)\cdots\phi^{i_k}(2k)\cdots\phi^{i_t}(2t)$. We have

$$\begin{aligned}
|w'| &= |\phi^{i_0}(0)\phi^{i_1}(2)\cdots\phi^{i_k}(2k)\cdots\phi^{i_t}(2t)| \\
&= |\phi^{i_0}(0)| + |\phi^{i_1}(2)| + \cdots + |\phi^{i_k}(2k)| + \cdots + |\phi^{i_t}(2t)| \\
&= |\phi^{i_0}(0)| + |\phi^{i_1}(0)| + \cdots + |\phi^{i_k}(0)| + \cdots + |\phi^{i_t}(0)| \qquad \text{(by Corollary 2)} \\
&= L_{i_0+2} + L_{i_1+2} + \cdots + L_{i_t+2} = n.
\end{aligned}$$

So we only need to show that $w' \lhd \mathbf{F}$.

By the definition of $i_0$, $i_1$, $\cdots$, $i_t$ and (3), we have $i_j > i_{j+1} + 1$ for $0 \leqslant j \leqslant t-1$. Thus for all $0 \leqslant j \leqslant t-1$,

$$(2j+2)(2j+3) \lhd \phi^{i_j - i_{j+1} - 1}(2j+2)$$

and

$$\begin{aligned}
\phi^{i_j}(2j+1) = \phi^{i_j-1}(2j+2) &= \phi^{i_{j+1}}(\phi^{i_j - i_{j+1} - 1}(2j+2)) \\
&= \phi^{i_{j+1}}(2j+2)\phi^{i_{j+1}}(2j+3)\cdots \qquad (7)
\end{aligned}$$

Therefore

$$\begin{aligned}
\phi^{i_0+1}(0) &= \phi^{i_0}(0)\phi^{i_0}(1) \\
&= \phi^{i_0}(0)\phi^{i_1}(2)\phi^{i_1}(3)\cdots && \text{by (7)} \\
&= \phi^{i_0}(0)\phi^{i_1}(2)\phi^{i_2}(4)\phi^{i_2}(5)\cdots && \text{by (7)} \\
&\cdots \\
&= \phi^{i_0}(0)\phi^{i_1}(2)\phi^{i_2}(4)\cdots\phi^{i_t}(2t)\cdots
\end{aligned}$$

which gives $w' \lhd \phi^{i_0+1}(0) \lhd \mathbf{F}$. The proof is completed. □

## 5 Singular words decomposition

In this section we present three decompositions (factorisations) of $\mathbf{F}$. These are given in Theorems 16, 17 and 19. We will state the theorems first and introduce some notation, then give the three proofs.

**Theorem 16.** $\mathbf{F} = \Pi_{j=-2}^{\infty}\phi^j(2)$*, where we define* $\phi^{-2}(2)$ *to be* 0 *and* $\phi^{-1}(2)$ *to be* 1*.*

To state the second decomposition theorem, we introduce the singular words. For all $n \geqslant 1$ and $i \geqslant 0$, the word

$$\alpha\phi^n(2i)\beta^{-1} =: S_n^{(2i)}$$

is called a *singular word* of order $n$ (of letter $2i$), where $\alpha$ and $\beta$ are two letters satisfying $\alpha\beta \rhd \phi^n(i)$. In addition, we define

$$S_{-1}^{(2i+2)} := (2i) \text{ and } S_0^{(2i+2)} := (2i+1).$$

For example, since $\phi^3(2) = 23445$, then $\alpha = 4$, $\beta = 5$ and $S_3^{(2)} = 42344$.

**Theorem 17.** *The infinite Fibonacci sequence* $\mathbf{F}$ *has the following decomposition using singular words of letter* 2*:*

$$\mathbf{F} = \prod_{j=-1}^{\infty} S_j^{(2)} = \underline{0}\ \underline{1}\ \underline{22}\ \underline{323}\ \underline{42344}\ \underline{52344545}\ \underline{6234454564566}\cdots$$

In the above two decompositions, the lengths of the components are unbounded. In the following, we will decompose $\mathbf{F}$ into singular words of a fixed order and gaps between them; and the gaps have only two different lengths.

Before stating the next result, we shall say some words on the positions of $p$-th occurrence of letter 0 and 1 in Fibonacci sequence $\mathbf{f}$. Those positions turn out to be very useful in locating singular words in $\mathbf{F}$. Let $\Lambda_0$ (resp. $\Lambda_1$) be the set of positions of letter 0 (resp. 1) in $\mathbf{f}$. Namely,

$$\begin{aligned}
\Lambda_0 &:= \{n \in \mathbb{N} : f_n = 0\} = \{\lambda^0(i)\}_{i\geqslant 1}, \\
\Lambda_1 &:= \{n \in \mathbb{N} : f_n = 1\} = \{\lambda^1(i)\}_{i\geqslant 1},
\end{aligned}$$

where $\lambda^0(i)$ (resp. $\lambda^1(i)$) is the position of $i$-th 0 (resp. 1) in $\mathbf{f}$. In fact, $\Lambda_0$ and $\Lambda_1$ are the sequences A022342 and A003622 in OEIS respectively (see [17]). Moreover, the closed form of the sequence A022342 is given in [17], i.e.,

$$\lambda^0(i) = \left\lfloor \frac{\sqrt{5}-1}{2} \cdot i \right\rfloor + i - 1, \quad \forall\, i \geqslant 1. \tag{8}$$

Note that the indices of elements in $\mathbf{f}$ starts from 0. Thus $\lambda^0(1) = 0$. Let $m^f := \{m^f(i)\}_{i \geqslant 0}$ where for all $i \geqslant 0$,

$$m^f(i) := F_{\lambda^0(i+1)} - f_{\lambda^0(i+1)}.$$

Of course $f_{\lambda^0(i)} = 0$ for all $i$, but writing the definition in this way will be useful in what follows.

**Proposition 18.** $m^f = \{F_n - f_n\}_{n \in \Lambda_0} = \{F_n - f_n\}_{n \in \Lambda_1}$.

*Proof.* The first equality is the definition. For the second one, we only need to show that for any $i \geqslant 1$, $F_{\lambda^0(i)} - f_{\lambda^0(i)} = F_{\lambda^1(i)} - f_{\lambda^1(i)}$. By the definition of $\lambda^0(i)$ and $\lambda^1(i)$, we have $f_{\lambda^0(i)} = 0$ and $f_{\lambda^1(i)} = 1$. So $f_{\lambda^1(i)} - f_{\lambda^0(i)} = 1$. In Fibonacci sequence $\mathbf{f}$, the $i$-th 1 is generated by iterating the $i$-th 0, and $\phi(F_{\lambda^0(i)}) = F_k F_{\lambda^1(i)}$ for some $k$ satisfying $F_k = F_{\lambda^0(i)}$. Thus, by the definition of $\phi$, $F_{\lambda^1(i)} - F_{\lambda^0(i)} = 1$. Hence, $F_{\lambda^1(i)} - F_{\lambda^0(i)} = f_{\lambda^1(i)} - f_{\lambda^0(i)}$ for all $i \geqslant 1$. □

Fix a $k \geqslant 1$. For any $p \geqslant 1$, let $S_{k,p}$ be the $p$-th occurrence of singular words of order $k$, and let $G_{k,p}$ be the *gap word* between two consecutive singular words $S_{k,p}$ and $S_{k,p+1}$. In addition, $G_{k,0}$ is defined to be the prefix of $\mathbf{F}$ before $S_{k,1}$. Then, we have the following decomposition theorem:

**Theorem 19.** *For any $k \geqslant 1$,*

$$\mathbf{F} = G_{k,0} S_{k,1} G_{k,1} S_{k,2} G_{k,2} S_{k,3} G_{k,3} \cdots = G_{k,0} \prod_{p=1}^{\infty} S_{k,p} G_{k,p},$$

*where $S_{k,p} = S_k^{(2)} \oplus m^f(p-1)$, and $G_{k,p} = \widetilde{X}_{k,p} \widetilde{Y}_{k,p}$ satisfying $|\widetilde{Y}_{k,p}| = L_{k+3} - 1$,*

$$\widetilde{X}_{k,p} = \begin{cases} \widetilde{X}_{k,1} \oplus m^f(p-1), & \text{if } p+1 \in \Lambda_0, \\ \widetilde{X}_{k,2} \oplus m^f(p-1), & \text{if } p+1 \in \Lambda_1, \end{cases}$$

*and*

$$\widetilde{Y}_{k,p} = \widetilde{Y}_{k,1} \oplus (F_{\lambda^0(\lambda^1(p)+2)} - 2),$$

*with initial values $G_{k,0} = \phi^k(0) k^{-1}$, $\widetilde{X}_{k,1} = (k+2)\phi^{k+1}(2)(k+3)$, $\widetilde{X}_{k,2} = F_k + L_{k+2} + 1$ and $\widetilde{Y}_{k,1} = \phi^{k+1}(2)(k+3)^{-1}$.*

For example, when $k = 2$, we have the following initial values

$$S_2^{(2)} = 323,\ G_{2,0} = 0122,\ \widetilde{X}_{2,1} = 423445,\ \widetilde{X}_{2,2} = 6,\ \widetilde{Y}_{2,1} = 2344,$$

and the decomposition

$$\mathbf{F} = 0122\mathbf{323}4234452344\mathbf{545}62344\mathbf{545}6456672344\mathbf{545}645667456\mathbf{767}\cdots$$

where $S_{2,*}$, $\widetilde{X}_{2,*}$, $\widetilde{Y}_{2,*}$ are marked in red, blue and green respectively.

### 5.1 Proof of the first two decomposition theorems

*Proof of Theorem 16.* We only need to show

$$\phi^n(0) = \Pi_{j=-2}^{n-2}\phi^j(2) \tag{9}$$

for all $n \geqslant 0$. When $k = 0$, $\phi^0(0) = 0 = \phi^{-2}(2)$. Assume that it is true for all $k < n$. Since

$$\begin{aligned}
\phi^n(0) &= \phi^{n-1}(0)F_{L_{n+1}}F_{L_{n+1}+1}\cdots F_{L_{n+2}-1} \\
&= \left(\Pi_{j=-2}^{n-3}\phi^j(2)\right)\left(F_{L_{n+1}}F_{L_{n+1}+1}\cdots F_{L_{n+2}-1}\right) \qquad \text{(by induction)} \\
&= \left(\Pi_{j=-2}^{n-3}\phi^j(2)\right)\phi^{n-2}(2) \qquad \text{(by Proposition 7)} \\
&= \Pi_{j=-2}^{n-2}\phi^j(2),
\end{aligned}$$

the formula (9) holds for $k = n$. So, by induction, the result follows. □

Now, we will prove Theorem 17.

*Proof of Theorem 17.* Firstly, we show $\phi^n(0)n^{-1} = S_{-1}^{(2)}S_0^{(2)}\cdots S_{n-2}^{(2)}$ for all $n \geqslant 1$. When $n = 1$, $S_{-1}^{(2)} = 0 = \phi^1(0)1^{-1}$. Assume that it is true for all $k < n$. Then

$$\begin{aligned}
\phi^n(0)n^{-1} &= \phi^{n-1}(0)\phi^{n-2}(2)n^{-1} \\
&= \phi^{n-1}(0)(n-1)^{-1}(n-1)\phi^{n-2}(2)n^{-1} \\
&= S_{-1}^{(2)}S_0^{(2)}\cdots S_{n-3}^{(2)}(n-1)\phi^{n-2}(2)n^{-1} \qquad \text{(by induction)} \\
&= S_{-1}^{(2)}S_0^{(2)}\cdots S_{n-3}^{(2)}S_{n-2}^{(2)}
\end{aligned} \tag{10}$$

where the last equality follows from the definition of singular words and Proposition 4. Since for all $n \geqslant 0$, $\phi^n(0)n^{-1}$ is a prefix of $\mathbf{F}$, and $|\phi^n(0)n^{-1}| \to \infty$ when $n \to \infty$, we have

$$\mathbf{F} = \lim_{n\to\infty}\phi^n(0)n^{-1} = \prod_{j=-1}^{\infty} S_j^{(2)}.$$ □

### 5.2 Proof of Theorem 19

In [27], Wen and Wen define the $k$-th singular word of $\mathbf{f}$ to be $s_k = \alpha\sigma^k(0)\beta^{-1}$, where $\alpha\beta \rhd \sigma^k(0)$, and they proved the following result.

**Lemma 20** ([27, Lemma 2]). *For any $k \geqslant 1$, $u_1, u_2 \in \{0,1\}$,*

$$s_k \prec \sigma^k(u_1u_2) \text{ if and only if } u_1u_2 = 10.$$

*Moreover, $s_k$ occurs only once in $\sigma^k(10)$ and*

$$s_k = \sigma^k(10)[L_{k+1}, L_{k+3}-1].$$

We have a similar characterization of the singular words of order $k$ in $\mathbf{F}$.

**Lemma 21.** *For any $i \geqslant 1$, $k \geqslant 1$ and $u_1u_2 \in \mathbb{N}^2$,*

$$S_k^{(2i)} \prec \phi^k(u_1u_2) \text{ if and only if } u_1u_2 = (2i-1)(2i).$$

*Moreover, $S_k^{(2i)}$ occurs only once in $\phi^k((2i-1)(2i))$ and*

$$S_k^{(2i)} = \phi^k((2i-1)(2i))[L_{k+1}, L_{k+3}-1].$$

*Proof.* From the definition of singular words, we know that

$$S_k^{(2i)} = \alpha\phi^k(2i)\beta^{-1}$$

where $\alpha\beta \rhd \phi^k(2i)$. By Proposition 4, we have $\alpha = 2i+k-1$, $\beta = 2i+k$ and $\alpha$ is the last digit of $\phi^k(2i-1)$. So $S_k^{(2i)} \prec \alpha\phi^k(2i) \prec \phi^k((2i-1)(2i))$.

Now suppose $S_k^{(2i)} \prec \phi^k(u_1u_2)$. Note that

$$S_k^{(2i)} \equiv s_k \pmod 2 \text{ and } \phi^k(u_1u_2) \equiv \sigma^k(v_1v_2) \pmod 2$$

where $u_i \equiv v_i \in \{0,1\} \pmod 2$ for $i = 0, 1$. By Lemma 20, we have $v_1v_2 = 10$ and

$$S_k^{(2i)} \text{ occurs only once in } \phi^k(u_1u_2) \text{ at the position } \phi^k(u_1u_2)[L_{k+1}, L_{k+3}-1].$$

Therefore, the first and the last digit of $S_k^{(2i)}$ are the last digit of $\phi^k(u_1)$ and the penultimate digit of $\phi^k(u_2)$ respectively. Combining this fact and Proposition 4, we have

$$2i+k-1 = k+u_1 \text{ and } 2i+k-1 = k+u_2-1$$

which imply $u_1 = 2i-1$ and $u_2 = 2i$. □

*Remark* 22. By Lemma 21, for any $k, i \geqslant 1$,

$$S_k^{(2i)} \prec \phi^k((2i-1)(2i)) \prec \phi^{k+2}(2i-2). \tag{11}$$

Remark 22 shows that there is a (unique) singular word of order $k$ in the $(k+2)$-th iteration of any even number $(2i)$. The following two lemmas will show that this is the only place that singular words of order $k$ can occur.

**Lemma 23.** *All factors of length two in $\mathbf{F}$ are of the following forms:*

$$(2j)(2j+1),\ (2i+2)(2j+2) \text{ and } (2i+1)(2j+2)$$

*where $0 \leqslant j \leqslant i$.*

*Proof.* Since $\mathbf{F} \equiv \mathbf{f} \pmod 2$ and $11 \not\prec \mathbf{f}$, $(2i+1)(2j+1) \not\prec \mathbf{F}$ for any $i, j \geqslant 0$. For $j \geqslant 0$, by the definition of $\phi$, we know every odd number $2j+1$ is generated by iterating the unique even number $2j$. So, in $\mathbf{F}$, every odd number $2j+1$ is preceded by $2j$, i.e., $(2j)(2j+1) \prec \mathbf{F}$.

Next, we will prove that $(2i+1, 2j+2)$ $(0 \leqslant i < j)$ does not occur in $\mathbf{F}$. Suppose on the contrary that $(2i+1)(2j+2) \prec \mathbf{F}$. We firstly show the following fact:

$$(2i+1)(2j+2) \prec \mathbf{F} \Rightarrow (2i)(2j+2) \prec \mathbf{F} \Rightarrow (2i-1)(2j+2) \prec \mathbf{F}. \qquad \text{(Fact 1)}$$

In fact, since $(2i+1)(2j+2)$ can not be generated by applying $\phi$ to any number in $\mathbb{N}$, we have $(2i+1)(2j+2) \prec \phi(u_1u_2)$ where $u_1u_2$ is a factor of length two in $\mathbf{F}$. Therefore, $2i+1 \rhd \phi(u_1)$ and $2j+2 \lhd \phi(u_2)$. So $u_1u_2$ is either $(2i)(2j+2)$ or $(2i)(2j+1)$. However, from the previous discussion, $(2i)(2j+1) \not\prec \mathbf{F}$ which implies $(2i)(2j+2) = u_1u_2 \prec \mathbf{F}$. The second "$\Rightarrow$" of (Fact 1) follows in the same way.

Applying (Fact 1) $i$ times, we have $1(2j+2) \prec \mathbf{F}$ for $j \geqslant 1$, which contradicts with the fact that 12 is only factor of length 2 in $\mathbf{F}$ with a leading 1.

Now, we will prove $(2i+2, 2j+2)$ $(0 \leqslant i < j)$ does not occur in $\mathbf{F}$. Suppose on the contrary that $(2i+2)(2j+2) \prec \mathbf{F}$. Applying (Fact 1), we have $(2i+1)(2j+2) \prec \mathbf{F}$ which is a contradiction. □

**Lemma 24.** *Let $u_1u_2$ be a factor of length two in $\mathbf{F}$. We have the following:*
(1) *If $u_1 \not\equiv u_2 \mod 2$, then singular words of order $k$ occur in $\phi^{k+2}(u_1u_2)$ only once;*
(2) *If $u_1 \equiv u_2 \mod 2$, then singular words of order $k$ occur in $\phi^{k+2}(u_1u_2)$ only twice.*

*Proof.* By (11), we know singular words of order $k$ occur in $\phi^{k+2}(u_1u_2)$, where $u_1u_2$ is a factor of length two in $\mathbf{F}$ and at least one of $u_1$ and $u_2$ is an even number.

(1) Suppose $u_1 \not\equiv u_2 \mod 2$. Since, by Lemma 23, $u_1u_2$ must be one of the following two forms: $(2j)(2j+1)$ and $(2i+1)(2j+2)$ where $0 \leqslant j \leqslant i$. So we shall discuss the following two cases:
**Case 1.** When $u_1u_2 = (2j)(2j+1)$ for some $j \geqslant 0$. Then

$$\phi^{k+2}(u_1u_2) = \phi^{k+2}((2j)(2j+1)) = \phi^{k}((2j)(2j+1)(2j+2)(2j+2)(2j+3)).$$

By Lemma 21, we know that singular words of order $k$ only occur in $\phi^k((2j+1)(2j+2))$. Therefore, singular words of order $k$ occur in $\phi^{k+2}((2j)(2j+1))$ only once.
**Case 2.** When $u_1u_2 = (2i+1)(2j+2)$ for some $0 \leqslant j \leqslant i$. Then

$$\phi^{k+2}(u_1u_2) = \phi^{k+2}((2i+1)(2j+2)) = \phi^{k}((2i+2)(2i+3)(2j+2)(2j+3)(2j+4)).$$

By Lemma 21, we know that singular words of order $k$ occur in $\phi^k((2j+3)(2j+4))$ and possibly in $\phi^k((2i+3)(2j+2))$. However, if there is a singular word of order $k$ that occurs in $\phi^k((2i+3)(2j+2))$, then according to Lemma 21, we have $k+2i+3 = k+2j+1$ which implies $j = i+1$. This contradicts with the assumption $0 \leqslant j \leqslant i$. Therefore, singular words of order $k$ occur in $\phi^{k+2}((2i+1)(2j+2))$ only once.

(2) If $u_1 \equiv u_2 \mod 2$, then $u_1u_2 = (2i+2)(2j+2)$ for some $i \geqslant j \geqslant 0$. Then

$$\phi^{k+2}(u_1u_2) = \phi^{k+2}((2i+2)(2j+2)) = \phi^{k}((2i+2)(2i+3)(2i+4)(2j+2)(2j+3)).$$

By Lemma 21, we know that singular words of order $k$ only occur in $\phi^k((2i+3)(2i+4))$ and $\phi^k((2j+2)(2j+3))$. Therefore, singular words of order $k$ occur in $\phi^{k+2}((2i+2)(2j+2))$ only twice. □

In the following, we will discuss the properties of the gap sequence $\{G_{k,p}\}_{p\geqslant 1}$ for any $k \geqslant 1$.

**Lemma 25.** *For any $k \geqslant 1$, the sequence $\{|G_{k,p}|\}_{p\geqslant 1}$ is the Fibonacci sequence over the alphabet $\{2L_{k+3}, L_{k+3}\}$.*

*Proof.* Combining (11) and Lemma 24, we know that for any $p \geqslant 1$, the singular word $S_{k,p} \prec \phi^{k+2}(F_{\lambda^0(p)})$. Then, by Lemma 21,

$$S_{k,p} = \phi^{k+2}(F_{\lambda^0(p)})[L_{k+3}, L_{k+4} - 1]. \tag{12}$$

For any $p \geqslant 1$, $F_{\lambda^0(p)}$ and $F_{\lambda^0(p+1)}$ either occur in $\mathbf{F}$ consecutively or separate by only one odd number. Then

$$G_{k,p} \prec \phi^{k+2}(F_{\lambda^0(p)}F_{\lambda^0(p+1)}) \text{ or } G_{k,p} \prec \phi^{k+2}(F_{\lambda^0(p)}F_tF_{\lambda^0(p+1)})$$

for some $t \in \Lambda^1$. In the first case, by (12),

$$G_{k,p} = \phi^{k+2}(F_{\lambda^0(p)}F_{\lambda^0(p+1)})[L_{k+4}, L_{k+5} - 1]. \tag{13}$$

In the second case, by (12),

$$G_{k,p} = \phi^{k+2}(F_{\lambda^0(p)}F_tF_{\lambda^0(p+1)})[L_{k+4}, L_{k+3} + L_{k+5} - 1]. \tag{14}$$

So for any $p \geqslant 1$,

$$|G_{k,p}| = \begin{cases} L_{k+3}, & \text{if } \lambda^0(p+1) - \lambda^0(p) = 1; \\ 2L_{k+3}, & \text{if } \lambda^0(p+1) - \lambda^0(p) = 2. \end{cases}$$

By (8),

$$\{\lambda^0(p+1) - \lambda^0(p)\}_{p\geqslant 1} = \left\{ \left\lfloor (p+1) \cdot \frac{\sqrt{5}-1}{2} \right\rfloor - \left\lfloor p \cdot \frac{\sqrt{5}-1}{2} \right\rfloor \right\}_{p\geqslant 1} + 1$$

which is the Fibonacci sequence over $\{1, 2\}$. This implies that $\{|G_{k,p}|\}_{p\geqslant 1}$ is the Fibonacci sequence over $\{2L_{k+3}, L_{k+3}\}$. □

Now we will give the explicit expression of the gap words. We denote gap words of length $2L_{k+3}$ and $L_{k+3}$ by $G^L_{k,*}$ and $G^S_{k,*}$ where the superscript 'L' and 'S' stand for long and short. Then

$$\{G^L_{k,p}\}_{p\geqslant 1} := \{G_{k,\lambda^0(p)+1}\}_{p\geqslant 1} \text{ and } \{G^S_{k,p}\}_{p\geqslant 1} := \{G_{k,\lambda^1(p)+1}\}_{p\geqslant 1}.$$

Combining (13) and (14), we have

$$\begin{aligned} G^S_{k,p} &= \phi^{k+2}(F_{\lambda^0(p)}F_{\lambda^0(p+1)})[L_{k+4}, L_{k+5} - 1], \\ G^L_{k,p} &= \phi^{k+2}(F_{\lambda^0(p)}F_tF_{\lambda^0(p+1)})[L_{k+4}, L_{k+3} + L_{k+5} - 1]. \end{aligned}$$

That is

$$G^L_{k,p} = X^L_{k,p}Y^L_{k,p} \text{ and } G^S_{k,p} = X^S_{k,p}Y^S_{k,p},$$

where $|Y^L_{k,p}| = |Y^S_{k,p}| = L_{k+3} - 1$, $|X^L_{k,p}| = 1 + L_{k+3}$ and $|X^S_{k,p}| = 1$. In detail,

$$\begin{aligned} X^S_{k,p} &= \phi^{k+2}(F_{\lambda^0(p)}F_{\lambda^0(p+1)})[L_{k+4}, L_{k+4}],\\ Y^S_{k,p} &= \phi^{k+2}(F_{\lambda^0(p)}F_{\lambda^0(p+1)})[L_{k+4}+1, L_{k+5}-1],\\ X^L_{k,p} &= \phi^{k+2}(F_{\lambda^0(p)}F_tF_{\lambda^0(p+1)})[L_{k+4}, L_{k+5}],\\ Y^L_{k,p} &= \phi^{k+2}(F_{\lambda^0(p)}F_tF_{\lambda^0(p+1)})[L_{k+5}+1, L_{k+3}+L_{k+5}-1]. \end{aligned}$$

Moreover,

$$\{G_{k,p}\}_{p\geqslant 1} = X^L_{k,1}Y^L_{k,1}\, X^S_{k,1}Y^S_{k,1}\, X^L_{k,2}Y^L_{k,2}\, X^L_{k,3}Y^L_{k,3}\, X^S_{k,2}Y^S_{k,2}\, X^L_{k,4}Y^L_{k,4}\cdots$$

For example, if we choose the singular word $S_2 = 323$, then it is easy to see that $G^L_{2,1} = 4234452344$ with $X^L_{2,1} = 423445$, $Y^L_{2,1} = 2344$ and $G^S_{2,1} = 62344$ with $X^S_{2,1} = 6$, $Y^S_{2,1} = 2344$. Here, we have

$$\begin{array}{lllllllllll} \mathbf{F} & = & 0122 & 323 & 423445 & 2344 & 545 & 6 & 2344 & 545 & \cdots \\ \mathbf{F} & = & G_{2,0} & S_{2,1} & X^L_1 & Y^L_1 & S_{2,2} & X^S_1 & Y^S_1 & S_{2,2} & \cdots \end{array}$$

*Proof of Theorem 19.* The proof is composed by the following three steps. Fix a $k \geqslant 1$.

**Step 1.** We will show that $S_{k,p} = S^{(2)}_k \oplus m^f(p-1)$, for any $p \geqslant 1$. In fact, combining (11) and Lemma 24, we know that for any $p \geqslant 1$, the singular word $S_{k,p} \prec \phi^{k+2}(F_{\lambda^0(p)})$. So by (12) and the definition of $\{m^f(p)\}_{p\geqslant 1}$, we have $S_{k,p} = S_{k,1} \oplus m^f(p-1)$, where $S_{k,1} = S^{(2)}_k$.

**Step 2.** We will show that

$$\widetilde{X}_{k,p} = \begin{cases} \widetilde{X}_{k,1} \oplus m^f(p-1), & \text{if } p+1 \in \Lambda_0,\\ \widetilde{X}_{k,2} \oplus m^f(p-1), & \text{if } p+1 \in \Lambda_1, \end{cases}$$

with initial values $\widetilde{X}_{k,1} = (k+2)\phi^{k+1}(2)(k+3)$ and $\widetilde{X}_{k,2} = F_k + L_{k+2} + 1$.

It is easy to see that $\widetilde{X}_{k,1} = X^L_{k,1} = (k+2)\phi^{k+1}(2)(k+3)$ and $\widetilde{X}_{k,2} = X^S_{k,1} = F_k + L_{k+2} + 1$. Combining Lemma 25 and Proposition 18, we have $X^L_{k,p} = X^L_{k,1} \oplus m^f(p-1)$ and $X^S_{k,p} = X^S_{k,1} \oplus m^f(p-1)$.

**Step 3.** We will show that

$$\widetilde{Y}_{k,p} = \widetilde{Y}_{k,1} \oplus \big(F_{\lambda^0(\lambda^1(p)+2)} - 2\big)$$

with the initial value $\widetilde{Y}_{k,1} = \phi^{k+1}(2)(k+3)^{-1}$. By Lemma 25, we only need to show $\widetilde{Y}_{k,p} \lhd \phi^{k+2}(F_{\lambda^0(p+1)})$ and $Y^L_{k,p} = Y^S_{k,p} = \widetilde{Y}_{k,1} \oplus \big(F_{\lambda^0(\lambda^1(p)+2)} - 2\big)$.

Apparently, $\widetilde{Y}_{k,1} = \phi^{k+1}(2)(k+3)^{-1}$. Since $\widetilde{Y}_{k,p}$ is a prefix of the word generating by iterating $k+2$ times of the second even number in $\mathbf{F}$, we have

$$\widetilde{Y}_{k,p} \lhd \phi^{k+2}(F_{\lambda^0(p+1)}).$$

Then by Lemma 25 and Proposition 18, we have

$$Y^L_{k,p} = \widetilde{Y}_{\lambda^0(p)+1} \lhd \phi^{k+2}(F_{\lambda^0(\lambda^1(p)+2)}), \quad Y^S_{k,p} = \widetilde{Y}_{\lambda^1(p)+1} \lhd \phi^{k+2}(F_{\lambda^0(\lambda^1(p)+2)})$$

and $|Y^L_{k,p}| = |Y^S_{k,p}|$. So $Y^L_{k,p} = Y^S_{k,p} = \widetilde{Y}_{k,1} \oplus (F_{\lambda^0(\lambda^1(p)+2)} - 2)$. □

## 6 Miscellaneous

By Theorem 19, we know that the structure of singular words will strongly affect the infinite Fibonacci sequence **F** itself. So we also study the structure of singular words, and we have the following results.

**Proposition 26.** *There are three constructions of singular words in the following:*
(1) *For any* $n \geqslant 1$,

$$S_{n+1}^{(0)} = nS_{-1}^{(2)}S_0^{(2)} \cdots S_{n-1}^{(2)}. \tag{15}$$

(2) *For any* $n \geqslant 1$, *if* $\alpha\beta \triangleright \phi^n(0)$, *then* $S_{n+1}^{(0)} = \beta\alpha^{-1}S_n^{(0)}S_{n-1}^{(2)}$.
(3) *For* $n \geqslant -1$, $S_{n+3}^{(2)} = UVU'$, *where* $V = S_n^{(2)} \oplus 2$, $U' = S_{n+1}^{(2)} \oplus 2$ *and* $U = (n+4)\phi^{n+1}(2)(n+3)^{-1}$.

*Proof.* (1) Since $S_{n+1}^{(0)} = n\phi^{n+1}(0)(n+1)^{-1}$, by (10), we have

$$S_{n+1}^{(0)} = nS_{-1}^{(2)}S_0^{(2)} \cdots S_{n-1}^{(2)}.$$

(2) When $n = 1$, $\phi^1(0) = 01, \alpha = 0$, so $\beta\alpha^{-1}S_1^{(0)}S_0^{(2)} = 10^{-1}001 = 101 = S_2^{(0)}$. Assume that it is true for all $k < n$. Then,

$$\begin{aligned} n(n-1)^{-1}S_n^{(0)}S_{n-1}^{(2)} &= n(n-1)^{-1}(n-1)(n-2)^{-1}S_{n-1}^{(0)}S_{n-2}^{(2)}S_{n-1}^{(2)} \\ &= n(n-2)^{-1}S_{n-1}^{(0)}S_{n-2}^{(2)}S_{n-1}^{(2)} \\ &= \cdots \\ &= n0^{-1}S_1^{(0)}S_0^{(2)}S_1^{(2)} \cdots S_{n-1}^{(2)} \\ &= nS_{-1}^{(2)}S_0^{(2)}S_1^{(2)} \cdots S_{n-1}^{(2)} \\ &= S_{n+1}^{(0)} \qquad \text{by (15).} \end{aligned}$$

(3) Since $\phi^n(2) = F_0^{(2)}F_1^{(2)} \cdots F_{L_{n+2}-2}^{(2)}F_{L_{n+2}-1}^{(2)}$, we have

$$S_n^{(2)} = F_{L_{n+2}-2}^{(2)}F_0^{(2)}F_1^{(2)} \cdots F_{L_{n+2}-2}^{(2)}.$$

Then

$$\begin{aligned} U &= \left(F_{L_{n+3}-2}^{(2)} + 2\right) F_0^{(2)}L_1^{(2)} \cdots F_{L_{n+3}-2}^{(2)}, \\ V &= \left(F_{L_{n+2}-2}^{(2)} + 2\right)\left(F_0^{(2)} + 2\right) \cdots \left(F_{L_{n+2}-2}^{(2)} + 2\right), \\ U' &= \left(F_{L_{n+3}-2}^{(2)} + 2\right)\left(F_0^{(2)} + 2\right) \cdots \left(F_{L_{n+3}-2}^{(2)} + 2\right). \end{aligned}$$

By Proposition 4, we have

$$F_{L_{n+3}-2}^{(2)} + 2 = F_{L_{n+5}-2}^{(2)} = n+6, \quad F_{L_{n+3}-1}^{(2)} = F_{L_{n+2}-2}^{(2)} + 2 = n+5$$

and $F^{(2)}_{L_{n+4}-1} = n+6$. Moreover,

$$\begin{aligned} F^{(2)}_{L_{n+3}} F^{(2)}_{L_{n+3}+1} F^{(2)}_{L_{n+3}+2} \cdots F^{(2)}_{L_{n+4}-1} &= \phi^n(2) \oplus 2 \\ &= \left(F^{(2)}_0 + 2\right)\left(F^{(2)}_1 + 2\right) \cdots \left(F^{(2)}_{L_{n+2}-1} + 2\right), \\ F^{(2)}_{L_{n+4}} F^{(2)}_{L_{n+4}+1} F^{(2)}_{L_{n+4}+2} \cdots F^{(2)}_{L_{n+5}-1} &= \phi^{n+1}(2) \oplus 2 \\ &= \left(F^{(2)}_0 + 2\right)\left(F^{(2)}_1 + 2\right) \cdots \left(F^{(2)}_{L_{n+3}-1} + 2\right). \end{aligned}$$

Thus

$$\begin{aligned} S^{(2)}_{n+3} = &\underbrace{F^{(2)}_{L_{n+5}-2} F^{(2)}_0 F^{(2)}_1 \cdots F^{(2)}_{L_{n+3}-2}}_{U} \underbrace{F^{(2)}_{L_{n+3}-1} F^{(2)}_{L_{n+3}} \cdots F^{(2)}_{L_{n+4}-2}}_{V} \\ &\underbrace{F^{(2)}_{L_{n+4}-1} F^{(2)}_{L_{n+4}} \cdots F^{(2)}_{L_{n+5}-2}}_{U'}. \end{aligned}$$

□

The next result completely characterizes palindromes in $\mathbf{F}$. Recall that a word $w = w_0 w_1 \cdots w_p$ is a palindrome if $w_p w_{p-1} \cdots w_1 w_0 = w$.

**Proposition 27.** *The palindromes in* $\mathbf{F}$ *are of the forms* $(22 \oplus 2i)$, $(232 \oplus 2i)$ *and* $(323 \oplus 2i)$ *where* $i \geqslant 0$.

*Proof.* $(i)$ Since $\mathbf{F} \equiv \mathbf{f} \mod 2$ and $11 \not\prec \mathbf{f}$, we have $(2i+1, 2i+1) \not\prec \mathbf{F}$. So the palindromes of length 2 in $\mathbf{F}$ are of the form $(2i, 2i)$ where $i > 0$.

$(ii)$ The palindromes of length 3 are generated by iterating words of length 2. By Lemma 23 and the fact that 000 does not occur in $\mathbf{f}$, we know that the palindromes of length 3 are $(2i, 2i+1, 2i)$ and $(2i+1, 2i, 2i+1)$.

$(iii)$ If there are palindromes of length 4 in $\mathbf{F}$, then they are of the form $u_1 u_2 u_2 u_1$. Since $\mathbf{F} \equiv \mathbf{f} \mod 2$, $11 \not\prec \mathbf{f}$ implies $u_2 = 2j$ for some $j \geqslant 1$, and $0000 \not\prec \mathbf{f}$ implies $u_1 = 2i+1$ for some $i \geqslant 0$. By Lemma 23, $u_2 u_1 \prec \mathbf{F}$ gives $i = j$. Thus $u_1 u_2 u_2 u_1 = (2i+1)(2i)(2i)(2i+1) \prec \mathbf{F}$. Since $\phi(\mathbf{F}) = \mathbf{F}$, the pre-image of $(2i+1)(2i)(2i)(2i+1)$ must be a factor of $\mathbf{F}$. Note that $(2i+1)(2i)(2i)(2i+1)$ has a unique pre-image under $\phi$, which is $(2i)(2i-1)(2i)$. So $(2i)(2i-1)(2i) \prec \mathbf{F}$ which contradicts $(ii)$. Therefore, there are no palindromes of length 4 in $\mathbf{F}$.

$(vi)$ Suppose that there are palindromes of length 5 in $\mathbf{F}$. Then by $(ii)$, they are of the following forms: $v_1(2i)(2i+1)(2i)v_1$ and $v_2(2i+1)(2i)(2i+1)v_2$.

- Note that $v_1(2i)(2i+1)(2i)v_1 \pmod 2$ is either 00100 or 10101. But $10101 \not\prec \mathbf{f}$, we have $v_1 = 2j$ for some $j \geqslant 1$. The pre-image of $(2j)(2i)(2i+1)(2i)(2j)$ is $(2j-1)(2i)(2i-1)z$ where $z = 2j-1$ or $2j$. In any case, $(2j-1)(2i)(2i-1) \prec \mathbf{F}$. However, by Lemma 23, $(2i)(2i-1) \not\prec \mathbf{F}$ which is a contradiction. Thus $(2j)(2i)(2i+1)(2i)(2j) \not\prec \mathbf{F}$.

- In the second case, $v_2 = 2j$ for some $j \geqslant 1$ since $11 \not\prec \mathbf{f}$. By Lemma 23, $(2j)(2i+1) \prec \mathbf{F}$ yields that $j = i$. Thus $(2i)(2i+1)(2i)(2i+1)(2i) \prec \mathbf{F}$, and its pre-image is $(2i)(2i)(2i-1)$ or $(2i)(2i)(2i)$. However, $(2i)(2i-1) \not\prec \mathbf{F}$ implies $(2i)(2i)(2i-1) \not\prec \mathbf{F}$, and $000 \not\prec \mathbf{f}$ implies $(2i)(2i)(2i) \not\prec \mathbf{F}$. Hence $(2i)(2i+1)(2i)(2i+1)(2i) \not\prec \mathbf{F}$.

So there are no palindrome of length 5 in $\mathbf{F}$.

($v$) If there are palindromes of length larger than or equal to 6, then we can find palindromes of length 4 or 5. However, according to ($iii$) and ($vi$), there are no palindromes of length 4 and 5, so there are no palindromes of length larger than or equal to 6 in $\mathbf{F}$. $\square$

In the last part, we give some observations of the sequence $\mathbf{F} \pmod{k}$ where $k \geqslant 2$. When $k$ is an even number, the sequence $\mathbf{F} \pmod{k}$ is a pure substitution sequence which is generated by the substitution $\phi \pmod{k}$, i.e.,

$$\begin{cases} 0 \to 01, \\ 1 \to 2, \end{cases} \begin{cases} 2 \to 23, \\ 3 \to 4, \end{cases} \cdots \begin{cases} k-2 \to (k-2)(k-1), \\ k-1 \to 0. \end{cases}$$

When $k$ is an odd number, the sequence $\mathbf{F} \pmod{k}$ is a substitution sequence under a coding where the substitution $\phi \pmod{2k}$, i.e.,

$$\begin{cases} 0 \to 01, \\ 1 \to 2, \end{cases} \begin{cases} 2 \to 23, \\ 3 \to 4, \end{cases} \cdots \begin{cases} 2k-2 \to (2k-2)(2k-1), \\ 2k-1 \to 0, \end{cases}$$

and the coding is $\tau : \ i \ (i+k) \to i$ for every $0 \leqslant i < k$.

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