Small feedback vertex sets in planar digraphs

Louis Esperet* Laetitia Lemoine Frédéric Maffray†

G-SCOP, CNRS & Univ. Grenoble Alpes
Grenoble, France

{louis.esperet, laetitia.lemoine, frederic.maffray}@grenoble-inp.fr

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Abstract

Let G be a directed planar graph on n vertices, with no directed cycle of length less than $g \geqslant 4$. We prove that G contains a set X of vertices such that $G - X$ has no directed cycle, and $|X| \leqslant \frac{5n-5}{9}$ if $g = 4$, $|X| \leqslant \frac{2n-5}{4}$ if $g = 5$, and $|X| \leqslant \frac{2n-6}{g}$ if $g \geqslant 6$. This improves recent results of Golowich and Rolnick.

A directed graph G (or digraph, in short) is said to be acyclic if it does not contain any directed cycle. The *digirth* of a digraph G is the minimum length of a directed cycle in G (if G is acyclic, we set its digirth to $+\infty$). A *feedback vertex set* in a digraph G is a set X of vertices such that $G - X$ is acyclic, and the minimum size of such a set is denoted by $\tau(G)$. In this short note, we study the maximum $f_g(n)$ of $\tau(G)$ over all planar digraphs G on n vertices with digirth g. Harutyunyan [1, 4] conjectured that $f_3(n) \leqslant \frac{2n}{5}$ for all n. This conjecture was recently refuted by Knauer, Valicov and Wenger [5] who showed that $f_g(n) \geqslant \frac{n-1}{g-1}$ for all $g \geqslant 3$ and infinitely many values of n. On the other hand, Golowich and Rolnick [3] recently proved that $f_4(n) \leqslant \frac{7n}{12}$, $f_5(n) \leqslant \frac{8n}{15}$, and $f_g(n) \leqslant \frac{3n-6}{g}$ for all $g \geqslant 6$ and n. Harutyunyan and Mohar [4] proved that the vertex set of every planar digraph of digirth at least 5 can be partitioned into two acyclic subgraphs. This result was very recently extended to planar digraphs of digirth 4 by Li and Mohar [6], and therefore $f_4(n) \leqslant \frac{n}{2}$.

This short note is devoted to the following result, which improves all the previous upper bounds for $g \geqslant 5$ (although the improvement for $g = 5$ is rather minor). Due to the very recent result of Li and Mohar [6], our result for $g = 4$ is not best possible (however its proof is of independent interest and might lead to further improvements).

Theorem 1. *For all $n \geqslant 3$ we have $f_4(n) \leqslant \frac{5n-5}{9}$, $f_5(n) \leqslant \frac{2n-5}{4}$ and for all $g \geqslant 6$, $f_g(n) \leqslant \frac{2n-6}{g}$.*

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In a planar graph, the degree of a face F, denoted by $d(F)$, is the sum of the lengths (number of edges) of the boundary walks of F. In the proof of Theorem 1, we will need the following two simple lemmas.

Lemma 2. *Let H be a planar bipartite graph, with bipartition (U, V), such that all faces of H have degree at least 4, and all vertices of V have degree at least 2. Then H contains at most $2|U| - 4$ faces of degree at least 6.*

Proof. Assume that H has n vertices, m edges, f faces, and f_6 faces of degree at least 6. Let N be the sum of the degrees of the faces of H, plus twice the sum of the degrees of the vertices of V. Observe that $N = 4m$, so, by Euler's formula, $N \leqslant 4n + 4f - 8$. The sum of degrees of the faces of H is at least $4(f - f_6) + 6f_6 = 4f + 2f_6$, and since each vertex of V has degree at least 2, the sum of the degrees of the vertices of V is at least $2|V|$. Therefore, $4f + 2f_6 + 4|V| \leqslant 4n + 4f - 8$. It follows that $f_6 \leqslant 2|U| - 4$, as desired. $\square$

Lemma 3. *Let G be a connected planar graph, and let $S = \{F_1, \ldots, F_k\}$ be a set of k faces of G, such that each F_i is bounded by a cycle, and these cycles are pairwise vertex-disjoint. Then $\sum_{F \notin S}(3d(F) - 6) \geqslant \sum_{i=1}^{k}(3d(F_i) + 6) - 12$, where the first sum varies over faces F of G not contained in S.*

Proof. Let n, m, and f denote the number of vertices, edges, and faces of G, respectively. It follows from Euler's formula that the sum of $3d(F) - 6$ over all faces of G is equal to $6m - 6f = 6n - 12 \geqslant 6\sum_{i=1}^{k} d(F_i) - 12$. Therefore, $\sum_{F \notin S}(3d(F) - 6) \geqslant 6\sum_{i=1}^{k} d(F_i) - 12 - \sum_{i=1}^{k}(3d(F_i) - 6) = \sum_{i=1}^{k}(3d(F_i) + 6) - 12$, as desired. $\square$

We are now able to prove Theorem 1.

Proof of Theorem 1. We prove the result by induction on $n \geqslant 3$. Let G be a planar digraph with n vertices and digirth $g \geqslant 4$. We can assume without loss of generality that G has no multiple arcs, since $g \geqslant 4$ and removing one arc from a collection of multiple arcs with the same orientation does not change the value of $\tau(G)$. We can also assume that G is connected, since otherwise we can consider each connected component of G separately and the result clearly follows from the induction (since $g \geqslant 4$, connected components of at most 2 vertices are acyclic and can thus be left aside). Finally, we can assume that G contains a directed cycle, since otherwise $\tau(G) = 0 \leqslant \min\{\frac{5n-5}{9}, \frac{2n-5}{4}, \frac{2n-6}{g}\}$ (since $n \geqslant 3$).

Let $\mathcal{C}$ be a maximum collection of arc-disjoint directed cycles in G. Note that $\mathcal{C}$ is non-empty. Fix a planar embedding of G. For a given directed cycle C of $\mathcal{C}$, we denote by $\overline{C}$ the closed region bounded by C, and by $\mathring{C}$ the interior of $\overline{C}$. It follows from classical uncrossing techniques (see [2] for instance), that we can assume without loss of generality that the directed cycles of $\mathcal{C}$ are pairwise non-crossing, i.e. for any two elements $C_1, C_2 \in \mathcal{C}$, either $\mathring{C}_1$ and $\mathring{C}_2$ are disjoint, or one is contained in the other. We define the partial order $\preceq$ on $\mathcal{C}$ as follows: $C_1 \preceq C_2$ if and only if $\mathring{C}_1 \subseteq \mathring{C}_2$. Note that $\preceq$ naturally defines a rooted forest $\mathcal{F}$ with vertex set $\mathcal{C}$: the roots of each of the components of $\mathcal{F}$ are the maximal elements of $\preceq$, and the children of any given node $C \in \mathcal{F}$ are the maximal elements $C' \preceq C$ distinct from C (the fact that $\mathcal{F}$ is indeed a forest follows from the non-crossing property of the elements of $\mathcal{C}$).

Consider a node C of $\mathcal{F}$, and the children $C_1, \dots, C_k$ of C in $\mathcal{F}$. We define the closed region $\mathcal{R}_C = \overline{C} - \bigcup_{1 \leqslant i \leqslant k} \mathring{C}_i$. Let ϕ_C be the sum of $3d(F) - 6$, over all faces F of G lying in $\mathcal{R}_C$.

Claim 4. *Let C_0 be a node of $\mathcal{F}$ with children $C_1, \dots, C_k$. Then $\phi_{C_0} \geqslant \frac{3}{2}(g-2)k + \frac{3}{2}g$. Moreover, if $g \geqslant 6$, then $\phi_{C_0} \geqslant \frac{3}{2}(g-2)k + \frac{3}{2}g + 3$.*

Assume first that the cycles $C_0, \dots, C_k$ are pairwise vertex-disjoint. Then, it follows from Lemma 3 that $\phi_{C_0} \geqslant (k+1)(3g+6) - 12$. Note that since $g \geqslant 4$, we have $(k+1)(3g+6) - 12 \geqslant \frac{3}{2}(g-2)k + \frac{3}{2}g$. Moreover, if $g \geqslant 6$, $(k+1)(3g+6) - 12 \geqslant \frac{3}{2}(g-2)k + \frac{3}{2}g + 3$, as desired. As a consequence, we can assume that two of the cycles $C_0, \dots, C_k$ intersect, and in particular, $k \geqslant 1$.

Consider the following planar bipartite graph H: the vertices of the first partite set of H are the directed cycles $C_0, C_1, \dots, C_k$, the vertices of the second partite set of H are the vertices of G lying in at least two cycles among $C_0, C_1, \dots, C_k$, and there is an edge in H between some cycle C_i and some vertex v if and only if $v \in C_i$ in G (see Figure 1). Observe that H has a natural planar embedding in which all internal faces have degree at least 4. Since $k \geqslant 1$ and at least two of the cycles $C_0, \dots, C_k$ intersect, the outerface also has degree at least 4. Note that the faces $F_1, \dots, F_t$ of H are in one-to-one correspondence with the maximal subsets $\mathcal{D}_1, \dots, \mathcal{D}_t$ of $\mathcal{R}_{C_0}$ whose interior is connected. Also note that each face of $G \cap \mathcal{R}_{C_0}$ is in precisely one region $\mathcal{D}_i$ and each arc of $\bigcup_{i=0}^{k} C_i$ (i.e. each arc on the boundary of $\mathcal{R}_{C_0}$) is on the boundary of precisely one region $\mathcal{D}_i$. For each region $\mathcal{D}_i$, let ℓ_i be the number of arcs on the boundary of $\mathcal{D}_i$, and observe that $\sum_{i=1}^{t} \ell_i = \sum_{j=0}^{k} |C_j|$. Let $\phi_{\mathcal{D}_i}$ be the sum of $3d(F) - 6$, over all faces F of G lying in $\mathcal{D}_i$. It follows from Lemma 3 (applied with $k = 1$) that $\phi_{\mathcal{D}_i} \geqslant 3\ell_i - 6$, and therefore $\phi_{C_0} = \sum_{i=1}^{t} \phi_{\mathcal{D}_i} \geqslant \sum_{i=1}^{t} (3\ell_i - 6)$.

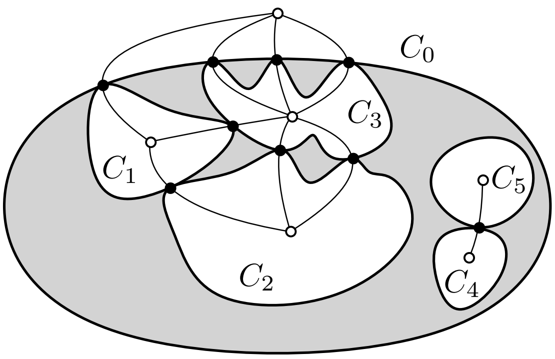

Figure 1: The region $\mathcal{R}_{C_0}$ (in gray) and the planar bipartite graph H.

A region $\mathcal{D}_i$ with $\ell_i \geqslant 4$ is said to be of *type 1*, and we set $T_1 = \{1 \leqslant i \leqslant t \,|\, \mathcal{D}_i \text{ is of type 1}\}$. Since for any $\ell \geqslant 4$ we have $3\ell - 6 \geqslant \frac{3\ell}{2}$, it follows from the paragraph above that the regions $\mathcal{D}_i$ of type 1 satisfy $\phi_{\mathcal{D}_i} \geqslant \frac{3\ell_i}{2}$. Let $\mathcal{D}_i$ be a region that is not of type 1. Since G is simple, $\ell_i = 3$. Assume first that $\mathcal{D}_i$ is bounded by (parts of) two directed cycles of $\mathcal{C}$ (in other words, $\mathcal{D}_i$ corresponds to a face of degree four in the graph H). In this case we say that $\mathcal{D}_i$ is of *type 2* and we set $T_2 = \{1 \leqslant i \leqslant t \mid \mathcal{D}_i \text{ is of type 2}\}$. Then the boundary of $\mathcal{D}_i$ consists in two consecutive arcs e_1, e_2 of some directed cycle C^+ of $\mathcal{C}$, and one arc e_3 of some directed cycle C^- of $\mathcal{C}$. Since $g \geqslant 4$, these three arcs do not form a directed cycle, and therefore their orientation is transitive. It follows that $|C^+| \geqslant g+1$, since otherwise the directed cycle obtained from C^+ by replacing e_1, e_2 with e_3 would have length $g-1$, contradicting that G has digirth at least g. Consequently, $\sum_{i=0}^{k} |C_i| \geqslant (k+1)g + |T_2|$. If a region $\mathcal{D}_i$ is not of type 1 or 2, then $\ell_i = 3$ and each of the 3 arcs on the boundary of $\mathcal{D}_i$ belongs to a different directed cycle of $\mathcal{C}$. In other words, $\mathcal{D}_i$ corresponds to some face of degree 6 in the graph H. Such a region $\mathcal{D}_i$ is said to be of *type 3*, and we set $T_3 = \{1 \leqslant i \leqslant t \mid \mathcal{D}_i \text{ is of type 3}\}$. It follows from Lemma 2 that the number of faces of degree at least 6 in H is at most $2(k+1)-4$. Hence, we have $|T_3| \leqslant 2k-2$.

Using these bounds on $|T_2|$ and $|T_3|$, together with the fact that for any $i \in T_2 \cup T_3$ we have $\phi_{\mathcal{D}_i} \geqslant 3\ell_i - 6 = 3 = \frac{3\ell_i}{2} - \frac{3}{2}$, we obtain:

$$\begin{aligned}
\phi_{C_0} &= \sum_{i\in T_1} \phi_{\mathcal{D}_i} + \sum_{i\in T_2} \phi_{\mathcal{D}_i} + \sum_{i\in T_3} \phi_{\mathcal{D}_i} \\
&\geqslant \sum_{i=1}^{t} \tfrac{3\ell_i}{2} - \tfrac{3}{2}|T_2| - \tfrac{3}{2}|T_3| \\
&\geqslant \tfrac{3}{2} \sum_{i=0}^{k} |C_i| - \tfrac{3}{2}|T_2| - \tfrac{3}{2}(2k-2) \\
&\geqslant \tfrac{3}{2}(k+1)g - 3k + 3 \;=\; \tfrac{3}{2}(g-2)k + \tfrac{3}{2}g + 3,
\end{aligned}$$

as desired. This concludes the proof of Claim 4. $\square$

Let $C_1, \ldots, C_{k_\infty}$ be the k_∞ maximal elements of $\preceq$. We denote by $\mathcal{R}_\infty$ the closed region obtained from the plane by removing $\bigcup_{i=1}^{k_\infty} \mathring{C}_i$. Note that each face of G lies in precisely one of the regions $\mathcal{R}_C$ ($C \in \mathcal{C}$) or $\mathcal{R}_\infty$. Let ϕ_∞ be the sum of $3d(F)-6$, over all faces F of G lying in R_∞. A proof similar to that of Claim 4 shows that $\phi_\infty \geqslant \frac{3}{2}k_\infty(g-2)+3$, and if $g \geqslant 6$, then $\phi_\infty \geqslant \frac{3}{2}k_\infty(g-2)+6$.

We now compute the sum ϕ of $3d(F)-6$ over all faces F of G. By Claim 4,

$$\begin{aligned}
\phi &= \phi_\infty + \sum_{C\in\mathcal{F}} \phi_C \\
&\geqslant \tfrac{3}{2}k_\infty(g-2) + 3 + (|\mathcal{C}| - k_\infty)\tfrac{3}{2}(g-2) + |\mathcal{C}| \cdot \tfrac{3}{2}g \\
&\geqslant (3g-3)|\mathcal{C}| + 3.
\end{aligned}$$

If $g \geqslant 6$, a similar computation gives $\phi \geqslant 3g|\mathcal{C}| + 6$. On the other hand, it easily follows from Euler's formula that $\phi = 6n - 12$. Therefore, $|\mathcal{C}| \leqslant \frac{2n-5}{g-1}$, and if $g \geqslant 6$, then $|\mathcal{C}| \leqslant \frac{2n-6}{g}$.

Let A be a set of arcs of G of minimum size such that $G - A$ is acyclic. It follows from the Lucchesi-Younger theorem [7] (see also [3]) that $|A| = |\mathcal{C}|$. Let X be a set of vertices covering the arcs of A, such that X has minimum size. Then $G - X$ is acyclic. If $g = 5$ we have $|X| \leqslant |A| = |\mathcal{C}| \leqslant \frac{2n-5}{4}$ and if $g \geqslant 6$, we have $|X| \leqslant |A| = |\mathcal{C}| \leqslant \frac{2n-6}{g}$, as desired. Assume now that $g = 4$. In this case $|A| = |\mathcal{C}| \leqslant \frac{2n-5}{3}$. It was observed by Golowich and Rolnick [3] that $|X| \leqslant \frac{1}{3}(n + |A|)$ (which easily follows from the fact that any graph on n vertices and m edges contains an independent set of size at least $\frac{2n}{3} - \frac{m}{3}$), and thus, $|X| \leqslant \frac{5n-5}{9}$. This concludes the proof of Theorem 1. $\square$

Final remark

A natural problem is to determine the precise value of $f_g(n)$, or at least its asymptotical value as g tends to infinity. We believe that $f_g(n)$ should be closer to the lower bound of $\frac{n-1}{g}$, than to our upper bound of $\frac{2n-6}{g}$.

For a digraph G, let $\tau^*(G)$ denote the the infimum real number x for which there are weights in $[0, 1]$ on each vertex of G, summing up to x, such that for each directed cycle C, the sum of the weights of the vertices lying on C is at least 1. Goemans and Williamson [2] conjectured that for any planar digraph G, $\tau(G) \leqslant \frac{3}{2}\tau^*(G)$. If a planar digraph G on n vertices has digirth at least g, then clearly $\tau^*(G) \leqslant \frac{n}{g}$ (this can be seen by assigning weight $1/g$ to each vertex). Therefore, a direct consequence of the conjecture of Goemans and Williamson would be that $f_g(n) \leqslant \frac{3n}{2g}$.

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