A new bijective proof
of Babson and Steingrímsson's conjecture

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Abstract

Babson and Steingrímsson introduced generalized permutation patterns and showed that most of the Mahonian statistics in the literature can be expressed by the combination of generalized pattern functions. Particularly, they defined a new Mahonian statistic in terms of generalized pattern functions, which is denoted $stat$. Given a permutation π, let $des(\pi)$ denote the descent number of π and $maj(\pi)$ denote the major index of π. Babson and Steingrímsson conjectured that $(des, stat)$ and (des, maj) are equidistributed on S_n. Foata and Zeilberger settled this conjecture using q-enumeration, generating functions and Maple packages ROTA and PERCY. Later, Burstein provided a bijective proof of a refinement of this conjecture. In this paper, we give a new bijective proof of this conjecture.

Keywords: Euler-Mahonian; bijection; involution

1 Introduction

In this paper, we give a new bijective proof of a conjecture of Babson and Steingrímsson [1] on Euler-Mahonian statistics.

Let S_n denote the set of all the permutations of $[n] = \{1, 2, \cdots, n\}$. Given a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$, a descent of π is a position $i \in [n-1]$ such that $\pi_i > \pi_{i+1}$, where π_i and π_{i+1} are called a descent top and a descent bottom, respectively. An ascent of π is a position $i \in [n-1]$ such that $\pi_i < \pi_{i+1}$, where π_i is called an ascent bottom and π_{i+1} is called an ascent top. The descent set and the ascent set of π are given by

$$Des(\pi) = \{i \colon \pi_i > \pi_{i+1}\},$$

$$Asc(\pi) = \{i \colon \pi_i < \pi_{i+1}\}.$$

The set of the inversions of π is

$$Inv(\pi) = \{(i,j) \colon 1 \leqslant i < j \leqslant n, \pi_i > \pi_j\}.$$

Let $des(\pi)$, $asc(\pi)$ and $inv(\pi)$ be the descent number, the ascent number and the inversion number of π, which are defined by $des(\pi) = |Des(\pi)|$, $asc(\pi) = |Asc(\pi)|$ and $inv(\pi) = |Inv(\pi)|$, respectively. The major index of π, denoted $maj(\pi)$, is given by

$$maj(\pi) = \sum_{i \in Des(\pi)} i.$$

Suppose that st_1 is a statistic on the object Obj_1 and st_2 is a statistic on the object Obj_2. If

$$\sum_{\sigma \in Obj_1} q^{st_1(\sigma)} = \sum_{\sigma \in Obj_2} q^{st_2(\sigma)},$$

we say that the statistic st_1 over Obj_1 is equidistributed with the statistic st_2 over Obj_2.

A statistic on S_n is said to be Eulerian if it is equidistributed with the statistic des on S_n. While a statistic on S_n is said to be Mahonian if it is equidistributed with the statistic inv on S_n. It is well-known that

$$\sum_{\pi \in S_n} q^{inv(\pi)} = \sum_{\pi \in S_n} q^{maj(\pi)} = [n]_q!,$$

where $[n]_q = 1 + q + \cdots + q^{n-1}$ and $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$. Thus, the major index maj is a Mahonian statistic. A pair of statistics on S_n is said to be Euler-Mahonian if it is equidistributed with the joint distribution of the descent number and the major index.

In [1], Babson and Steingrímsson introduced generalized permutation patterns, where they allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. Let $\mathcal{A}$ be the alphabet $\{a, b, c, \ldots\}$ with the usual ordering. We write patterns as words in $\mathcal{A}$, where two adjacent letters may or may not be separated by a dash. Two adjacent letters without a dash in a pattern indicates that the corresponding letters in the permutation must be adjacent. Given a generalized pattern τ and a permutation π, we say a subsequence of π is an occurrence (or instance) of τ in π if it is order-isomorphic to τ and satisfies the above dash conditions. Let $(\tau)\pi$ denote the number of occurrences of τ in π. Here, we see (τ) as a generalized pattern function. For example, an occurrence of the generalized pattern $b\text{-}ca$ in a permutation $\pi = \pi_1\pi_2 \cdots \pi_n$ is a subsequence $\pi_i\pi_j\pi_{j+1}$ such that $i < j$ and $\pi_{j+1} < \pi_i < \pi_j$. For $\pi = 4753162$, we have $(b\text{-}ca)\pi = 4$.

Further, Babson and Steingrímsson [1] showed that almost all of the Mahonian permutation statistics in the literature can be written as linear combinations of generalized patterns. We list some of them below.

$$maj = (a\text{-}cb) + (b\text{-}ca) + (c\text{-}ba) + (ba),$$

$$stat = (ac\text{-}b) + (ba\text{-}c) + (cb\text{-}a) + (ba).$$

They conjectured that the statistic $(des, stat)$ is Euler-Mahonian.

Conjecture 1. The distribution of the bistatistic $(des, stat)$ is equal to that of (des, maj).

In 2001, D. Foata and D. Zeilberger [4] gave a proof of this conjecture using q-enumeration and generating functions and an almost completely automated proof via Maple packages ROTA and PERCY.

Given a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$, let $F(\pi) = \pi_1$ be the first letter of π and

$$adj(\pi) = |\{i \colon 1 \leqslant i \leqslant n \text{ and } \pi_i - \pi_{i+1} = 1\}|,$$

where $\pi_{n+1} = 0$. Burstein [2] provided a bijective proof of the following refinement of Conjecture 1 as follows.

Theorem 2. *The statistics $(adj, des, F, maj, stat)$ and $(adj, des, F, stat, maj)$ are equidistributed over S_n for all n.*

In this paper, we will give a new bijective proof of Conjecture 1, which does not preserve the statistic adj.

2 A new bijective proof of Conjecture 1

In this section, we recall a particular bijection φ on S_n that maps the inversion number to the major index, which is due to Carlitz [3] and stated more clearly in [6] and [7]. Based on this, we give an analogous bijection which proves Conjecture 1.

To give a description of φ, we first recall two labeling schemes for permutations. It involves accounting for the effects of inserting a new largest element into a permutation, and so it is known as the insertion method.

Given a permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{n-1}$ in S_{n-1}. To obtain a permutation $\pi \in S_n$, we can insert n in n spaces, namely, immediately before σ_1 or immediately after σ_i for $1 \leqslant i \leqslant n$. In order to keep track of how the insertion of n affects the inversion number and the major index, we may define two labelings of the n inserting spaces.

The inv-labeling of σ is given by numbering the spaces from right to left with $0, 1, \ldots, n-1$. The maj-labeling of σ is obtained by labeling the space after σ_{n-1} with 0, labeling the descents from right to left with $1, 2, \ldots, des(\sigma)$ and labeling the remaining spaces from left to right with $des(\sigma) + 1, \ldots, n$. As an example, let $\sigma = 13287546$, the inv-labeling of σ is

$$_8 1 _7 3 _6 2 _5 8 _4 7 _3 5 _2 4 _1 6 _0,$$

while the maj-labeling of σ is given by

$$_5 1 _6 3 _4 2 _7 8 _3 7 _2 5 _1 4 _8 6 _0.$$

For $n \geqslant 2$, we define the map

$$\phi_{inv,n} \colon \{0, 1, \ldots, n-1\} \times S_{n-1} \to S_n$$

i	$\pi^{(i-1)}$	m_i
8	$_4 1_5 3_3 2_6 7_2 5_1 4_7 6_0$	6
7	$_3 1_4 3_2 2_5 5_1 4_6 6_0$	5
6	$_3 1_4 3_2 2_5 5_1 4_0$	0
5	$_2 1_3 3_1 2_4 4_0$	4
4	$_2 1_3 3_1 2_0$	0
3	$_1 1_2 2_0$	2
2	$_1 1_0$	0

Table 1: The computation of the major index table of 13287546.

by setting $\phi_{inv,n}(i,\sigma)$ to be the permutation obtained by inserting n in the space labeled i in the inv-labeling of σ. By changing the inv-labeling to maj-labeling, we obtain the map $\phi_{maj,n}$. As an example,

$$\phi_{inv,9}(3, 13287546) = 132879546 \quad \text{and} \quad \phi_{maj,9}(3, 13287546) = 132897546.$$

For maps $\phi_{inv,n}$ and $\phi_{maj,n}$, we have the following two lemmas.

Lemma 3. *For $\sigma \in S_{n-1}$ and $i \in \{0, 1, \ldots, n-1\}$, we have*

$$inv(\phi_{inv,n}(i,\sigma)) = inv(\sigma) + i.$$

Lemma 4. *For $\sigma \in S_{n-1}$ and $i \in \{0, 1, \ldots, n-1\}$, we have*

$$maj(\phi_{maj,n}(i,\sigma)) = maj(\sigma) + i.$$

Lemma 3 is easy to verified. For a detailed proof of Lemma 4, see [5].

Given a permutation $\pi \in S_n$, let $\pi^{(i)}$ be the restriction of π to the letters $1, 2, \ldots, i$ for $1 \leqslant i \leqslant n$. For $2 \leqslant i \leqslant n$, let

$$(c_i, \pi^{(i-1)}) = \phi_{inv,i}^{-1}(\pi^{(i)}),$$

$$(m_i, \pi^{(i-1)}) = \phi_{maj,i}^{-1}(\pi^{(i)}).$$

By setting $c_1 = 0$ and $m_1 = 0$, we obtain two sequences $c_1 c_2 \cdots c_n$ and $m_1 m_2 \cdots m_n$ in E_n, where

$$E_n = \{w = w_1 \cdots w_n | w_i \in [0, i-1], 1 \leqslant i \leqslant n\}.$$

Define $\gamma(\pi) = c_1 c_2 \cdots c_n$ and $\mu(\pi) = m_1 m_2 \cdots m_n$. It is not hard to check that both γ and μ are bijections. The sequence $c_1 c_2 \cdots c_n$ is called the inversion table of π, while $m_1 m_2 \cdots m_n$ is called the major index table of π. Moreover, we have $\sum_{i=1}^{n} c_i = inv(\pi)$ and $\sum_{i=1}^{n} m_i = maj(\pi)$. As an example, let $\pi = 13287546$, then $\pi^{(8)} = \pi$ and $\mu(\pi) = 00204056$ as computed in Table 1.

Now, we can define the bijection φ that maps the inversion number to the major index by letting $\varphi = \mu^{-1}\gamma$. Clearly, φ is a bijection which proves that statistics inv and maj are equidistributed over S_n.

In the following of this section, we will construct a bijection ρ to prove Conjecture 1, which is, to some extent, an analogue of the above bijection φ.

First, we define a stat-labeling of $\sigma \in S_n$. Label the descents of σ and the space after σ_n by $0, 1, \ldots, des(\sigma)$ from left to right. Label the space before σ_0 by $des(\sigma) + 1$. The ascents of σ are labeled from right to left by $des(\sigma) + 2, \ldots, n$. As an example, for $\sigma = 13287546$, we have the stat-labeling of σ as follows

$$_5 1 _8 3 _0 2 _7 8 _1 7 _2 5 _3 4 _6 6 _4.$$

Based on the stat-labeling, we define the map $\phi_{stat,n}$ for $n \geqslant 2$. For a permutation $\sigma = \sigma_1 \sigma_2 \ldots \sigma_{n-1}$ and $0 \leqslant i \leqslant n-1$, define $\phi_{stat,n}(i, \sigma)$ to be the permutation obtained by inserting n in the position labeled i in the stat-labeling of σ. For instance, $\phi_{stat,9}(7, 13287546) = 132987546$.

It should be noted that unlike the properties of $\phi_{inv,n}$ and $\phi_{maj,n}$ stated in Lemma 3 and Lemma 4, we deduce that $\phi_{stat,n}(i, \sigma)) \neq stat(\sigma) + i$ for $i = des(\sigma) + 1$. For the map $\phi_{stat,n}$, we have the following property.

Lemma 5. *For $\sigma \in S_{n-1}$ and $i \in \{0, 1, \ldots, des(\sigma), des(\sigma) + 2, \ldots, n - 1\}$, we have*

$$stat(\phi_{stat,n}(i, \sigma)) = stat(\sigma) + i.$$

Proof. First, we recall that $stat = (ac\text{-}b) + (ba\text{-}c) + (cb\text{-}a) + (ba)$. To prove this lemma, we have to consider the changes of the statistic $stat$ brought by inserting n into σ. Assume that $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{n-1}$, there are three cases for us to consider.

- Case 1 : n is inserted into the space after σ_{n-1}.

  Write $\pi = \sigma n$. Clearly, the insertion of n does not bring new $ac\text{-}b$, $cb\text{-}a$ and ba patterns. While n can form new $ba\text{-}c$ patterns of π with the ba patterns of σ. It follows that $stat(\pi) - stat(\sigma) = (ba)\sigma = des(\sigma)$. Notice that the label of the space after σ_{n-1} is $des(\sigma)$. Hence, in this case the lemma holds.

- Case 2 : n is inserted into a descent.

  Suppose that τ is the permutation obtained by inserting n to the position i and $\sigma_i > \sigma_{i+1}$. Moreover, let this descent be the k-th descent from left to right. We claim that $stat(\tau) - stat(\sigma) = k - 1$.

  The insertion of n forms some new $ac\text{-}b$ patterns, and the number of these new patterns is $|\{j, \ j > i \text{ and } \sigma_j > \sigma_i\}|$. Moreover, n forms $k - 1$ new $ba\text{-}c$ patterns with the former $k - 1$ descents, while it destroys the $ba\text{-}c$ patterns of σ by the number $|\{j, \ j > i \text{ and } \sigma_j > \sigma_i\}|$. It is easy to verify the functions $(cb\text{-}a)$ and (ba) do not change. Hence, we conclude that $stat(\tau) - stat(\sigma) = k - 1$. The claim is verified. Notice that the label of the k-th descent from left to right is also $k - 1$. It follows that the lemma holds for this case.

- Case 3 : n is inserted into an ascent.

i	$\pi^{(i-1)}$	s_i
8	$_3 5_0 2_7 7_1 1_6 3_5 4_4 6_2$	6
7	$_3 5_0 2_1 1_6 3_5 4_4 6_2$	1
6	$_3 5_0 2_1 1_5 3_4 4_2$	2
5	$_2 2_0 1_4 3_3 4_1$	2
4	$_2 2_0 1_3 3_1$	1
3	$_2 2_0 1_1$	1
2	$_1 1_0$	1

Table 2: The computation of the stat table of 52718346.

Suppose that p is the permutation obtained by inserting n into the position i, where $\sigma_i < \sigma_{i+1}$. Moreover, we assume that there are k descents to the left of position i. We claim that $stat(p) - stat(\sigma) = k + n - i + 1$.

Now we proceed to prove this claim. The insertion of n to σ brings new ac-b patterns, the number of which is $|\{j,\ j \geqslant i+1 \text{ and } \sigma_j \geqslant \sigma_{i+1}\}|$. Moreover, the insertion of n brings k new ba-c patterns with the former k descents. Also, $n\sigma_{i+1}\sigma_l$, where $l > i+1$ and $\sigma_l < \sigma_{i+1}$, forms a new cb-a pattern of p. Notice that $(ba)p - (ba)\sigma = 1$. By combining all above, we see that $stat(\tau) - stat(\sigma) = k + n - i + 1$. The claim is verified. By the stat-labeling of σ, we see that the label of this position is also $k + n - i + 1$. Hence, in this case the lemma holds.

By combining the three cases above, we compete the proof. □

Based on the stat-labeling, we define the stat table of a permutation. Given $\pi \in S_n$, for $2 \leqslant i \leqslant n$, let

$$(s_i, \pi^{(i-1)}) = \phi_{stat,i}^{-1}(\pi^{(i)}).$$

Set $s_1 = 0$ and $\nu(\pi) = s_1 s_2 \cdots s_n$. It is easily checked that ν is a bijection from S_n to E_n. As an example, $\nu(52718346) = 01112216$, which is computed in Table 2.

Now we can give the definition of the map ρ which proves Conjecture 1. Given $\pi \in S_n$, let $\sigma = \rho(\pi)$, where σ can be constructed as follows. Assume that $F(\pi) = k$ and $\pi^{(k)} = kp_2 \cdots p_k$. Then, let $\sigma^{(k)} = k(k-p_k)(k-p_{k-1}) \cdots (k-p_2)$. Assume that $s = s_1 s_2 \cdots s_n = \nu(\pi)$. For $k+1 \leqslant i \leqslant n$, let $\sigma^{(i)} = \phi_{maj,i}(s_i, \sigma^{(i-1)})$. Clearly, $\sigma = \sigma^{(n)}$ can be constructed by the procedure above. As an example, we compute $\rho(\pi)$, where $\pi = 52718346$. It is straightforward to see that $k = 5$ and $\pi^{(5)} = 52134$. By the computation in Table 3, we see that $\rho(52718346) = 56128473$. Notice that in this example, the descent number and the first letter of both of the preimage and image of ρ are the same. In fact, these properties always holds.

Lemma 6. *For $\pi \in S_n$, we have $F(\pi) = F(\rho(\pi))$ and $des(\pi) = des(\rho(\pi))$.*

Proof. Suppose that $\sigma = \rho(\pi)$ and $k = F(\pi)$. We proceed to show that $F(\sigma^{(i)}) = F(\pi^{(i)}) = k$ and $des(\sigma^{(i)}) = des(\pi^{(i)})$ for $k \leqslant i \leqslant n$ by induction. Let $\pi^{(k)} = kp_2 \cdots p_k$,

i	s_i	$\sigma^{(i)}$
5	2	$_3 5_2 1_4 2_5 4_1 3_0$
6	2	$_3 5_4 6_2 1_5 2_6 4_1 3_0$
7	1	$_3 5_4 6_2 1_5 2_6 4_7 7_1 3_0$
8	6	56128473

Table 3: The computation of $\rho(52718346)$.

then we have $\sigma^{(k)} = k(k - p_k)(k - p_{k-1}) \cdots (k - p_2)$. It is routine to check that $F(\sigma^{(k)}) = F(\pi^{(k)}) = k$ and $des(\sigma^{(k)}) = des(\pi^{(k)})$, hence, we omit the details here.

Now assume that $des(\sigma^{(l)}) = des(\pi^{(l)}) = d$ and $F(\sigma^{(l)}) = F(\pi^{(l)}) = k$ for $k \leqslant l \leqslant n-1$. We proceed to show that $des(\sigma^{(l+1)}) = des(\pi^{(l+1)})$ and $F(\sigma^{(l+1)}) = F(\pi^{(l+1)}) = k$.

Let $s = s_1 s_2 \cdots s_n = \nu(\pi)$. By the constructions of ν and ρ, we may see that $\pi^{(l+1)} = \phi_{stat,l+1}(s_{l+1}, \pi^{(l)})$ and $\sigma^{(l+1)} = \phi_{maj,l+1}(s_{l+1}, \sigma^{(l)})$. Notice that both in the maj-labeling of $\sigma^{(l)}$ and the stat-labeling of $\pi^{(l)}$, the descents and the space after the last letter are labeled by $\{0, 1, \ldots, d\}$, the space before the first element is labeled by $d+1$, and the ascents are labeled by $\{d+2, \ldots, l\}$. Since $F(\pi^{(l)}) = k = F(\pi)$, it is easy to see that $s_{l+1} \neq d+1$. It follows that $F(\sigma^{(l+1)}) = F(\pi^{(l+1)}) = k$.

If $s_{l+1} < d+1$, then $l+1$ is inserted into the descents or the space after the last element of $\pi^{(l)}$ and $\sigma^{(l)}$. Hence, we deduce that $des(\sigma^{(l+1)}) = des(\pi^{(l+1)}) = d$. If $s_{l+1} > d+1$, then $l+1$ is inserted into the ascents of $\pi^{(l)}$ and $\sigma^{(l)}$. Hence, we deduce that $des(\sigma^{(l+1)}) = des(\pi^{(l+1)}) = d+1$. Combining the two cases above, we have $des(\sigma^{(l+1)}) = des(\pi^{(l+1)})$. Notice that $\sigma^{(n)} = \sigma$ and $\pi^{(n)} = \pi$, we complete the proof. $\qquad \square$

Base on the construction of ρ and Lemma 6, we have the following theorem.

Theorem 7. *The map ρ is an involution on S_n.*

Proof. Given a permutation $\pi \in S_n$, it suffices for us to show that $\rho^2(\pi) = \pi$. That is, writing $\sigma = \rho(\pi)$, we need to show that $\rho(\sigma) = \pi$.

Let $F(\pi) = k$, then by Lemma 6, we know that $k = F(\pi) = F(\sigma)$. Write $\pi^{(k)} = kp_2 \cdots p_k$, then we have $\sigma^{(k)} = k(k - p_k) \cdots (k - p_2)$. Assume that $\nu(\pi) = s = s_1 s_2 \cdots s_n$ and $\mu(\sigma) = m_1 m_2 \cdots m_n$. By the construction of ρ, we have $\sigma^{(i)} = \phi_{maj,i}(s_i, \sigma^{(i-1)})$ for $k+1 \leqslant i \leqslant n$. It follows that $m_i = s_i$ for $k+1 \leqslant i \leqslant n$.

Suppose that $\alpha = \rho(\sigma)$, in the following, we proceed to show that $\alpha^{(i)} = \pi^{(i)}$ for $k \leqslant i \leqslant n$ by induction. By the definition of ρ, we have $\alpha^{(k)} = kp_2 \cdots p_k = \pi^{(k)}$. Assume that $\alpha^{(i)} = \pi^{(i)}$ holds for $k \leqslant i \leqslant n-1$, we aim to show that $\alpha^{(i+1)} = \pi^{(i+1)}$. To achieve this, we need to mention the following property of the maj-labeling and the stat-labeling of a single permutation.

For a permutation $p \in S_n$, assume the maj-labeling of p is $f_0 f_1 \cdots f_n$, while the stat-

labeling of p is $h_0 h_1 \cdots h_n$. Then it is easily checked that

$$f_i + h_i = \begin{cases} des(p), & \text{if } i \text{ is a descent of } p \text{ or } i = n, \\ n + des(p) + 2, & \text{if } i \text{ is an ascent of } p, \\ 2des(p) + 2, & \text{if } i = 0. \end{cases} \tag{1}$$

Let $\nu(\sigma) = l_1 l_2 \cdots l_n$ and $d = des(\alpha^{(i)})$. Then, we have

$$\alpha^{(i+1)} = \phi_{maj,i+1}(l_{i+1}, \alpha^{(i)}) = \begin{cases} \phi_{stat,i+1}(d - l_{i+1}, \alpha^{(i)}), & \text{if } 0 \leqslant l_{i+1} \leqslant d, \\ \phi_{stat,i+1}(n + d + 2 - l_{i+1}, \alpha^{(i)}), & \text{if } d + 2 \leqslant l_{i+1} \leqslant n. \end{cases}$$

By the proof of Lemma 6, we see that $des(\sigma^{(i)}) = des(\alpha^{(i)}) = d$. Recall that $\nu(\sigma) = l_1 l_2 \cdots l_n$ and $\mu(\sigma) = m_1 m_2 \cdots m_n$. Hence, it follows from (1) that

$$\begin{aligned} \alpha^{(i+1)} &= \phi_{stat,i+1}(m_{i+1}, \alpha^{(i)}) \\ &= \phi_{stat,i+1}(s_{i+1}, \pi^{(i)}) \\ &= \pi^{(i+1)}. \end{aligned}$$

Notice that $\alpha = \alpha^{(n)}$ and $\pi = \pi^{(n)}$. Hence, we have $\pi = \rho(\sigma)$, namely, $\rho^2(\pi) = \pi$. This completes the proof. $\qquad\square$

Indeed, the involution ρ also preserves some addtional statistics, which is stated in the following proposition.

Proposition 8. *For any $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$, we have $maj(\rho(\pi)) = stat(\pi)$ and $stat(\rho(\pi)) = maj(\pi)$.*

Proof. Write $\sigma = \rho(\pi)$. Let $k = F(\pi)$ and $\pi^{(k)} = kp_2 \cdots p_k$. Then we have $\sigma^{(k)} = k(k - p_k) \cdots (k - p_2)$. In the following, we will show that $maj(\sigma^{(i)}) = stat(\pi^{(i)})$ for $k \leqslant i \leqslant n$ by induction.

First, we show that $maj(\sigma^{(k)}) = stat(\pi^{(k)})$. By the proof of Lemma 6, we have $des(\pi^{(k)}) = des(\sigma^{(k)})$. Hence $(ba)\pi^{(k)} = (ba)\sigma^{(k)}$. It suffices for us to show that

$$(ac - b)\pi^{(k)} + (ba - c)\pi^{(k)} + (cb - a)\pi^{(k)} = (a - cb)\sigma^{(k)} + (b - ca)\sigma^{(k)} + (c - ba)\sigma^{(k)} \tag{2}$$

Given a pattern $p = p_1 p_2 \cdots p_k$, by putting a line under p_1 (resp. p_k), we mean that an instance of p must begin (resp. end) with the leftmost (resp. rightmost) letter of the permutation. By putting a dot under p_1, we mean that an instance of p must not begin with the leftmost letter. For instance, $(\underline{c}b\text{-}a)$ is a function which maps a permutation, say $\tau = \tau_1 \tau_2 \cdots \tau_n$, to $|\{i, \tau_1 \tau_2 \tau_i \text{ forms a 321 pattern of } \tau\}|$.

By the definition of ρ, we know that

$$\begin{aligned} &(ac - b)\pi^{(k)} + (ba - c)\pi^{(k)} + (cb - a)\pi^{(k)} \\ =&(ac - b)\pi^{(k)} + (ba - c)\pi^{(k)} + (\underline{c}b - a)\pi^{(k)} + (\dot{c}b - a)\pi^{(k)} \end{aligned}$$

$$= (b - ac)\sigma^{(k)} + (a - cb)\sigma^{(k)} + (\underline{c} - b - \underline{a})\sigma^{(k)} + (\underline{c} - ba)\sigma^{(k)}$$

Hence, to prove (2), we need to prove that for any permutation $p \in S_k$ with $p_1 = k$,

$$(\underline{c} - ba)p + (b - ca)p = (\underline{c} - b - \underline{a})p + (b - ac)p. \tag{3}$$

We define sets $A(p), C(p)$ and multisets $B(p), D(p)$ as follows.

$$A(p) = \{p_i \colon p_1 p_i p_{i+1} \text{ forms a 321 pattern of } p\},$$
$$B(p) = \{p_i \colon p_i p_j p_{j+1} \text{ forms a 231 pattern of } p\},$$
$$C(p) = \{p_i \colon p_1 p_i p_m \text{ forms a 321 pattern of } p\},$$
$$D(p) = \{p_i \colon p_i p_j p_{j+1} \text{ forms a 213 pattern of } p\}.$$

To prove (3), it is enough to show that $A \cup B = C \cup D$, where the union operator is a multiset union. First, we show that $C \cup D \subseteq A \cup B$.

Let $p_k = a$, then we know that $C(p) = \{a + 1, a + 2, \ldots, k - 1\}$. If $p_i \in C(p)$ is a descent top, it is easy to see that $p_i \in A(p)$. If $p_i \in C(p)$ is an ascent bottom, we claim that $p_i \in B(p)$. This claim will be proved together with case 4 in the following.

Given an element p_i in $D(p)$, if the multiplicity of p_i is x, there exists a set

$$\{j_1, j_1 + 1, j_2, j_2 + 1, \ldots, j_x, j_x + 1\},$$

which is ordered by increasing order, satisfying that

$$p_i p_{j_1} p_{j_1+1}, \ \ p_i p_{j_2} p_{j_2+1}, \ \ \ldots, \ \ p_i p_{j_x} p_{j_x+1}$$

are instances of 213 pattern. We claim that there exists $j_1 + 1 \leqslant r_1 < j_2$ such that $p_i p_{r_1} p_{r_1+1}$ forms a 231 pattern.

Choose the smallest g_1 such that $j_1 + 1 \leqslant g_1 < j_2$ and g_1 is a descent. If $p_{g_1+1} < p_i$, then $p_i p_{g_1} p_{g_1+1}$ forms a 231 pattern, the claim is verified. Otherwise, we seek the smallest $g_1 + 1 \leqslant g_2 < j_2$ such that g_2 is a descent. If $p_{g_2+1} < p_i$, the claim is verified. If not, we repeat the above process. Since $p_i > p_{j_2}$, the process must be terminated. Hence, the claim is verified. By a similar means, we deduce that there exists $j_l + 1 \leqslant r_l < j_{l+1}$ where $1 \leqslant l \leqslant x - 1$ such that $p_i p_{r_l} p_{r_l+1}$ forms a 231 pattern. The claim is verified.

To analyze the element p_i in $D(p)$, we consider four cases.

- p_i is a descent top and $1 < p_i \leqslant a - 1$.

  Suppose that the multiplicity of p_i of this type in $D(p)$ is x. By the above statement, we see that $p_i p_{r_l} p_{r_l+1}$ forms a 231 pattern, where $j_l + 1 \leqslant r_l < j_{l+1}$ and $1 \leqslant l \leqslant x - 1$. Thus, we deduce that there are $x - 1$ p_is in $B(p)$. Notice that there is one p_i left in $B(p)$. Clearly, we can set this p_i to be the element of $A(p)$ consisting of p_1, p_i and p_{i+1}.

- p_i is an ascent bottom and $1 \leqslant p_i \leqslant a - 1$.

Suppose that the multiplicity of p_i of this type in $D(p)$ is x. Similarly, we know that $p_i p_{r_l} p_{r_l+1}$ forms a 231 pattern, where $j_l + 1 \leqslant r_l < j_{l+1}$ and $1 \leqslant l \leqslant x - 1$. Since $p_i < p_{i+1}$, we have $j_1 > i + 1$. Based on this, it can be easily seen that there exists r_0 such that $i + 1 \leqslant r_0 < j_1$ and $p_i p_{r_0} p_{r_0+1}$ forms a 231 pattern. Hence, we deduce that in this case there are x p_is in $B(p)$.

- p_i is a descent top and $a + 1 \leqslant p_i \leqslant k - 1$.

  Suppose that the multiplicity of p_i of this type in $D(p)$ is x. Similarly, we deduce that $p_i p_{r_l} p_{r_l+1}$ forms a 231 pattern, where $j_l + 1 \leqslant r_l < j_{l+1}$ and $1 \leqslant l \leqslant x - 1$. What's more, it follows from $p_i > a$ that there exists $j_x \leqslant r_x < m$ such that $p_i p_{r_x} a$ forms a 231 pattern. Hence, we deduce that p_i in $B(p)$ and its multiplicity is x.

- p_i is an ascent bottom and $a + 1 \leqslant p_i \leqslant k - 1$.

  Suppose that the multiplicity of p_i of this case in $D(p)$ is x. Notice that p_i is also an element of $C(p)$ with multiplicity equals 1. Hence, in this case, we have to prove that there are $x + 1$ p_is in $B(p)$.

  Similarly with the above cases, we deduce that $p_i p_{r_l} p_{r_l+1}$ forms a 231 pattern, where $j_l + 1 \leqslant r_l < j_{l+1}$ and $1 \leqslant l \leqslant x - 1$. Since $p_i > a$, there exists $j_x \leqslant r_x < m$ such that $p_i p_{r_x} a$ forms a 231 pattern. By $p_i < p_{i+1}$, we have $j_1 > i + 1$. Based on this, it can be easy seen that there exists r_0 such that $i + 1 \leqslant r_0 < j_1$ and $p_i p_{r_0} p_{r_0+1}$ forms a 231 pattern. Hence, we deduce that p_i in $B(p)$ and its multiplicity is $x + 1$.

Combining all above, we deduce that $C \cup D \subseteq A \cup B$. By a similar analysis, we can prove that $A \cup B \subseteq C \cup D$. We omit it here. As an example, if $p = 978452613$, we have $A(p) = \{5, 6, 8\}$, $B(p) = \{2, 4, 4, 5, 7\}$, $C(p) = \{4, 5, 6, 7, 8\}$ and $D(p) = \{2, 4, 5\}$. It can be verified that $A \cup B = C \cup D$. This proves that $maj(\sigma^{(k)}) = stat(\pi^{(k)})$.

Now assume that $maj(\sigma^{(i)}) = stat(\pi^{(i)})$ for $k \leqslant i \leqslant n - 1$, we proceed to show that $maj(\sigma^{(i+1)}) = stat(\pi^{(i+1)})$. Write $\nu(\pi) = s_1 s_2 \cdots s_n$. Then, by Lemma 4, Lemma 5 and the construction of ρ, we have

$$maj(\sigma^{(i+1)}) = s_{i+1} + maj(\sigma^{(i)})$$
$$= s_{i+1} + stat(\pi^{(i)})$$
$$= stat(\pi^{(i+1)}).$$

This proves $maj(\sigma^{(i)}) = stat(\pi^{(i)})$ for $k \leqslant i \leqslant n$. Notice that $\pi = \pi^{(n)}$ and $\sigma = \sigma^{(n)}$. We deduce that $maj(\sigma) = maj(\rho(\pi)) = stat(\pi)$. By Theorem 7, we see that ρ is an involution. This implies that $stat(\rho(\pi)) = maj(\pi)$. This completes the proof. $\square$

Combining Lemma 6, Theorem 7 and Proposition 8, we give a proof of Conjecture 1.

It should be mentioned that Burstein [2] provided a direct bijective proof of a refinement of Conjecture 1. The bijection χ is given as follows. Given a permutation $\pi \in S_n$ with $F(\pi) = k$. Let $\pi' = \chi(\pi)$ with $\pi'(1) = k$ and

$$\pi'(i) = \begin{cases} k - \pi(n + 2 - i), & \text{if } \pi(n + 2 - i) < k, \\ n + k + 1 - \pi(n + 2 - i), & \text{if if } \pi(n + 2 - i) > k. \end{cases} \tag{4}$$

In addition to preserving the statistics des and F, the bijection χ also preserves the statistic adj, while our bijection does not. As an example, set $\pi = 543617982$, then $\rho(\pi) = \sigma = 539784621$ and $\chi(\pi) = \pi' = 537684921$. It can be checked that $adj(\pi) = adj(\pi')$, while $adj(\pi) \neq adj(\sigma)$. Moreover, it is easily seen that for $\pi \in S_n$ with $F(\pi) = n$, we have $\rho(\pi) = \chi(\pi) = \sigma$. We note that in this case both Burstein and us have to prove $maj(\sigma) = stat(\pi)$. Different form our proof in Proposition 8, Burstein gave the following two relations, which implies that $maj(\sigma) = stat(\pi)$.

$$maj(\pi) + stat(\pi) = (n+1)des(\pi) - (F(\pi) - 1),$$

$$maj(\pi) + maj(\sigma) = (n+1)des(\pi) - (F(\pi) - 1).$$

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