Neighborhood reconstruction and cancellation of graphs

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Submitted: Dec 11, 2016; Accepted: Mar 29, 2017; Published: Apr 13, 2017
Mathematics Subject Classifications: 05C60, 05C76

Abstract

We connect two seemingly unrelated problems in graph theory.

Any graph G has a *neighborhood multiset* $\mathscr{N}(G) = \{N(x) \mid x \in V(G)\}$ whose elements are precisely the open vertex-neighborhoods of G. In general there exist non-isomorphic graphs G and H for which $\mathscr{N}(G) = \mathscr{N}(H)$. The *neighborhood reconstruction problem* asks the conditions under which G is uniquely reconstructible from its neighborhood multiset, that is, the conditions under which $\mathscr{N}(G) = \mathscr{N}(H)$ implies $G \cong H$. Such a graph is said to be *neighborhood-reconstructible.*

The *cancellation problem* for the direct product of graphs seeks the conditions under which $G \times K \cong H \times K$ implies $G \cong H$. Lovász proved that this is indeed the case if K is not bipartite. A second instance of the cancellation problem asks for conditions on G that assure $G \times K \cong H \times K$ implies $G \cong H$ for any bipartite K with $E(K) \neq \emptyset$. A graph G for which this is true is called a *cancellation graph.*

We prove that the neighborhood-reconstructible graphs are precisely the cancellation graphs. We also present some new results on cancellation graphs, which have corresponding implications for neighborhood reconstruction. We are particularly interested in the (yet-unsolved) problem of finding a simple structural characterization of cancellation graphs (equivalently, neighborhood-reconstructible graphs).

Keywords: graph neighborhood-reconstruction; graph cancellation; graph direct product

1 Preliminaries

For us, a graph G is a symmetric relation $E(G)$ on a finite *vertex set* $V(G)$. An an *edge* $(x, y) \in E(G)$ is denoted xy. A *loop* is a reflexive edge xx. The *open neighborhood* of a vertex $x \in V(G)$ is the set $N_G(x) = \{y \in V(G) \mid xy \in E(G)\}$, which we may denote as $N(x)$ when this is unambiguous. Notice that $x \in N_G(x)$ if and only if $xx \in E(G)$, that is, there is a loop at x.

In this paper we are careful to distinguish between graph equality and isomorphism. The statement $G = H$ means $V(G) = V(H)$ and $E(G) = E(H)$. By $G \cong H$ we mean that G and H are isomorphic. An isomorphism from G to itself is called an *automorphism* of G. The group of all automorphisms of G is denoted $\mathrm{Aut}(G)$. An automorphism of order 2 is called *involution*. A *homomorphism* $G \to H$ is a map $\varphi : V(G) \to V(H)$ for which $xy \in E(G)$ implies $\varphi(x)\varphi(y) \in E(H)$.

The *direct product* of graphs G and H is the graph $G \times H$ with vertices $V(G) \times V(H)$ and edges $E(G \times H) = \{(x, x')(y, y') \mid xy \in E(G) \text{ and } x'y' \in E(H)\}$. See Figure 1.

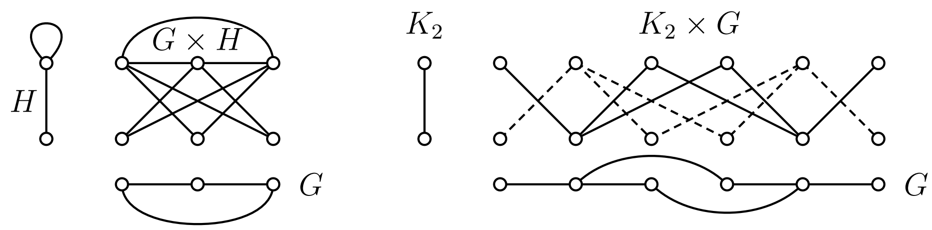

Figure 1: Examples of direct products

We assume our reader is at least somewhat familiar with direct products. See [3] for a survey. The direct product is associative, commutative, and distributive in the sense that $G \times (H + K) = G \times H + G \times K$, where $+$ represents disjoint union. Weichsel's theorem [3, Theorem 5.9] states that $G \times H$ is connected if and only if both G and H are connected and at least one of them has an odd cycle. If G and H are both connected and bipartite, then $G \times H$ has exactly two components. In particular, if G is bipartite, then $G \times K_2 = G + G$, as illustrated on the right of Figure 1.

2 Neighborhood reconstruction

Any graph G has an associated *neighborhood multiset* $\mathscr{N}(G) = \{N_G(x) \mid x \in V(G)\}$ whose elements are precisely the open neighborhoods of G. It is possible that $G \ncong H$ but nonetheless $\mathscr{N}(G) = \mathscr{N}(H)$, as illustrated in Figure 2.

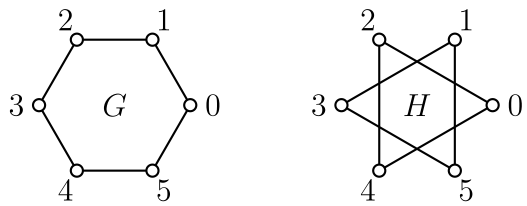

Figure 2: Here $G \ncong H$ but $\mathscr{N}(G) = \mathscr{N}(H) = \{\{0,2\}, \{2,4\}, \{0,4\}, \{1,3\}, \{3,5\}, \{1,5\}\}$

Two types of questions have been asked about neighborhood multisets. Given a set V and a multiset $\mathscr{N} = \{N_1, N_2, \ldots, N_n\}$ of subsets of V, we may ask if there is a graph G on V for which $\mathscr{N}(G) = \mathscr{N}$. Let us call this the *neighborhood realizability problem.* Aigner and Triesch [1] attribute this problem to Sós, and show that it is NP-complete.

On the other hand, the *neighborhood reconstruction problem* asks if a given graph G can be reconstructed from information in $\mathscr{N}(G)$, that is, whether $\mathscr{N}(G) = \mathscr{N}(H)$ implies $G \cong H$. If this is the case we say that G is *neighborhood reconstructible.* Figure 2 shows that the hexagon is not neighborhood reconstructible.

Aigner and Triesch [1] note that the problem of deciding whether a graph is neighborhood reconstructible is NP-complete.

We now adapt their approach to describe for given G all those graphs H for which $\mathscr{N}(G) = \mathscr{N}(H)$. Given a permutation α of $V(G)$ we define G^α to be the digraph on $V(G)$ with an arc directed from x to $\alpha(y)$ whenever $xy \in E(G)$. (In general, we denote an arc directed from u to v as an ordered list uv, with the understanding that it points from the left vertex u to the right vertex v. Thus the arc set of G^α is $E(G^\alpha) = \{x\,\alpha(y) \mid xy \in E(G)\}$.) Even though G is a graph (i.e. the edge relation is symmetric), G^α may not be a graph. In fact, G^α is a graph if and only if α has the property that $xy \in E(G) \iff \alpha(x)\alpha^{-1}(y) \in E(G)$. Indeed, if G^α is a graph, then

$$\begin{aligned} xy \in E(G) \iff yx \in E(G) &\iff y\alpha(x) \in E(G^\alpha) \\ &\iff \alpha(x)y \in E(G^\alpha) \iff \alpha(x)\alpha^{-1}(y) \in E(G). \end{aligned}$$

Conversely, if α obeys $xy \in E(G) \iff \alpha(x)\alpha^{-1}(y) \in E(G)$, then G^α is a graph because

$$\begin{aligned} xy \in E(G^\alpha) \iff x\alpha^{-1}(y) \in E(G) &\iff \alpha^{-1}(y)x \in E(G) \\ &\iff y\alpha^{-1}(x) \in E(G) \iff yx \in E(G^\alpha). \end{aligned}$$

A map α with the above properties is called an *anti-automorphism* in [3] and [4].

To summarize, an *anti-automorphism* of a graph G is a bijection $\alpha : V(G) \to V(G)$ for which $xy \in E(G)$ if and only if $\alpha(x)\alpha^{-1}(y) \in E(G)$. Given an anti-automorphism α of $V(G)$ we have a graph G^α on the same vertex set as G, but with

$$E(G^\alpha) = \{x\alpha(y) \mid xy \in E(G)\}.$$

Notice that this means $N_G(y) = N_{G^\alpha}\big(\alpha(y)\big)$, and therefore

$$\mathscr{N}(G) = \mathscr{N}(G^\alpha). \tag{1}$$

For example, consider the hexagon G in Figure 2, and let α be the antipodal map that rotates it 180° about its center. Then α is an anti-automorphism (it also happens to be an automorphism) and $G^\alpha = H$ is the union of two triangles shown on the right of Figure 2.

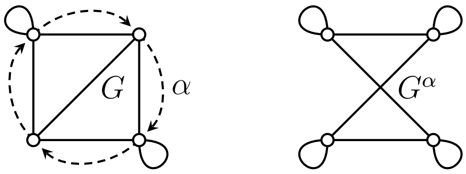

Figure 3: A graph G, an anti-automorphism α, and the corresponding graph G^α.

Figure 3 shows a second example. The $90°$ rotation α of $V(G)$ is an anti-automorphism, and G^α is shown on the right. Notice that $\mathscr{N}(G) = \mathscr{N}(G^\alpha)$. These examples illustrate our first proposition, which in essence was noted in [1], in the context of loopless graphs.[1]

Proposition 1. *If G and H are two graphs on the same vertex set, then $\mathscr{N}(G) = \mathscr{N}(H)$ if and only if $H = G^\alpha$ for some anti-automorphism of G.*

Proof: If $H = G^\alpha$ for some anti-automorphism α of G, then $\mathscr{N}(G) = \mathscr{N}(H)$ by Equation (1).

Conversely, let G and H have vertex set V, and suppose $\mathscr{N}(G) = \mathscr{N}(H)$. Then there is a permutation α of V with $N_G(x) = N_H\big(\alpha(x)\big)$ for all $x \in V$, so also $N_H(x) = N_G\big(\alpha^{-1}(x)\big)$. Note that α is an anti-automorphism of G because

$$\begin{aligned} xy \in E(G) &\Longleftrightarrow y \in N_G(x) \\ &\Longleftrightarrow y \in N_H\big(\alpha(x)\big) \\ &\Longleftrightarrow y\alpha(x) \in E(H) \\ &\Longleftrightarrow \alpha(x) \in N_H(y) \\ &\Longleftrightarrow \alpha(x) \in N_G\big(\alpha^{-1}(y)\big) \\ &\Longleftrightarrow \alpha(x)\alpha^{-1}(y) \in E(G). \end{aligned}$$

To verify $H = G^\alpha$, observe that

$$\begin{aligned} xy \in E(H) &\Longleftrightarrow x \in N_H(y) \\ &\Longleftrightarrow x \in N_G\big(\alpha^{-1}(y)\big) \\ &\Longleftrightarrow x\alpha^{-1}(y) \in E(G) \\ &\Longleftrightarrow x\alpha(\alpha^{-1}(y)) = xy \in E(G^\alpha). \end{aligned}$$ □

Let's pause to elaborate on the notion of neighborhood reconstructibility. As noted above, G is neighborhood-reconstructible if for any H with $\mathscr{N}(G) = \mathscr{N}(H)$ it necessarily

[1]The article [1] differs slightly from our current setting. What we here call an *anti-automorphism* plays the role of an *admissible map* in [1]. Admissible maps coincide with our anti-automorphisms, except that they have an additional condition that assures G^α is loopless. Thus the definition of an anti-automorphism is weaker than that of an admissible map.

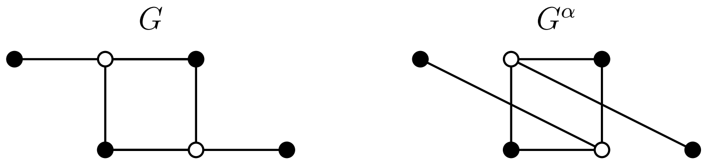

Figure 4: Rotation α of G by 180° is an involution, and hence also an anti-automorphism. Notice that $\mathscr{N}(G) = \mathscr{N}(G^\alpha)$. Here $G \neq G^\alpha$, but $G \cong G^\alpha$.

follows that $G \cong H$. Say G is *strongly neighborhood-reconstructible* if $\mathscr{N}(G) = \mathscr{N}(H)$ implies $G = H$.

The example in Figure 4 should clarify the distinction. Clearly $G \neq G^\alpha$, though $G \cong G^\alpha$ and $\mathscr{N}(G) = \mathscr{N}(G^\alpha)$. In fact, the results of Section 4 will show that this graph G is neighborhood-reconstructible. Hence it is neighborhood-reconstructible but not strongly neighborhood-reconstructible.

For a simple (but not completely trivial) example of a graph that is strongly neighborhood-reconstructible, let G be an edge ab with a loop at a. It is straightforward that G with $E(G) = \{ab, aa\}$ is the only graph that can be reconstructed from $\mathscr{N}(G)$.

We close this section with an immediate corollary of Proposition 1.

Corollary 2. *A graph G is neighborhood-reconstructible if and only if $G \cong G^\alpha$ for every anti-automorphism α of G.*

Note that Proposition 1 also implies that G is strongly neighborhood-reconstructible if and only if $G = G^\alpha$ for every anti-automorphism α of G. We can further refine this by forming an equivalence relation R on $V(G)$ by declaring xRy if and only if $N(x) = N(y)$. The proof of the next corollary is straightforward from definitions.

Corollary 3. *A graph G is strongly neighborhood-reconstructible if and only if its anti-automorphisms are precisely the permutations of $V(G)$ that preserve the R-equivalence classes of $V(G)$. (That is, $\alpha(X) = X$ for each R-equivalence class X.)*

Although Corollary 2 characterizes neighborhood-reconstructible graphs, we certainly cannot regard it as a simple characterization, as finding all anti-automorphisms of G promises to be quite difficult in general, let alone deciding if $G \cong G^\alpha$ for all of them. However, it does provide a link to cancellation, which we now explore.

3 Cancellation

Lovász [10, Theorem 9] proved that if a graph K has an odd cycle, then $G \times K \cong H \times K$ implies $G \cong H$. In such a situation we say that *cancellation holds*.

Cancellation may fail if K is bipartite. For example, consider graphs G and H from Figure 2. In Figure 5 we see that $G \times K_2 \cong H \times K_2$, as both products are isomorphic to two copies of a hexagon, but cancellation fails because $G \ncong H$. Recall that $H = G^\alpha$ where α is the antipodal map of G. Thus we have $G \times K_2 \cong G^\alpha \times K_2$.

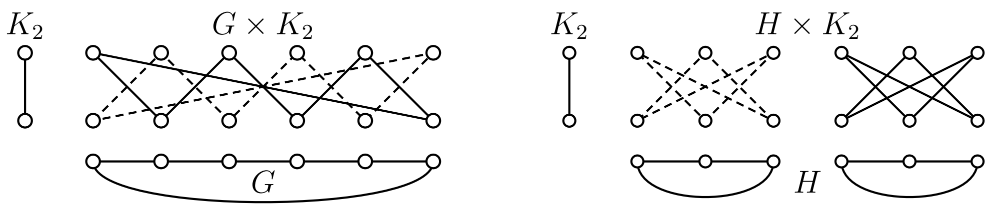

Figure 5: Failure of cancellation: $G \times K_2 \cong H \times K_2$ but $G \ncong H$.

For another example, take the graphs G and G^α from Figure 3. Again, Figure 6 reveals that $G \times K_2 \cong G^\alpha \times K_2$. (Each products is isomorphic to the three-dimensional cube.)

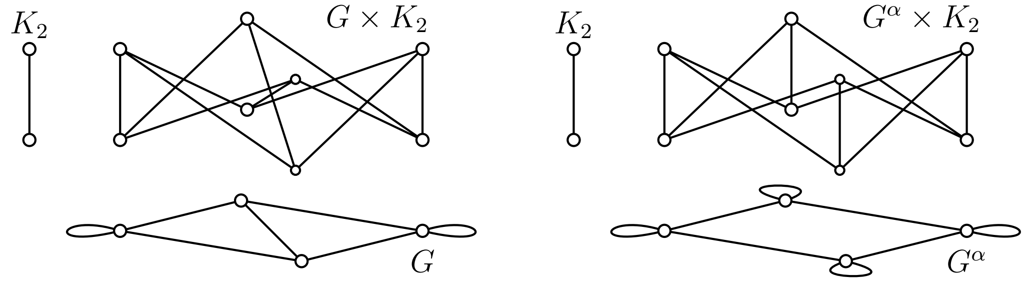

Figure 6: Failure of cancellation: $G \times K_2 \cong G^\alpha \times K_2$ but $G \ncong G^\alpha$.

These examples are instances of our next proposition, which was proved in [4] and also in [11]. For completeness we include an abbreviated proof.

Proposition 4. *Suppose a bipartite graph K has at least one edge. Then $G \times K \cong H \times K$ if and only if $H \cong G^\alpha$ for some anti-automorphism α of G.*

Proof. We use a result by Lovász [10, Theorem 6]: If there is a graph homomorphism $K' \to K$, then $G \times K \cong H \times K$ implies $G \times K' \cong H \times K'$.

Let $G \times K \cong H \times K$. As there is a homomorphism $K_2 \to K$, we have $G \times K_2 \cong H \times K_2$. Take an isomorphism $\varphi : G \times K_2 \to H \times K_2$. We easily check that we may assume φ has form $\varphi(g, k) = (\beta(g, k), k)$. (This is also a special instance of [10, Theorem 7].) Put $V(K_2) = \{0, 1\}$.

Define bijections $\mu, \lambda : G \to H$ as $\mu(g) = \beta(g, 0)$ and $\lambda(g) = \beta(g, 1)$. First we will show $\mu^{-1}\lambda$ is an anti-automorphism of G. Then we will show $G^{\mu^{-1}\lambda} \cong H$. Observe that

$$\begin{aligned} xy \in E(G) &\iff (x, 1)(y, 0) \in E(G \times K_2) \\ &\iff \varphi(x, 1)\varphi(y, 0) \in E(H \times K_2) \\ &\iff (\lambda(x), 1)(\mu(y), 0) \in E(H \times K_2) \\ &\iff \lambda(x)\mu(y) \in E(H). \end{aligned}$$

A similar argument gives $xy \in E(G) \iff \mu(x)\lambda(y) \in E(H)$. It follows that

$$xy \in E(G) \iff \lambda(x)\mu(y) \in E(H)$$

$$\begin{aligned} &\iff \mu^{-1}(\lambda(x))\lambda^{-1}(\mu(y)) \in E(G) \\ &\iff \mu^{-1}\lambda(x)(\mu^{-1}\lambda)^{-1}(y) \in E(G). \end{aligned}$$

Therefore $\mu^{-1}\lambda$ is an anti-automorphism of G.

Now, $\mu : G^{\mu^{-1}\lambda} \to H$ is an isomorphism because

$$\begin{aligned} xy \in E(G^{\mu^{-1}\lambda}) &\iff x(\mu^{-1}\lambda)^{-1}(y) \in E(G) \\ &\iff x\lambda^{-1}(\mu(y)) \in E(G) \\ &\iff \mu(x)\lambda(\lambda^{-1}\mu(y)) \in E(H) \\ &\iff \mu(x)\mu(y) \in E(H). \end{aligned}$$

Conversely, suppose $H \cong G^\alpha$ for an anti-automorphism α of G. Say K has partite sets X_0 and X_1. It suffices to show that $G \times K \cong G^\alpha \times K$. Define $\Theta : G \times K \to G^\alpha \times K$ to be

$$\Theta(g,k) = \begin{cases} (g,k) & \text{if } k \in X_0 \\ (\alpha(g),k) & \text{if } k \in X_1. \end{cases}$$

To prove Θ is an isomorphism take $(g,k)(g',k') \in E(G \times K)$. Then k and k' must be in different partite sets. Say $k \in X_0$ and $k' \in X_1$. Then

$$\begin{aligned} (g,k)(g',k') \in E(G \times K) &\iff gg' \in E(G) \text{ and } kk' \in E(K) \\ &\iff g\alpha(g') \in E(G^\alpha) \text{ and } kk' \in E(K) \qquad \text{(def. of } G^\alpha) \\ &\iff (g,k)(\alpha(g'),k') \in E(G^\alpha \times K) \\ &\iff \Theta(g,k)\Theta(g',k') \in E(G^\alpha \times K). \end{aligned}$$

Similarly, if $k \in X_1$ and $k' \in X_0$ we get $(g,k)(g',k') \in E(G \times K)$ if and only if $\Theta(g,k)\Theta(g',k') \in E(G^\alpha \times K)$. Thus Θ is an isomorphism. □

We define a graph G to be a *cancellation graph* if $G \times K \cong H \times K$ implies $G \cong H$ for all graphs K that have at least one edge. (We require at least one edge because if K is edgeless, then $G \times K \cong H \times K$ whenever $|V(G)| = |V(H)|$, as both products are also edgeless.)

Note that $G^{\mathrm{id}} = G$. By Lovász's result that $G \times K \cong H \times K$ implies $G \cong H$ when K has an odd cycle, we see that $G \times K \cong H \times K$ if and only if $H \cong G^{\mathrm{id}}$ when K is not bipartite. Combining this with our Proposition 4, we get a corollary.

Corollary 5. *A graph G is a cancellation graph if and only if $G \cong G^\alpha$ for all its anti-automorphisms α.*

Combining this with Corollary 2 yields a theorem.

Theorem 6. *A graph is a cancellation graph if and only if it is neighborhood-reconstructible.*

This implies a new cancellation result.

Theorem 7. *Suppose $G \times K \cong H \times K$. If G is neighborhood-reconstructible, then $G \cong H$.*

4 Further Results

This section seeks structural characterizations of cancellation (hence neighborhood reconstructible) graphs. We develop a sufficient condition for arbitrary graphs, and a characterization for the bipartite case.

Let $\mathrm{Ant}(G)$ be the set of all anti-automorphisms of G. By Corollary 2, Proposition 4 and Theorem 6, a graph is neighborhood-reconstructible and a cancellation graph if and only if $G \cong G^{\alpha}$ for each $\alpha \in \mathrm{Ant}(G)$. Therefore it is beneficial to determine the conditions under which $G \cong G^{\alpha}$, and, more generally, when $G^{\alpha} \cong G^{\beta}$.

To this end we adopt a construction from [9] and [6], and apply it to our current setting. Define the following set of pairs of permutations of $V(G)$. Let

$$\mathrm{Aut}^{\mathrm{TF}}(G) = \big\{(\lambda,\mu) \mid \lambda,\mu \text{ are permutations of } V(G) \text{ with } xy \in E(G) \Longleftrightarrow \lambda(x)\mu(y) \in E(G)\big\}.$$

Elements of $\mathrm{Aut}^{\mathrm{TF}}(G)$ are called *two-fold automorphisms* in [6, 7, 9]. R. Mizzi's Dissertation [6] applies them to the neighborhood reconstruction, and shows (for example) that there are only two graphs with the same neighborhood multiset as the Petersen graph.

Notice that $\mathrm{Aut}^{\mathrm{TF}}(G)$ is non-empty because it contains $(\mathrm{id},\mathrm{id})$. It is also a group under pairwise composition, and with $(\lambda,\mu)^{-1} = (\lambda^{-1},\mu^{-1})$. Observe that $(\lambda,\mu) \in \mathrm{Aut}^{\mathrm{TF}}(G)$ if and only if $\lambda\big(N_G(x)\big) = N_G\big(\mu(x)\big)$ for all $x \in V(G)$. Also $\alpha \in \mathrm{Ant}(G)$ if and only if $(\alpha,\alpha^{-1}) \in \mathrm{Aut}^{\mathrm{TF}}(G)$, and $\alpha \in \mathrm{Aut}(G)$ if and only if $(\alpha,\alpha) \in \mathrm{Aut}^{\mathrm{TF}}(G)$.

We can think of $\mathrm{Aut}^{\mathrm{TF}}(G)$ as follows: Suppose a bijection $\lambda : V(G) \to V(G)$ sends neighborhoods of G to neighborhoods of G, that is, it "permutes" the elements of $\mathscr{N}(G)$. Then there must be at least one bijection $\mu : V(G) \to V(G)$ with $\lambda\big(N_G(x)\big) = N_G\big(\mu(x)\big)$, and then $(\lambda,\mu) \in \mathrm{Aut}^{\mathrm{TF}}(G)$. If no two vertices of G have the same neighborhood, then there is a unique μ paired with any such λ, otherwise there will be more than one μ. [2]

The group $\mathrm{Aut}^{\mathrm{TF}}(G)$ acts on the set $\mathrm{Ant}(G)$ as $(\lambda,\mu)\cdot\alpha = \lambda\alpha\mu^{-1}$.

Proposition 8. *Suppose $\alpha,\beta \in \mathrm{Ant}(G)$. Then $G^{\alpha} \cong G^{\beta}$ if and only if α and β are in the same $\mathrm{Aut}^{\mathrm{TF}}(G)$-orbit. In particular, G is neighborhood reconstructible and a cancellation graph if and only if the $\mathrm{Aut}^{\mathrm{TF}}(G)$ action on $\mathrm{Ant}(G)$ is transitive.*

Proof: Suppose $\gamma : G^{\alpha} \to G^{\beta}$ is an isomorphism. Then

$$\begin{aligned} xy \in E(G) &\Longleftrightarrow x\alpha(y) \in E(G^{\alpha}) \\ &\Longleftrightarrow \gamma(x)\gamma(\alpha(y)) \in E(G^{\beta}) \\ &\Longleftrightarrow \gamma(x)\beta^{-1}(\gamma(\alpha(y))) \in E(G). \end{aligned}$$

Thus $(\gamma,\, \beta^{-1}\gamma\alpha) \in \mathrm{Aut}^{\mathrm{TF}}(G)$. Also $\gamma\alpha(\beta^{-1}\gamma\alpha)^{-1} = \beta$, so α and β are in the same $\mathrm{Aut}^{\mathrm{TF}}(G)$-orbit.

[2] We remark in passing that $\mathrm{Aut}^{\mathrm{TF}}(G)$ is similar to the so-called *factorial* $G!$ of a graph (or digraph) G, as defined in [5] and [3]. The vertex set of $G!$ is the set of permutations of $V(G)$, with an edge joining permutations λ and μ provided $xy \in E(G)$ implies $\lambda(x)\mu(y) \in E(G)$. Thus the edge set of $G!$ can be identified with $\mathrm{Aut}^{\mathrm{TF}}(G)$. The factorial is used in [5] to settle the general cancellation problem for digraphs. However, our present purposes do not require the graph structure of the factorial, so we will phrase the discussion in terms of two-fold automorphisms.

Conversely, let α and β be in the same $\mathrm{Aut}^{\mathrm{TF}}(G)$-orbit. Take $(\lambda, \mu) \in \mathrm{Aut}^{\mathrm{TF}}(G)$ with $\beta = (\lambda, \mu) \cdot \alpha = \lambda\alpha\mu^{-1}$, so $\alpha^{-1}\lambda^{-1} = \mu^{-1}\beta^{-1}$. Then $\lambda^{-1} : G^\beta \to G^\alpha$ is an isomorphism, as

$$\begin{aligned} xy \in E(G^\beta) &\iff x\beta^{-1}(y) \in E(G) \\ &\iff \lambda^{-1}(x)\mu^{-1}(\beta^{-1}(y)) \in E(G) && \text{(because } (\lambda^{-1}, \mu^{-1}) \in \mathrm{Aut}^{\mathrm{TF}}(G)) \\ &\iff \lambda^{-1}(x)\alpha^{-1}(\lambda^{-1}(y)) \in E(G) && \text{(because } \alpha^{-1}\lambda^{-1} = \mu^{-1}\beta^{-1}) \\ &\iff \lambda^{-1}(x)\lambda^{-1}(y) \in E(G^\alpha). && \square \end{aligned}$$

If $\alpha \in \mathrm{Ant}(G)$, then it is immediate that also $\alpha^k \in \mathrm{Ant}(G)$ for all integers k. Moreover, because $(\alpha, \alpha^{-1}) \in \mathrm{Aut}^{\mathrm{TF}}(G)$ and $(\alpha, \alpha^{-1}) \cdot \alpha = \alpha^3$, Proposition 8 yields $G^\alpha \cong G^{\alpha^3}$. Iterating, we get a proposition.

Proposition 9. *If $\alpha \in \mathrm{Ant}(G)$, then $G^\alpha \cong G^{\alpha^{1+2n}}$ for all integers n. In particular, if α has odd order, then $G^\alpha \cong G$.*

Now, if α has *even* order, we may write its order as $2^m(1 + 2n)$ for integers m and n. Then α^{1+2n} has order 2^m, and by Proposition 9, $G^\alpha \cong G^{\alpha^{1+2n}}$. Consequently we can get all G^α, up to isomorphism, with only those anti-automorphisms whose order is a power of 2. Of course this is little help in enumerating all G^α, but it does lead to a quick sufficient condition for a graph to be neighborhood-reconstructible.

Corollary 10. *If a graph has no involutions, then it is neighborhood-reconstructible, and thus also a cancellation graph.*

Proof: If G is not neighborhood-reconstructible, then there is some $\alpha \in \mathrm{Ant}(G)$ with $G^\alpha \not\cong G$. Proposition 9 says the order n of α is even, so $\alpha^{n/2}$ is an involution of G. $\square$

If G is bipartite, this corollary tightens to a characterization. As a preliminary to this we claim that any anti-automorphism α of a bipartite graph carries any partite set of a connected component of G bijectively to a partite set of a component of G. Indeed, suppose x_0 and x_0' both belong to the same partite set of a connected component of G. Then G has an even-length path $x_0, v_1, \ldots, v_{2n+1}, x_0'$. Thus the path $\alpha(x_0), \alpha^{-1}(v_1), \ldots, \alpha^{-1}(v_{2n+1}), \alpha(x_0')$ has even length, so $\alpha(x_0)$ and $\alpha(x_0')$ are in the same partite set of some component of G.

Theorem 11. *A bipartite graph is a cancellation graph (is neighborhood-reconstructible) if and only if it has no involution that reverses the bipartition of one of its components.*

Proof: Let G be bipartite. Suppose G has an involution α that reverses the partite sets of one of its components. Call that component H, and its partite sets X and Y. Select $x \in X$. Then $\alpha(x) \in Y$, and H has an odd path $x, x_1, x_2, x_3, \ldots, x_{2k-1}, x_{2k}, \alpha(x)$. Thus G^α has an odd walk $x, \alpha(x_1), x_2, \alpha(x_3), \ldots, \alpha(x_{2k-1}), x_{2k}, \alpha^2(x)$. But this odd walk begins and ends at x, so G^α is not bipartite. Consequently $G \not\cong G^\alpha$ so G is neither a cancellation graph nor neighborhood-reconstructible, by Corollaries 2 and 5.

Conversely, suppose G has no involutions that reverse the bipartition of a component. Say G has c components H_i, each with partite sets $V(H_i) = X_i \cup Y_i$, where $1 \leqslant i \leqslant c$.

Now, we noted above that any $\alpha \in \text{Ant}(G)$ permutes the set $\{X_1, Y_1, X_2, Y_2 \dots X_c, Y_c\}$. Notice that the α-orbit of a particular X_i cannot meet the α-orbit of the corresponding Y_i. The reason is that we'd then have $\alpha^k(X_i) = Y_i$ for some power k. From this we could concoct an involution σ of G that reverses the bipartition of H_i by simply declaring

$$\sigma(x) = \begin{cases} x & \text{if } x \in V(G) - V(H_i) \\ \alpha^k(x) & \text{if } x \in X_i \\ \alpha^{-k}(x) & \text{if } x \in Y_i. \end{cases}$$

Since no such involution exists, the α-orbit of a X_i never meets the α-orbit of Y_i.

Therefore, given any $\alpha \in \text{Ant}(G)$, we may assume (by interchanging the labels X_i and Y_i as appropriate) that α sends each X_i to some X_j, and it sends each Y_k to some Y_ℓ. Define a bipartition $V(G) = X \cup Y$, where $X = \bigcup X_i$, and $Y = \bigcup Y_i$. By construction we have $\alpha(X) = X$ and $\alpha(Y) = Y$. Now form a map $\mu : G \to G^\alpha$ as

$$\mu(x) = \begin{cases} \alpha(x) & \text{if } x \in X \\ x & \text{if } x \in Y. \end{cases}$$

That this is an isomorphism follows immediately from the definition of G^α and the anti-automorphism property of α. Consequently we have $G \cong G^\alpha$ for every $\alpha \in \text{Ant}(G)$, so G is neighborhood reconstructible and a cancellation graph by Corollaries 2 and 5. □

As an example of Proposition 11, the graph in Figure 4 is neighborhood reconstructible and a cancellation graph.

5 Conclusion

Our Theorem 6 states that the class of neighborhood-reconstructible graphs is the same as the class of cancellation graphs, and from this we have deduced a new cancellation law, Theorem 7.

Theorem 11 characterizes bipartite cancellation (neighborhood reconstructible) graphs as those that do not admit a certain kind of involution, namely one that reverses the bipartition of a component. Of course a bipartite graph admitting no such involutions may well still have involutions that preserve the bipartitions of the components – we just have to rule out the existence of reversing involutions.

We do not have a characterization of non-bipartite cancellation (neighborhood reconstructible) graphs. Corollary 10 implies that a non-bipartite graph is a cancellation graph if it has no involutions at all. However, this is not a characterization because some non-bipartite neighborhood-reconstructible graphs do admit involutions. For example, K_3 with loops added to each vertex is neighborhood reconstructible and admits involutions. It would be interesting to find a way to extend the sufficient condition of Corollary 10 to some kind of characterization, perhaps one that rules out only certain kinds of involutions. Exactly what kinds of involutions, we do not know. We leave this as an open problem.

Acknowledgments

We thank the referees for their prompt and careful reading, and for suggestions that have improved the paper. We especially appreciate one referee calling to our attention several works with which we were not familiar, in particular the dissertation [6]. Also the fact that our Theorem 11 bears some similarity to the result of [8], which, given a bipartite digraph G, shows that the digraphs H for which $G \times K_2 \cong H \times K_2$ are in bijective correspondence with the $\mathrm{Aut}(G \times K_2)$-conjugacy classes of the set of automorphisms of G that reverse a bipartition. We note that an independent result [2] for graphs also provides a means of constructing all such H.

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