Kazhdan-Lusztig polynomials of thagomizer matroids

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Abstract

We introduce thagomizer matroids and compute the Kazhdan-Lusztig polynomial of a rank $n+1$ thagomizer matroid by showing that the coefficient of t^k is equal to the number of Dyck paths of semilength n with k long ascents. We also give a conjecture for the S_n-equivariant Kazhdan-Lusztig polynomial of a thagomizer matroid.

Keywords: matroid theory; Kazhdan-Lusztig polynomials; generating functions; Schur functions

1 Introduction

The main objects of study in this paper are the Kazhdan-Lusztig polynomials of a particular family of matroids. The Kazhdan-Lusztig polynomial of a matroid was introduced by Elias, Proudfoot and Wakefield [EPW16]. In the appendix of that paper, the authors (along with Young) explicitly computed the coefficients of these polynomials for some uniform and braid matroids of small rank. Proudfoot, Wakefield and Young studied uniform matroids of rank $n-1$ on n elements [PWY16] and gave a combinatorial description for the coefficients of the associated Kazhdan-Lusztig polynomial.

Let M_n be the matroid associated with the graph obtained from the bipartite graph $K_{2,n}$ by adding an edge between the two distinguished vertices. We call M_n a **thagomizer matroid**[1]. The ground set of M_n has size $2n+1$ and the rank of M_n is $n+1$. We give a description of the flats of M_n in Section 3.

[1]The underlying graph is also called the complete tripartite graph $K_{1,1,n}$ or the fan graph $F_{n,2}$.

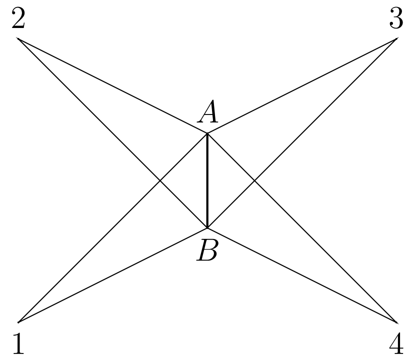

Figure 1: The underlying graph of M_4.

Let $P_n(t)$ be the **Kazhdan-Lusztig polynomial** of M_n and set

$$\Phi(t,u) := \sum_{n=0}^{\infty} P_n(t) u^{n+1}.$$

Let $c_{n,k}$ be the k-th coefficient of $P_n(t)$ and note that the degree of $P_n(t)$ is $\lfloor \frac{n}{2} \rfloor$. The following theorem is our main result.

Theorem 1. *The following (equivalent) statements hold.*

1. *For all n and k, $c_{n,k}$ is the number of Dyck paths of semilength n with k long ascents.*
2. *The generating function $\Phi(t,u)$ is equal to* $\dfrac{1-\sqrt{1-4u(1-u+tu)}}{2(1-u+tu)}$.

Remark 2. It is known that the number of Dyck paths of semilength n with k long ascents is equal to the quantity $\frac{1}{n+1}\binom{n+1}{k}\sum_{j=2k}^{n}\binom{j-k-1}{k-1}\binom{n+1-k}{n-j}$ (see [STT06] and sequence A091156 in [Slo16]).

Remark 3. The total number of Dyck paths of semilength n is equal to the n-th Catalan number $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$. Thus Theorem 1 implies that $P_n(1) = \mathcal{C}_n$ and Remark 2 implies that the leading coefficient of $P_{2n}(t)$ is $\mathcal{C}_n$. Interestingly, $\mathcal{C}_n$ also appears as the leading coefficient of the Kazhdan-Lusztig polynomial of the uniform matroid of rank $2n-1$ on $2n$ elements (see [EPW16] Appendix A and [PWY16]).

Remark 4. Prior to this paper, uniform matroids were the only infinite family of matroids for which the Kazhdan-Lusztig polynomial has been computed. For example, it is still an open problem to compute the Kazhdan-Lusztig polynomial of the braid matroid; see [EPW16] and [GPY17] for partial results.

We conclude this section with a description of the structure of the paper. In Section 2, we recall the definition of the Kazhdan-Lusztig polynomial of a matroid and review Dyck paths. Section 3 is dedicated to proving Theorem 1.

In Section 4, we recall the definition of the equivariant Kazhdan-Lusztig polynomial of a matroid and explore the S_n action on M_n which allows us to make a conjecture for the S_n-equivariant Kazhdan-Lusztig polynomial of M_n. This categorification of Kazhdan-Lusztig coefficients was first considered for a uniform matroid of rank $n-1$ on n elements by Proudfoot, Wakefield and Young [PWY16] where they were given by an irreducible representation of S_n. The equivariant Kazhdan-Lusztig polynomial for a general matroid was subsequently defined by the author, Proudfoot and Young [GPY17] where we further studied uniform matroids in this context and computed the S_n-equivariant Kazhdan-Lusztig polynomials of braid matroids of small rank.

2 Preliminaries

2.1 Kazhdan-Lusztig polynomials of a matroid

In this section we follow [EPW16] to define the non-equivariant Kazhdan-Lusztig polynomial of a matroid (which we simply refer to as the Kazhdan-Lusztig polynomial).

Let M be a matroid on the finite ground set E. Denote by $L(M)$ the **lattice of flats** of M and $\chi_M(t)$ the **characteristic polynomial** of M. The matroid M^F is called the **contraction** of M at F; it is the matroid on the ground set $E \setminus F$ whose lattice of flats is $L^F := \{G \setminus F \mid G \in L(M) \text{ and } G \geqslant F\}$. The matroid M_F is called the **localization** of M at F and is the matroid with ground set F whose lattice of flats is $L_F := \{G \in L(M) \mid G \leqslant F\}$.

The **Kazhdan-Lusztig polynomial** $P_M(t) \in \mathbb{Z}[t]$ is characterized by the following three properties [EPW16, Theorem 2.2].

- If $\operatorname{rk} M = 0$, then $P_M(t) = 1$.
- If $\operatorname{rk} M > 0$, then $\deg P_M(t) < \frac{1}{2} \operatorname{rk} M$.
- For every M, $t^{\operatorname{rk} M} P_M(t^{-1}) = \displaystyle\sum_F \chi_{M_F}(t) P_{M^F}(t)$.

2.2 Dyck paths

A **Dyck path** of semilength n is a lattice path in $\mathbb{N}^2$ beginning at $(0,0)$ and ending at $(2n,0)$ with up-steps of the form $u = (1,1)$ and down-steps of the form $d = (1,-1)$. Such a Dyck path may be expressed as a word $\alpha \in \{u,d\}^{2n}$.

A **long ascent** of a Dyck path is an ascent of length at least 2. Equivalently, a long ascent of a Dyck path α is a maximal subword consisting of at least two consecutive u's. The Dyck path given in Figure 2 has two long ascents.

Let $\mathcal{D}_n$ be the set of all Dyck paths of semilength n. We denote by $a_{n,k}$ the number of elements in $\mathcal{D}_n$ with exactly k long ascents. As noted in [STT06], $a_{n,k}$ is also the number of words $\alpha \in \mathcal{D}_n$ with k occurrences of the subword uud. Additional interpretations of $a_{n,k}$ are known; see sequence A091156 in [Slo16].

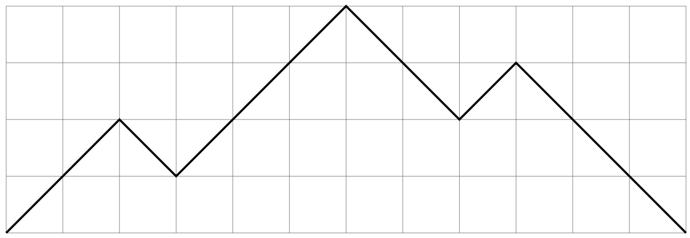

Figure 2: The Dyck path *uuduuudduddd*.

3 Main results

We begin this section with a description of the flats $F \in L(M_n)$ given by the underlying graph. Let AB be the distinguished edge. For any $j \in \{1, \dots, n\}$, we call the subgraph with edges Aj and Bj a **spike**.

If $\operatorname{rk} F = i$, then either

1. F contains exactly one edge from i distinct spikes, or

2. F is the union of $i-1$ spikes and AB.

For example, when $n = 4$, a rank 2 flat of the first type is given by $\{A1, B3\}$ and a rank 2 flat of the second type is given by $\{AB, A4, B4\}$ (see Figure 1).

In the first case, the localization $(M_n)_F$ yields a Boolean matroid of rank i, and the contraction M_n^F gives a matroid whose lattice of flats is isomorphic to that of M_{n-i}. In the second case, the localization $(M_n)_F$ gives a matroid whose lattice of flats is isomorphic to that of M_{i-1}, and the contraction M_n^F is a Boolean matroid of rank $n-i+1$.

The characteristic polynomial of a rank i Boolean matroid is equal to $(t-1)^i$. For thagomizer matroids, it is clear that

$$\chi_{M_i}(t) = (t-1)(t-2)^i$$

by a simple deletion/contraction argument.

Recall that we've set

$$P_n(t) := P_{M_n}(t) \quad \text{and} \quad \Phi(t,u) := \sum_{n=0}^{\infty} P_n(t) u^{n+1}.$$

We first turn our attention towards proving the following lemma.

Lemma 5. *We have the following (equivalent) equations.*

1. *For all n,* $t^{n+1} P_n(t^{-1}) = (t-1)^{n+1} + \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} (t-1)^{n-i} P_i(t).$

2. $\Phi(t^{-1}, tu) = \dfrac{ut-u}{1+u-tu} + \Phi\left(t, \dfrac{u}{1+2u-2tu}\right).$

Proof. There are $\binom{n}{i} \cdot 2^{n-i}$ flats of the first type of rank $n-i$ and $\binom{n}{i}$ flats of the second type of rank $i+1$. Note that for any Boolean matroid M, $P_M(t) = 1$ [EPW16, Corollary 2.10]. Then we have

$$\begin{aligned} t^{n+1} P_n(t^{-1}) &= \sum_{i=0}^{n} \binom{n}{i} \left(2^{n-i}(t-1)^{n-i} P_i(t) + (t-1)(t-2)^i \right) \\ &= (t-1)^{n+1} + \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}(t-1)^{n-i} P_i(t) \end{aligned} \tag{1}$$

which is the formula given in Lemma 5(1). Now our defining recursion tells us that

$$\begin{aligned} \Phi(t^{-1}, tu) &= \sum_{n=0}^{\infty} P_n(t^{-1}) t^{n+1} u^{n+1} \\ &= \sum_{n=0}^{\infty} (t-1)^{n+1} u^{n+1} + \sum_{n=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}(t-1)^{n-i} P_i(t) u^{n+1}. \end{aligned}$$

We let $m = n - i$ which allows us to write the second summand as

$$\sum_{i=0}^{\infty} P_i(t) u^{i+1} \sum_{m=0}^{\infty} 2^m \binom{m+i}{i} (t-1)^m u^m.$$

Recall the identity

$$\sum_{\ell=0}^{\infty} \binom{r+\ell}{r} x^\ell = \frac{1}{(1-x)^{r+1}}$$

and set $\ell = m$ and $x = 2u(t-1)$. This gives

$$\begin{aligned} \Phi(t^{-1}, tu) &= u(t-1) \sum_{n=0}^{\infty} (t-1)^n u^n + \sum_{i=0}^{\infty} \frac{P_i(t) u^{i+1}}{(1-2u(t-1))^{i+1}} \\ &= \frac{u(t-1)}{1-u(t-1)} + \sum_{i=0}^{\infty} P_i(t) \left(\frac{u}{1-2u(t-1)} \right)^{i+1} \\ &= \frac{ut-u}{1+u-tu} + \Phi\left(t, \frac{u}{1+2u-2tu} \right). \end{aligned}$$

This completes the proof of Lemma 5. ∎

Finally we are ready to prove Theorem 1. Let $a_{n,k}$ be as in Section 2.2, and set

$$F(t,u) := \sum_{n,k \geqslant 0} a_{n,k} t^k u^n.$$

It was shown in [STT06, Section 1] that $F(t,u)$ satisfies

$$u(1-u+tu) \cdot (F(t,u))^2 - F(t,u) + 1 = 0$$

which gives

$$F(t,u) = \frac{1-\sqrt{1-4u(1-u+tu)}}{2u(1-u+tu)}.$$

A priori, this formula should have a $\pm$ sign. However, a plus sign would not give $F(t,u)$ as a formal power series. Hence we use a negative sign instead.

Let $f(t,u) := u \cdot F(t,u)$. Since we'd like to show that $\Phi(t,u) = u \cdot F(t,u)$, we first check that $f(t,u)$ satisfies the functional equation in Lemma 5(2).

We have

$$f(t,u) = \frac{1-\sqrt{1-4u(1-u+tu)}}{2(1-u+tu)}$$

and hence

$$\begin{aligned}
f(t^{-1},tu) &= \frac{1-\sqrt{1-4tu(1-tu+u)}}{2(1-tu+u)} \\
&= \frac{ut-u}{1-tu+u} + \frac{1-2ut+2u-\sqrt{1-4tu(1-tu+u)}}{2(1-tu+u)} \\
&= \frac{ut-u}{1-tu+u} + \frac{1-\frac{1}{1+2u-2tu}\sqrt{1-4tu(1-tu+u)}}{\frac{2(1+u-tu)}{1+2u-2tu}} \\
&= \frac{ut-u}{1-tu+u} + \frac{1-\sqrt{1-\frac{4u(1+2u-2tu-u+tu)}{(1+2u-2tu)^2}}}{\frac{2(1+u-tu)}{1+2u-2tu}} \\
&= \frac{ut-u}{1-tu+u} + f\left(t, \frac{u}{1+2u-2tu}\right).
\end{aligned}$$

Lastly, we note that both $c_{n,k}$ and $a_{n,k}$ are zero if $n > 2k$ and that $f(t,0) = \Phi(t,0) = 1$. Then $f(t,u) = \Phi(t,u)$ which equivalently tells us that $c_{n,k} = a_{n,k}$. This completes the proof of Theorem 1. □

4 The S_n action

Recall the notation set in Section 2.1. That is, let M be a matroid on a finite ground set E. Given a flat $F \in L(M)$, let M^F denote the contraction of M at F and let M_F denote the localization of M at F.

Let W be a finite group acting on E and preserving M. We refer to the data $\{M, E, W\}$ as an **equivariant matroid** $W \curvearrowright M$. For any $F, G \in L(M)$, let $W_F \subseteq W$ be the stabilizer of F and let $W_{FG} := W_F \cap W_G$. Note that the action of W on M induces an action of W_F on both M_F and M^F. Let $\mathrm{VRep}(W)$ be the ring of isomorphism classes of virtual representations of W and set

$$\mathrm{grVRep}(W) := \mathrm{VRep}(W) \otimes_{\mathbb{Z}} \mathbb{Z}[t].$$

Let $OS^W_{M,i} \in \mathrm{Rep}(W)$ be the degree i part of the Orlik-Solomon algebra of M. The **equivariant characteristic polynomial** of M, $H^W_M(t)$, is given by

$$H^W_M(t) := \sum_{p=0}^{\operatorname{rk} M} (-1)^p t^{\operatorname{rk} M - p} OS^W_{M,p} \in \mathrm{grVRep}(W).$$

Note that the equivariant characteristic polynomial $H^W_M(t)$ is a categorified version of the usual characteristic polynomial $\chi_M(t)$. That is, we can recover $\chi_M(t)$ from $H^W_M(t)$ by taking the graded dimension.

The **equivariant Kazhdan-Lusztig polynomial** of $W \curvearrowright M$, denoted $\mathcal{P}^W_M(t)$, is a categorified version of the Kazhdan-Lusztig polynomial and is characterized by the following three properties [GPY17, Theorem 2.8].

- If $\operatorname{rk} M = 0$, $\mathcal{P}^W_M(t)$ is equal to the trivial representation in degree 0.
- If $\operatorname{rk} M > 0$, $\deg \mathcal{P}^W_M(t) < \frac{1}{2} \operatorname{rk} M$.
- For every M, $t^{\operatorname{rk} M} \mathcal{P}^W_M(t^{-1}) = \sum_{[F] \in L/W} \mathrm{Ind}^W_{W_F} \left(H^{W_F}_{M_F}(t) \otimes \mathcal{P}^{W_F}_{M^F}(t) \right).$

The polynomial $\mathcal{P}^W_M(t)$ is an element of $\mathrm{grVRep}(W)$ and we can recover $P_M(t)$ from $\mathcal{P}^W_M(t)$ by taking the graded dimension.

Now we turn our attention back to the thagomizer matroid M_n. Though the full automorphism group of M_n is $S_n \times S_2$ (unless $n = 1$ in which case it is S_3), here we only consider the action of the symmetric group S_n. Let

$$\mathcal{P}_n(t) := \mathcal{P}^{S_n}_{M_n}(t) \quad \text{and} \quad \phi(t,u) := \sum_{u=0}^{\infty} \mathcal{P}_n(t) u^{n+1}.$$

Let Υ_n be all partitions of n of the form $[a, n-a-2i-\eta, 2^i, \eta]$ where $\eta \in \{0,1\}$, $i \geqslant 0$ and $1 < a < n$. For any partition λ of n, we let V_λ be the irreducible representation of S_n indexed by λ.

For any partition λ, we set

$$\kappa(\lambda) = \begin{cases} \lambda_1 - \lambda_2 + 1 & \lambda \neq [n-1,1] \\ \lambda_1 - 1 & \text{otherwise} \end{cases}$$

and

$$\omega(\lambda) = \begin{cases} 1 & \lambda_{\ell(\lambda)} \neq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Conjecture 6. For all $n > 0$, we have

$$\mathcal{P}_n(t) = \sum_{\lambda \in \Upsilon_n} \kappa(\lambda) V_\lambda t^{\ell(\lambda)-1} (t+1)^{\omega(\lambda)} + V_{[n]}((n-1)t+1).$$

Remark 7. We have checked this conjecture for thagomizer matroids of rank at most 20 using SageMath [Dev16]. For our calculations, we worked in the symmetric function setting (see Proposition 10).

Remark 8. We know the coefficients of $\mathcal{P}_n(t)$ will be honest representations by [GPY17, Corollary 2.12] since M_n is S_n-equivariantly realizable.

Remark 9. Unlike the analogous statements for uniform matroids, Conjecture 6 is less enlightening than Theorem 1(1) (see [GPY17], Theorem 3.1 and Remark 3.4). That is, the coefficients of the Kazhdan-Lusztig polynomial of a uniform matroid are more cleanly expressed when given as the dimension of a certain representation of the symmetric group. This is not the case for thagomizer matroids.

The remainder of this section is devoted to understanding the results that allow us to derive the recursive formula and functional equation for the Frobenius characteristic of $\mathcal{P}_n(t)$. Let

$$W(t) := (t-1)\mathbb{C} \quad \text{and} \quad V(t) := (t-2)\mathbb{C}$$

as virtual vector spaces. Then $W(t)^{\otimes r}$ is the equivariant characteristic polynomial of a rank r Boolean matroid and $W(t) \otimes V(t)^{\otimes r}$ is the equivariant characteristic polynomial of M_r. Both $W(t)^{\otimes r}$ and $W(t) \otimes V(t)^{\otimes r}$ are virtual representations of S_r, where S_r acts by permuting the factors of the graded tensor product. Note that the equivariant Kazhdan-Lusztig polynomial of a Boolean matroid is the trivial representation in degree zero.

We'd like to categorify the recursive formula given in Lemma 5(1). Recall Equation 1

$$t^{n+1}P_n(t^{-1}) = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}(t-1)^{n-i}P_i(t) + \sum_{i=0}^{n} \binom{n}{i}(t-1)(t-2)^i.$$

The first sum is over flats of rank $n-i$ of the first type mentioned in Section 3. For flats of this type, summing over $[F] \in L(M_n)/S_n$ gives

$$\sum_{m+j+i=n} \mathrm{Ind}_{S_m \times S_j \times S_i}^{S_n} \left(W(t)^{\otimes m} \otimes W(t)^{\otimes j} \otimes \mathcal{P}_i(t)\right) \quad \in \mathrm{grVRep}(S_n) \tag{5}$$

where S_m permutes the vertices that are connected to A by an edge in F, S_j permutes the vertices that are connected to B by an edge in F, and S_i permutes the vertices that are not adjacent to any edge in F. Similarly, summing over flats of the second type gives

$$\sum_{i=0}^{n} \mathrm{Ind}_{S_i \times S_{n-i}}^{S_n} \left(W(t) \otimes V(t)^{\otimes i}\right) \quad \in \mathrm{grVRep}(S_n) \tag{6}$$

where S_{n-i} is acting trivially.

As in [GPY17, Section 3.1], we now translate to symmetric functions. We consider the **Frobenius characteristic**

$$\mathrm{ch} : \mathrm{grVRep}(S_n) \xrightarrow{\sim} \Lambda_n[t]$$

where Λ_n is the space of symmetric functions of degree n in infinitely many formal variables $\{x_i \mid i \in \mathbb{N}\}$.

Let $s[\lambda] := \operatorname{ch} V_\lambda$ be the Schur function corresponding to λ and set

$$p_n(t) := \operatorname{ch} \mathcal{P}_n(t), \quad w_n(t) := \operatorname{ch} W(t)^{\otimes n} \quad \text{and} \quad v_n(t) := \operatorname{ch} V(t)^{\otimes n}.$$

Applying Frobenius characteristic to Equations 5 and 6, we obtain

$$t^{n+1} p_n(t^{-1}) = (t-1) \sum_{\ell=0}^{n} v_\ell(t) s[n-\ell] + \sum_{i+j+m=n} p_i(t) w_j(t) w_m(t).$$

Finally, we pass to generating functions, working in the ring $\Lambda[[t,u]]$ of formal power series in the variables $\{t, u, x_1, x_2, \ldots\}$ that are symmetric in the x variables.We let

$$s(u) := \sum_n s[n] u^n, \quad w(t,u) := \sum_n w_n(t) u^n$$

and

$$v(t,u) := \sum_n v_n(t) u^n.$$

Note that

$$w(t,u) = \frac{s(tu)}{s(u)}$$

by [GPY17, Proposition 3.9]. The results of this section can be summarized in the following proposition.

Proposition 10. *We have the following (equivalent) equations.*

1. *For $n > 0$, $t^{n+1} p_n(t^{-1}) = (t-1) \sum_{\ell=0}^{n} v_\ell(t) s[n-\ell] + \sum_{i+j+m=n} p_i(t) w_j(t) w_m(t)$.*

2. *$\phi(t^{-1}, tu) = (t-1) u s(u) v(t,u) + w(t,u)^2 \phi(t,u)$.*

Remark 8. In [GPY17], we were able to compute the equivariant Kazhdan-Lusztig polynomial for uniform matroids by showing that our "guess" satisfied a recursion analogous to the one found in Proposition 10(2). That case was much simpler; we only had to consider singular applications of the Pieri rule. In this case, $w(t,u)^2$ requires multiple applications of the Pieri rule while $v_n(t) = s[n]\big[v_1(t)\big]$ involves a plethysm. This makes proving Conjecture 6 much more difficult.

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