On the structure of the power graph and the enhanced power graph of a group

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Abstract

Let G be a group. The *power graph* of G is a graph with the vertex set G, having an edge between two elements whenever one is a power of the other. We characterize nilpotent groups whose power graphs have finite independence number. For a bounded exponent group, we prove its power graph is a perfect graph and we determine its clique/chromatic number. Furthermore, it is proved that for every group G, the clique number of the power graph of G is at most countably infinite. We also measure how close the power graph is to the *commuting graph* by introducing a new graph which lies in between. We call this new graph the *enhanced power graph*. For an arbitrary pair of these three graphs we characterize finite groups for which this pair of graphs are equal.

Keywords: power graph, clique number, chromatic number, independence number, group.

1 Introduction

We begin with some standard definitions from graph theory and group theory.

Let G be a graph with vertex set $V(G)$. If $x \in V(G)$, then the number of vertices adjacent to x is called the *degree* of x, and denoted by $\deg(x)$. The *distance* between two vertices in a graph is the number of edges in a shortest path connecting them. The *diameter* of a connected graph G, denoted by $\operatorname{diam}(G)$, is the maximum distance between any pair of vertices of G. If G is disconnected, then $\operatorname{diam}(G)$ is defined to be infinite. A *star* is a graph in which there is a vertex adjacent to all other vertices, with no further edges. The *center* of a star is a vertex that is adjacent to all other vertices. Let $U \subseteq V(G)$. The induced subgraph on U is denoted by $\langle U \rangle$. An *independent set* is a set of vertices in a graph, no two of which are adjacent; that is, a set whose induced subgraph is null. The *independence number* of a graph G is the cardinality of the largest independent set and is denoted by $\alpha(G)$. A subset S of the vertex set of G is called a *dominating set* if for every vertex v of G, either $v \in S$ or v is adjacent to a vertex in S. The minimum size of dominating sets of G, denoted by $\gamma(G)$, is called the *domination number* of G. A *clique* in a graph is a set of pairwise adjacent vertices. The supremum of the sizes of cliques in G, denoted by $\omega(G)$, is called the *clique number* of G. By $\chi(G)$, we mean the *chromatic number* of G, i.e., the minimum number of colours which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colours.

The cyclic group of order n is denoted by C_n. A group G is called *periodic* if every element of G has finite order. For every element $g \in G$, the order of g is denoted by $o(g)$. If there exists an integer n such that for all $g \in G$, $g^n = e$, where e is the identity element of G, then G is said to be of *bounded exponent.* If G is of bounded exponent, then the *exponent* of G is the least common multiple of the orders of its elements; that is, the least n for which $g^n = e$ for all $g \in G$. A group G is said to be *torsion-free* if apart from the identity every element of G has infinite order. Let p be a prime number. The *p-quasicyclic group* (known also as the *Prüfer group*) is the p-primary component of $\mathbb{Q}/\mathbb{Z}$, that is, the unique maximal p-subgroup of $\mathbb{Q}/\mathbb{Z}$. It is denoted by C_{p^∞}. The *center* of a group G, denoted by $Z(G)$, is the set of elements that commute with every element of G. A group G is called *locally finite* if every finitely generated subgroup of G is finite. A group is *locally cyclic* if any finitely generated subgroup is cyclic. Other concepts will be defined when needed.

Now, we define the object of interest to us in this paper. Let G be a group. The *power graph* of G, denoted by $\mathcal{G}(G)$, is the graph whose vertex set is G, two elements being adjacent if one is a power of the other.

The concept of a power graph was first introduced by Kelarev and Quinn [17]. Note that in [17], a power graph is directed. The full automorphism group of a power graph of a finite group was characterized in [11]. Also, see [1] for a survey of results and open questions on power graphs. Other studies include [9] for semigroups and [8, 7] for groups. In the last of these papers, it was shown that, for a finite group, the undirected power graph determines the directed power graph up to isomorphism. As a consequence, two finite groups which have isomorphic undirected power graphs have the same number of elements of each order.

Our results about the power graph fall into four classes.

- In Section 2.1, we consider the independence number $\alpha(\mathcal{G}(G))$. We show that if the independence number is finite then G is a locally finite group whose centre has finite index. Using this we are able to give precise characterizations of nilpotent groups G for which $\alpha(\mathcal{G}(G))$ is finite – such a group (if infinite) is the direct product of a p-quasicyclic group and a nilpotent p'-group.
- In Section 2.2, we show that the power graph of every group has clique number at most countable. We note here that Shitov [21] answered a question in an earlier version of this paper by showing that the chromatic number of the power graph of every group is also at most countable.
- A group with finite clique number must be of bounded exponent. Hence we obtain a structure theorem for abelian groups with this property, as well as showing that it passes to subgroups and supergroups of finite index.
- We show that, if G has bounded exponent, then $\mathcal{G}(G)$ is perfect.
- Finally, in Section 2.3 there are some miscellaneous results. A group is periodic if and only if its power graph is connected, and in this case its diameter must be at most 2. Also we show that, if all vertex degrees in $\mathcal{G}(G)$ are finite, then G is finite.

In the recent paper [11], the authors prove that the power graph of every finite group is perfect. We acknowledge that our result on the perfectness along with all results in the Section 2 were proved independently in 2011.

Another well-studied graph associated to a group G is the *commuting graph* of G. This graph appears to be first studied by Brauer and Fowler in 1955 in [6] as a part of classification of finite simple groups. As the elements of the centre are adjacent to all other vertices, usually the vertices are assumed to be non-central. For more information on the commuting graph, see [3, 15, 24] and the references therein.

In Section 3 we relate the power graph to the commuting graph and characterize when they are equal for finite groups. A new graph pops up while considering these graphs, a graph whose vertex set consists of all group elements, in which two vertices x and y are adjacent if they generate a cyclic group. We call this graph as the *enhanced power graph* of G and we denote it by $\mathcal{G}_e(G)$. The enhanced power graph contains the power graph and is a subgraph of the commuting graph. We further study some properties of this graph in the Section 3.

We characterize the finite groups for which equality holds for any two of these three graphs, and the solvable groups for which the power graph is equal to the commuting graph. Other results are as follows:

- If the power graphs of G and H are isomorphic, then their enhanced power graphs are isomorphic.
- A maximal clique in the enhanced power graph is either a cyclic or a locally cyclic subgroup.

- $\mathcal{G}_e(G)$ has finite clique number if and only if G has finite exponent; if this holds, then the clique number of $\mathcal{G}_e(G)$ is equal to the largest order of an element of G. Also, for any group G, the clique number of $\mathcal{G}_e(G)$ is at most countable.

2 Power graphs of groups

2.1 Independent sets in power graphs

In this section we provide some results on the finiteness of the independence number of the power graphs. In the proof of the our first theorem, we need the following definition. Let G be a group and associate with G a graph $\Gamma(G)$ as follows: the vertices of $\Gamma(G)$ are the elements of G and two vertices g and h of $\Gamma(G)$ are joined by an edge if and only if g and h do not commute, see [2] and [18] for more details. Now, we have the following result.

Theorem 1. *Let G be a group and $\alpha(\mathcal{G}(G)) < \infty$. Then*

(i) $[G : Z(G)] < \infty$.

(ii) *G is locally finite.*

Proof. (i) First we note that if x and y are adjacent in $\Gamma(G)$, then x and y are not adjacent in $\mathcal{G}(G)$. Thus $\omega(\Gamma(G)) \leqslant \alpha(\mathcal{G}(G)) < \infty$. Hence [18, Theorem 6] implies that $[G : Z(G)] < \infty$.
(ii) Let H be a finitely generated subgroup of G. Then by (i) and [19, 1.6.11], $Z(H)$ is finitely generated, too. So by the fundamental theorem for finitely generated abelian groups we find that $Z(H) \cong \mathbb{Z}^n \times C_{q_1} \times \cdots \times C_{q_k}$, where n and k are non-negative integers and every q_i, $1 \leqslant i \leqslant k$, is a power of a prime number. Since $\alpha(\mathcal{G}(\mathbb{Z})) = \infty$, we deduce that H is a finite group and so the proof is complete. □

Now, we characterize those abelian groups whose power graphs have finite independence number. First we need the following theorem.

Theorem 2. ([19, 4.3.11]) *If G is an abelian group which is not torsion-free, then it has a non-trivial direct summand which is either cyclic or quasicyclic.*

Theorem 3. *Let G be an abelian group such that $\alpha(\mathcal{G}(G)) < \infty$. Then either G is finite or $G \cong C_{p^\infty} \times H$, where H is a finite group and $p \nmid |H|$.*

Proof. If G is torsion-free, then G contains $\mathbb{Z}$ and so $\alpha(\mathcal{G}(G)) \geqslant \alpha(\mathcal{G}(\mathbb{Z})) = \infty$, a contradiction. Thus by Theorem 2, $G = G_1 \oplus H_1$, where G_1 is either cyclic or quasicyclic. If H_1 is trivial, then we are done. Otherwise, $\alpha(\mathcal{G}(H_1)) < \infty$ implies that $H_1 = G_2 \oplus H_2$, where G_2 is either cyclic or quasicyclic. So $G = G_1 \oplus G_2 \oplus H_2$. By repeating this procedure and using $\alpha(\mathcal{G}(G)) < \infty$, we deduce that there exists a positive integer n such that $G \cong \bigoplus_{i=1}^{n} G_i$, where every G_i is either cyclic or quasicyclic. We show that at most one G_i is quasicyclic. By the contrary, suppose that G contains the group $C_{p^\infty} \times C_{q^\infty}$. It is not hard to see that for every positive integer n, $I_n = \{(1/p^i + \mathbb{Z}, 1/q^{n-i+1} + \mathbb{Z}) : 1 \leqslant i \leqslant n\}$ is an independent set of size n, a contradiction. So either $G \cong C_{p^\infty} \times \prod_{i=1}^{n} C_{p_i^{\alpha_i}}$ or $G \cong \prod_{i=1}^{n} C_{p_i^{\alpha_i}}$, where p and p_i are prime numbers. Now, suppose that the first case occurs. To complete the proof, we show that $p \neq p_i$, for every i, $1 \leqslant i \leqslant n$. By contrary, suppose that $p = p_i$, for some i. Then $C_{p^\infty} \times C_p$ is a subgroup of G. Since $C_{p^\infty} \times \{1\}$ is an independent set in $\mathcal{G}(C_{p^\infty} \times C_p)$, we get a contradiction. So, the proof is complete. □

Theorem 4. *Let p be a prime number and G be a p-group such that $\alpha(\mathcal{G}(G)) < \infty$. Then either G is finite or $G \cong C_{p^\infty}$.*

Proof. Since $\alpha(\mathcal{G}(G)) < \infty$, we deduce that $\alpha(\mathcal{G}(Z(G))) < \infty$. Thus by Theorem 3, either $Z(G)$ is finite or $Z(G) \cong C_{p^\infty}$, for some prime number p. If $Z(G)$ is finite, then by Theorem 1, G is finite. Now, suppose that $Z(G) \cong C_{p^\infty}$. To complete the proof, we show that G is abelian. To the contrary, suppose that there exists $a \in G \setminus Z(G)$. Let $H = \langle Z(G) \cup \{a\} \rangle$. Clearly, H is an abelian p-subgroup of G and $\alpha(\mathcal{G}(H)) < \infty$. So, by Theorem 3, $H \cong C_{p^\infty} \cong Z(G)$. Since every proper subgroup of C_{p^∞} is finite, we get a contradiction. Hence G is abelian and $G = Z(G) \cong C_{p^\infty}$. □

Now, we exploit Theorem 4 to extend Theorem 3 to nilpotent groups.

Remark 5. Let H and K be two subgroups of G. If $H \cap K = \{e\}$, $G = HK$ and $H \subseteq Z(G)$, then $G \cong H \times K$.

Theorem 6. *Let G be an infinite nilpotent group. Then $\alpha(\mathcal{G}(G)) < \infty$ if and only if $G \cong C_{p^\infty} \times H$, for some prime number p, where H is a finite group and $p \nmid |H|$.*

Proof. First suppose that $G \cong C_{p^\infty} \times H$, where H is a finite group and $p \nmid |H|$. Suppose to the contrary, $\{(s_n/p^{\alpha_n} + \mathbb{Z}, g_n) : n \geqslant 1, s_n \in \mathbb{Z}, p \nmid s_n \text{ and } g_n \in H\}$ is an infinite independent set of $\mathcal{G}(G)$. Since H is a finite group, there exists $g \in H$ such that the infinite set

$$\{(s_n/p^{\alpha_n} + \mathbb{Z}, g) : n \geqslant 1, s_n \in \mathbb{Z} \text{ and } p \nmid s_n\}$$

forms an independent set. Since $o(g) < \infty$, there exist α_i and α_j such that $p^{\alpha_i} \equiv p^{\alpha_j} (\bmod\, o(g))$ and $\alpha_i > \alpha_j$. On the other hand, we know that $\gcd(s_i, p) = 1$. So, let t_i be the multiplicative inverse of s_i in $C_{p^{\alpha_j}}$. Thus by Chinese Reminder Theorem, there exists a positive integer x such that $x \equiv t_i s_j (\bmod\, p^{\alpha_j})$ and $x \equiv p^{\alpha_j - \alpha_i} (\bmod\, o(g))$. Therefore, we have

$$p^{\alpha_i - \alpha_j} x \frac{s_i}{p^{\alpha_i}} - \frac{s_j}{p^{\alpha_j}} = \frac{s_i x - s_j}{p^{\alpha_j}} \in \mathbb{Z}, \qquad g^{x p^{\alpha_i - \alpha_j}} = g.$$

Thus, $(s_i/p^{\alpha_i} + \mathbb{Z}, g)$ and $(s_j/p^{\alpha_j} + \mathbb{Z}, g)$ are adjacent, a contradiction.

Conversely, suppose that $\alpha(\mathcal{G}(G)) < \infty$. Then by Theorem 1, $[G : Z(G)] < \infty$ and so $G = Z(G)H$, where H is a finitely generated subgroup of G. Now, Theorem 1 implies that H is finite. By Theorem 3, $Z(G) = AB$, where $A \cong C_{p^\infty}$ and B is a finite group such that $p \nmid |B|$. Also, since H is nilpotent, we have $H \cong H_p H_{p_1} \cdots H_{p_t}$, where H_p and H_{p_i} $(1 \leqslant i \leqslant t)$ are sylow p-subgroup and sylow p_i-subgroup of H, respectively. We show that $H_p \subseteq A$. To the contrary, suppose that $x \in H_p \setminus A$. Then $\langle A, x \rangle$ is a

p-group and so by Theorem 4, $\langle A, x\rangle \cong C_{p^\infty} \cong \langle A\rangle$. Since every proper subgroup of C_{p^∞} is finite, we get a contradiction. Thus $H_p \subseteq Z(G)$ and so $G = ABH_{p_1} \cdots H_{p_t}$. Since G is nilpotent, $BH_{p_1} \cdots H_{p_t}$ is a finite subgroup of G and $p \nmid |BH_{p_1} \cdots H_{p_t}|$. Hence by Remark 5, $G \cong A \times BH_{p_1} \cdots H_{p_t}$, as desired. □

2.2 The colouring of power graphs

Let G be a group. In this section, we first show that the chromatic number of the power graph of G is finite if and only if the clique number of the power graph of G is finite and this statement is also equivalent to that the exponent of G is finite. Then it is proved that the clique number of the power graph of G is at most countable. Finally, it is shown that the power graph of every bounded exponent group is perfect.

Lemma 7. *Let G be a group. If $\omega(\mathcal{G}(G))$ is finite, then G is of bounded exponent.*

Proof. By the contrary, suppose that G is not of bounded exponent. Then for every positive integer k, there is an element $g_k \in G$ such that $o(g_k) > 2^k$. So one can easily show that $\{g_k^{2^i} \mid 0 \leqslant i \leqslant k\}$ is a clique of size $k+1$ in $\mathcal{G}(G)$. This implies that $\omega(\mathcal{G}(G)) = \infty$, a contradiction. The proof is complete. □

Remark 8. The proof uses the Axiom of Choice for families of finite sets.

Corollary 9. *Let G be an abelian group and $\omega(\mathcal{G}(G)) < \infty$. Then there are some positive integers r and n_i and sets I_i, $1 \leqslant i \leqslant r$ such that*

$$G \cong \prod_{i=1}^{r} \prod_{I_i} C_{n_i}.$$

Proof. Since $\omega(\mathcal{G}(G)) < \infty$, by Lemma 7, G is bounded exponent. So the assertion follows from Prüfer-Baer Theorem (see [19, 4.3.5]). □

Theorem 10. *The clique number of the power graph of any group is at most countably infinite.*

Proof. Let C be a clique in the power graph of G, and take $x \in C$. Then the remaining vertices y of C are of two types:

- $y = x^n$ for some n;
- $x = y^n$ for some n.

Clearly, there are at most countably many of the first type. We denote the set of vertices of the second type by $C(n)$. We show that $C(n)$ is at most countably infinite. If there is only one y in $C(n)$, then there is nothing to prove; so suppose there are at least two elements in $C(n)$. We claim that every element in $C(n)$ has finite order. Choose $y, y' \in C(n)$. With no loss of generality, one can assume that $y' = y^k$, for some positive integer k. So $y^{(k-1)n} = 1$. This implies that the orders of both y and y' are finite. Thus the claim is proved. Now, for every positive integer k, define $C(n,k) = \{y \in C(n) \mid o(y) = k\}$. By the claim, $C(n) = \bigcup_{k \geqslant 1} C(n,k)$. It is not hard to show that for every $a, b \in C(n,k)$, $\langle a\rangle = \langle b\rangle$ and so $C(n,k)$ is finite. Therefore, $C(n)$ is at most countably infinite. □

In an earlier version of this paper (on the arXiv), we asked whether it is the case that the chromatic number of the power graph of any group is at most countable. We proved that this was the case for several classes of groups (abelian groups, periodic groups and free groups). In the meantime, Shitov [21], acknowledging our paper, has proved that the answer to our question is affirmative, more generally for all power-associative magmas.

2.2.1 Perfectness of the power graph

A graph G is called *perfect* if for every finite induced subgraph H of G, $\chi(H) = \omega(H)$. The Strong Perfect Graph Theorem states that a finite graph G is perfect if and only if neither G nor $\overline{G}$ (the complement of G) contains an induced odd cycle of length at least 5, see [5, Theorem 14.18]. However, this is a deep theorem, and we do not need it to prove our results.

Utilizing Lemma 7 to colour the power graph with a finite set of colours we require the group to be bounded exponent. Here we show that for such groups the resulting power graph is always perfect and can be finitely coloured. To prove this result we facilitate the concepts of comparability graph.

Let $\leqslant$ be a binary relation on the elements of a set P. If $\leqslant$ is reflexive and transitive, then $(P, \leqslant)$ is called a *pre-ordered set*. All partially ordered sets are pre-ordered. The *comparability graph* of a pre-ordered set $(P, \leqslant)$ is the simple graph $\Upsilon(P)$ with the vertex set P and two distinct vertices x and y are adjacent if and only if either $x \leqslant y$ or $y \leqslant x$ (or both).

Theorem 11. *Let m be a positive integer and P be a pre-ordered set (not necessarily finite) whose maximum chain size is m. Then the comparability graph $\Upsilon(P)$ is perfect and*

$$\omega(\Upsilon(P)) = \chi(\Upsilon(P)) = m.$$

Proof. The result is well known for comparability graphs of partial orders; our proof is a slight extension of this. Since the class of comparability graphs is closed under taking induced subgraphs, it is enough to prove that the comparability graph of P has equal clique number and chromatic number. Clearly, a clique in a $\Upsilon(P)$ is a chain in P, while a colouring is a partition into antichains.

First we show that $\omega(\Upsilon(P)) = m$. Let C be a clique in $\Upsilon(P)$. Then C is a chain in P, and so $|C| \leqslant m$. Thus $\omega(\Upsilon(P)) = m$.

Now, we show that $\chi(\Upsilon(P)) \leqslant m$. We form a directed graph by putting an arc from x to y whenever $x \leqslant y$ but $y \nleqslant x$; and, if C is an equivalence class of the relation $\equiv$ defined by $x \equiv y$ if $x \leqslant y$ and $y \leqslant x$, then take an arbitrary directed path on the elements of C. Clearly, the longest directed path contains m vertices. Let P_i be the set of elements x for which the longest directed path ending at x contains i vertices. It is easy to see that P_i is an independent set; these sets partition P into m classes. □

Now, we show that the power graph of a group is the comparability graph of a pre-ordered set. First, we define some notations. Let n be a positive integer and $\mathcal{D}(n)$ be the set of all divisors of n in $\mathbb{N}$. Define a relation $\preceq$ on $\mathcal{D}(n)$ by $r \preceq s$ if and only if $r \mid s$.

Clearly, $(\mathcal{D}(n), \preceq)$ is a partially ordered set. Denote the set of all chains of $(\mathcal{D}(n), \preceq)$ by $\mathcal{C}(n)$. Using this convention we are able to determine the clique/chromatic number of the power graph of a group of bounded exponent (see Lemma 7).

Theorem 12. *Let G be a group of exponent n. Then $\mathcal{G}(G)$ is a perfect graph and*

$$\chi(\mathcal{G}(G)) = \omega(\mathcal{G}(G)) = \max\left\{ \sum_{\substack{d\in C\\ G_d\neq\emptyset}} \phi(d) \ :\ C \in \mathcal{C}(n) \right\} \leqslant n,$$

where G_d is the set of elements of G of order d, for some divisor d of n.

Proof. First we consider two following facts:

Fact 1. Suppose that d is a divisor of n and $G_d \neq \emptyset$. If $g, h \in G_d$ and g and h are adjacent in the power graph, then g and h generate the same cyclic group. So $\langle G_d\rangle$ is a disjoint union of cliques of size $\phi(d)$. Therefore; if x is an element of a clique H of $\langle G_d\rangle$ adjacent to an element y of a clique K of $\langle G_{d'}\rangle$, then every element of H is adjacent to every element of K and moreover, $d \mid d'$ or $d' \mid d$.

Fact 2. Note that if z is an element of order d, then for each divisor d' of d, $z^{\frac{d}{d'}}$ is of order d'. So for each clique T of $\langle G_d\rangle$, every element of T is adjacent to every element of a clique S of $\langle G_{d'}\rangle$.

Since $\{G_d : G \text{ has an element of order } d\}$ forms a partition for G, Fact 1 implies that every maximal clique of $\mathcal{G}(G)$ has the form $Cl_1 \cup \cdots \cup Cl_m$, where Cl_i is a clique of $\langle G_{d_i}\rangle$ of size $\phi(d_i)$ and $\{d_1, \ldots, d_m\}$ is a chain of length m belonging to $\mathcal{C}(n)$. Moreover; by Fact 2, we deduce that for every chain $\{d_1, \ldots, d_m\}$ in $\mathcal{C}(n)$, there exists a clique for $\mathcal{G}(G)$ of this form. Now, by $|Cl_i| = \phi(d_i)$, we conclude that

$$\omega(\mathcal{G}(G)) = \max\left\{ \sum_{\substack{d\in C\\ G_d\neq\emptyset}} \phi(d) \ :\ C \in \mathcal{C}(n) \right\} \leqslant \sum_{d\mid n} \phi(d) = n.$$

Define a pre-ordering $\leqslant$ on G by $x \leqslant y$ if and only if x is a power of y. Clearly, the power graph of G is the comparability graph of $\leqslant$ and so by Theorem 11, the power graph of G is perfect. Thus $\chi(\mathcal{G}(G)) = \omega(\mathcal{G}(G))$ and the proof is complete. □

The next two corollaries are direct consequences of Lemma 7 and Theorem 12.

Corollary 13. *For every group G, the following statements are equivalent:*

(i) $\chi(\mathcal{G}(G)) < \infty$*;*

(ii) $\omega(\mathcal{G}(G)) < \infty$*;*

(iii) *G is bounded exponent.*

Moreover, the chromatic number of $\mathcal{G}(G)$ does not exceed the exponent of G.

Corollary 14. *Let G be an abelian group of exponent n. Then*

$$\chi(\mathcal{G}(G)) = \omega(\mathcal{G}(G)) = \max\left\{\sum_{d\in C}\phi(d) \,:\, C\in\mathcal{C}(n)\right\}.$$

Corollary 15. *Let H be a subgroup of G and $[G:H]<\infty$. Then $\omega(\mathcal{G}(H))<\infty$ if and only if $\omega(\mathcal{G}(G))<\infty$.*

The following example shows that a similar assertion does not hold for the independence number.

Example 16. Let $G = C_2\times C_{2^\infty}$ and $H=\{0\}\times C_{2^\infty}$. Thus $[G:H]=2$. Since $\mathcal{G}(H)$ is a complete graph, $\alpha(H)=1$. Clearly, the set $\{1\}\times C_{2^\infty}$ is independent and so $\alpha(G)=\infty$.

2.3 Miscellaneous properties

We conclude this section with three miscellaneous properties of the power graph of a group.

Theorem 17. *If $\mathcal{G}(G)$ is a triangle-free graph, then G is isomorphic to a direct product of C_2 and $\mathcal{G}(G)$ is a star.*

Proof. First we show that the order of every element of G is at most 2. Let $a\in G$. If $o(a)\geqslant 3$, then $\{e,a,a^2\}$ is a triangle, a contradiction. So G is an elementary abelian 2-group. Therefore, by Prüfer-Baer Theorem, G is isomorphic to a direct product of C_2 and so $\mathcal{G}(G)$ is a star with the center e. $\square$

The following theorem characterizes those groups whose power graphs are connected.

Theorem 18. *Let G be a group. The following statements are equivalent.*

(i) *$\mathcal{G}(G)$ is connected;*

(ii) *G is periodic;*

(iii) *$\gamma(\mathcal{G}(G))=1$;*

(iv) $\operatorname{diam}(\mathcal{G}(G))\leqslant 2$.

Proof. (i)$\Longrightarrow$(ii) Let x ($x\neq e$) be a vertex of $\mathcal{G}(G)$. We show that x is of finite order in G. Since $\mathcal{G}(G)$ is connected, there is a path from x to e. Let y be the adjacent vertex to e in this path. So the order of y is finite. Now, suppose that t is the adjacent vertex to y in this path. Then the order of t is finite, too. By repeating this procedure, we deduce that the order of x is finite. So G is periodic.
(ii)$\Longrightarrow$(iii) Since every element in G has a finite order, $\{e\}$ is a dominating set.
The parts (iii)$\Longrightarrow$(iv) and (iv)$\Longrightarrow$(i) are clear. $\square$

Theorem 19. *If* $\deg(g) < \infty$*, for every* $g \in G$*, then* G *is a finite group.*

Proof. Let $g \in G$. Since $\deg(g) < \infty$, g has a finite order in G. Thus G is a periodic group and so e is adjacent to every other vertices of $\mathcal{G}(G)$. Since $\deg(e) < \infty$, we deduce that G is finite. $\square$

Remark 20. Let $G \cong \prod_{i \geqslant 1} C_2$. Then $\mathcal{G}(G)$ is an infinite star with the center 0. This shows that in the previous theorem the condition $\deg(g) < \infty$, for every $g \in G$ is necessary.

3 Power graph and commuting graph

Let G be a group. If the vertices x and y are joined in the power graph of G, then they are joined in the commuting graph; so the power graph is a spanning subgraph of the commuting graph.

Question 21. For which groups is it the case that the power graph is equal to the commuting graph?

The identity is joined to all others in the commuting graph; so if the two graphs are equal, then G is a periodic group.

Theorem 22. *Let* G *be a finite group with power graph equal to commuting graph. Then one of the following holds:*

- G *is a cyclic* p*-group;*
- G *is a semidirect product of* C_{p^a} *by* C_{q^b}*, where* p *and* q *are primes with* $a, b > 0$*,* $q^b \mid p-1$ *and* C_{q^b} *acts faithfully on* C_{p^a}*;*
- G *is a generalized quaternion group.*

Proof. Let G have power graph equal to commuting graph; that is, if two elements commute, then one is a power of the other. Then G contains no subgroup isomorphic to $C_p \times C_q$, where p and q are primes, since this group fails the condition.

A theorem of Burnside [16, Theorem 12.5.2] says that a p-group containing no $C_p \times C_p$ subgroup is cyclic or generalized quaternion. So all Sylow subgroups of G are of one of these two types.

Suppose that all Sylow subgroups are cyclic. Then G is metacyclic [16, Theorem 9.4.3]. The cyclic normal subgroup of G has order divisible by one prime only, say p. Its centraliser in G is a Sylow p-subgroup P of G, since it contains no elements of order coprime to p. Hence G is a semidirect product of P and a cyclic group Q of order coprime to p, necessarily a cyclic q-group for some prime q. If $|Q| = q^b$, then we have $q^b \mid p-1$.

So we may suppose that G has generalized quaternion Sylow 2-subgroups. By Glauberman's Z*-Theorem [12], $G/O(G)$ has a central involution, where $O(G)$ is the maximal normal subgroup of G of odd order. This involution must act fixed-point-freely on $O(G)$, so $O(G)$ is abelian, and hence cyclic of prime power order. But a generalized quaternion group cannot act faithfully on such a group. So $O(G) = 1$. Then the involution in G is central, so G is a 2-group, necessarily a generalized quaternion group. $\square$

Remark 23. In the infinite case, there are other examples, such as *Tarski monsters*, which are infinite groups whose non-trivial proper subgroups are all cyclic of prime order p. Probably no classification is possible.

In the next theorem, we extend Theorem 22 to solvable groups.

Theorem 24. *Let G be a solvable group with power graph equal to commuting graph. Then one of the following holds:*

- *G is a cyclic p-group;*
- *G is a semidirect product of C_{p^a} by C_{q^b}, where p and q are primes with $a, b > 0$, $q^b \mid p - 1$ and C_{q^b} acts faithfully on C_{p^a};*
- *G is a generalized quaternion group;*
- *G is the p-quasicyclic group C_{p^∞};*
- *G is a semidirect product of p-quasicyclic group C_{p^∞} and a finite cyclic group.*

Proof. We use this fact that every finitely generated periodic solvable group is finite. We know that G is periodic. We show that there are no three elements whose order are distinct primes. Suppose that $o(a) = p$, $o(b) = q$ and $o(c) = r$, where p, q and r are distinct primes. Let H be the subgroup generated by a, b and c. Then H is finite. Clearly, the power graph and the commuting graph of H are equal. This contradicts Theorem 22. Thus the order of every finite subgroup of G is $p^\alpha q^\beta$, for some non-negative integers α and β. By the second part of Theorem 22, we may assume that β is bounded, because $q^\beta | p-1$. Also, by Theorem 22, the order of every element of G is a p-power or a q-power, because every cyclic subgroup is a p-group or a q-group. If $\alpha, \beta > 0$, then by the second part of Theorem 22, $\langle a, b\rangle$ is semidirect product of $\langle a\rangle$ and $\langle b\rangle$. So, $\langle a\rangle$ and $\langle b\rangle$ are both cyclic (even in the case $q = 2$). Let N be the set of all elements of G whose orders are p-power. We show that N is an abelian normal subgroup of G. To see this first we show that if x and y are two elements of N, then $xy = yx$. Let S be the subgroup generated by x and y. Then S is a finite group of order $p^\alpha q^\beta$. If $\beta = 0$, then by Theorem 22, S is a cyclic p-group and we are done. So assume that $\beta > 0$. Let N_1 and Q be Sylow p-subgroup and Sylow q-subgroup of S, respectively. Then by Theorem 22 both are cyclic. Now, by [20, Theorem 6.2.11], Q has a normal complement. So, $N_1 \lhd S$. This implies that $x, y \in N_1$ and so $xy = yx$. Thus we conclude that N is an abelian p-subgroup of G. Now, by the definition of N, $N \lhd G$.

Now, let Q be a q-subgroup of G which has maximum size. We prove that $G = NQ$.

Let $a \in G$ be an element of G whose order is q-power. Let M be the subgroup generated by a and Q. Then $M = P_1Q_1$, where P_1 and Q_1 are Sylow p-subgroup and Sylow q-subgroup of M, respectively and $P_1 \lhd M$. But Q_1 is a conjugate of Q and so $M = P_1Q$. Since $a \in M$, we have $a = bc$, where $b \in P_1$ and $c \in Q$. But $P_1 \subseteq P$ and this implies that $a \in PQ$, as desired. So $G = NQ$.

Since N is an abelian group, the commuting graph of N and so the power graph of N is a complete graph. Now, Theorem 4 yields that N is finite or $N = Z_{p^\infty}$. ☐

4 The enhanced power graph

4.1 Definition and properties

In the Section 2 we investigated some properties of power graphs of groups. In Theorem 22, we characterized finite groups for which the power graph is the same as the commuting graph. Now, it is natural to ask if they are not equal, how close these graphs are. To tackle this problem we introduce an intermediate graph. This graph can be regarded as a measurement for this difference. Given a group G, the *enhanced power graph* of G denoted by $\mathcal{G}_e(G)$ is the graph with vertex set G, in which x and y are joined if and only if there exists an element z such that both x and y are powers of z.

The power graph and commuting graph behave well on restriction to a subgroup (that is, if $H \leqslant G$, then the power graph of H is the induced subgraph of the power graph of G on the set H, and similarly for the commuting graph). Because of the existential quantifier in the definition, it is not obvious that the same holds for the enhanced graph. That this is so is a consequence of the fact that x and y are joined in the enhanced power graph if and only if $\langle x, y\rangle$ is cyclic. Note that

- the power graph is a spanning subgraph of the enhanced power graph;
- the enhanced power graph is a spanning subgraph of the commuting graph.

In the next remark, we use the concept of graph squares. For a graph H, the *square* of H is a graph with the same vertex set as H in which two vertices are adjacent if their distance in H is at most two.

Remark 25. If we assume that the (undirected or directed) power graph has a loop at each vertex, then the enhanced power graph lies between the power graph and its square. We already saw that it contains the power graph (as a spanning subgraph). Now, let x and y be two vertices joined by a path (x, z, y) of length 2 in the power graph. There are four cases in the directed power graph $D = \vec{\mathcal{G}}(G)$:

- $(x, z), (z, y) \in E(D)$. Then x is a power of z, and z a power of y; so x is a power of y, and $(x, y) \in E(D)$.
- $(z, x), (y, z) \in E(D)$. Dual to the first case.
- $(x, z), (y, z) \in E(D)$. Then x and y are powers of z, so they are joined in the square of the power graph.
- $(z, x), (z, y) \in E(D)$. In this case there is nothing we can say.

Also the following holds:

Theorem 26. *Let G and H be finite groups. If the power graphs of G and H are isomorphic, then their enhanced power graphs are also isomorphic.*

Proof. Note that x and y are joined in the enhanced power graph if and only if there is a vertex z which dominates both in the directed power graph. So the theorem follows from the main theorem of [7]. □

4.2 Comparing to the power graph and commuting graph

Question 27. For which (finite) groups is the power graph equal to the enhanced power graph?

This question connects with another graph associated with a finite group, the *prime graph*, defined by Gruenberg and Kegel [14]: the vertices of the prime graph of G are the prime divisors of $|G|$, and vertices p and q are joined if and only G contains an element of order pq. To state the next result we need a definition. The group G is a 2-*Frobenius group* if it has normal subgroups F_1 and F_2 such that $F_1 < F_2$, F_2 is a Frobenius group with Frobenius kernel F_1, and G/F_1 is a Frobenius group with Frobenius kernel F_2/F_1.

In the statement of the following theorem, p and q denote distinct primes.

Theorem 28. *For a finite group G, the following conditions are equivalent:*

(a) the power graph of G is equal to the enhanced power graph;

(b) every cyclic subgroup of G has prime power order;

(c) the prime graph of G is a null graph.

A group G with these properties is one of the following: a p-group; a Frobenius group whose kernel is a p-group and complement a q-group; a 2-Frobenius group where F_1 and G/F_2 are p-groups and F_2/F_1 is a q-group; or G has a normal 2-subgroup with quotient group H, where $S \leqslant H \leqslant \mathrm{Aut}(S)$ and $S \cong A_5$ or A_6.

All these types of group exist. Examples include S_3 and A_4 (Frobenius groups); S_4 (a 2-Frobenius group); A_5, A_6 and $2^4 : A_5$.

Proof. Let p and q be distinct primes. The cyclic group of order pq does not have property (a); so a group satisfying (a) must also satisfy (b). Conversely, suppose that (b) holds. If x and y are adjacent in the enhanced power graph, then $\langle x, y\rangle$ is cyclic, necessarily of prime power order; so it must be generated by one of x and y, and so x and y are adjacent in the power graph.

Clearly, (b) and (c) are equivalent.

Now, let G be a group satisfying these conditions. Either G is a p-group for some prime p, or the prime graph of G is disconnected. Now, we use the result of Gruenberg and Kegel [14] (stated and proved in Williams [23]), asserting that a finite group with disconnected prime graph is Frobenius or 2-Frobenius, simple, π_1 by simple, simple by π_1-solvable, or π_1 by simple by π_1. Here π_1 is the set of primes in the connected component of the prime graph containing 2, assuming that $|G|$ is even; and a 2-Frobenius group is a group G with normal subgroups $F_1 < F_2$ such that F_2 is a Frobenius group with kernel F_1, and G/F_1 is a Frobenius group with kernel F_2/F_1.

It follows from the work of Frobenius that a Frobenius complement either has all Sylow subgroups cyclic (and so is metacyclic) or has $\mathrm{SL}(2, 3)$ or $\mathrm{SL}(2, 5)$ as a normal subgroup. These last two cases cannot occur, since the central involution commutes with elements of order 3. In the first case, the results of Gruenberg and Kegel (see the first corollary in Williams [23]) show that the Frobenius complement has only one prime divisor.

In the case of a 2-Frobenius group, an element of the Frobenius complement in the top group centralises some element of F_1; so F_1 and G/F_2 must be p-groups for the same prime p.

In the remaining cases, it can be read off from the tables in Williams [23] that the simple group can only be A_5 or A_6, and the conclusions of the theorem follow since $\pi_1 = \{2\}$. □

Question 29. For which (finite) groups is the enhanced power graph equal to the commuting graph?

Again, we have a lot of information about such a group.

Theorem 30. *For a finite group G, the following conditions are equivalent:*

(a) the enhanced power graph of G is equal to its commuting graph;

(b) G has no subgroup $C_p \times C_p$ for p prime;

(c) the Sylow subgroups of G are cyclic or (for $p = 2$) generalized quaternion.

A group satisfying these conditions is either a cyclic p-group for some prime p, or satisfies the following: if $O(G)$ denotes the largest normal subgroup of G of odd order, then $O(G)$ is metacyclic, $H = G/O(G)$ is a group with a unique involution z, and $H/\langle z\rangle$ is a cyclic or dihedral 2-group, a subgroup of $\mathrm{P\Gamma L}(2, q)$ containing $\mathrm{PSL}(2, q)$ for q an odd prime power, or A_7.

An example of a group for the second case is the direct product of the Frobenius group of order 253 by $\mathrm{SL}(2, 5)$.

Proof. The group $C_p \times C_p$ has commuting graph not equal to its enhanced power graph, so cannot be a subgroup of a group satisfying (a); thus (a) implies (b). Conversely, suppose that (b) holds. Let x and y be elements of G which are adjacent in the commuting graph. Then $\langle x, y\rangle$ is abelian, and hence is the direct product of two cyclic groups, say $C_r \times C_s$. Under hypothesis (b), we must have $\gcd(r, s) = 1$, and so $\langle x, y\rangle \cong C_{rs}$; thus x and y are joined in the enhanced power graph.

Conditions (b) and (c) are equivalent by a theorem of Burnside [16, Theorem 12.5.2].

Suppose that a Sylow 2-subgroup P of G is cyclic or generalized quaternion. If P is cyclic, then by Burnside's transfer theorem [16, Section 14.3], G has a normal 2-complement: that is, if $O(G)$ is the largest normal subgroup of G of odd order, then $G/O(G) \cong P$. If P is generalized quaternion, then by Glauberman's Z^*-Theorem, $H = G/O(G)$ has a unique central involution z. Put $Z = \langle z\rangle$. Then the Sylow 2-subgroups Q of H/Z are dihedral; the Gorenstein–Walter theorem [13] shows that H/Z is isomorphic to a subgroup of $\mathrm{P\Gamma L}(2, q)$ containing $\mathrm{PSL}(2, q)$ (for odd q), or to the alternating group A_7, or to Q. For any such group $H^* = H/Z$, an argument of Glauberman (which can be found in [4]) shows that there is a unique double cover H with a single involution. □

Question 31. What can be said about the difference of the enhanced power graph and the power graph, or the difference of the commuting graph and the enhanced power graph? In particular, for which groups is either of these graphs connected?

4.3 Maximal cliques in the enhanced power graph

We will now look at maximal cliques in the enhanced power graph. This requires a lemma which looks trivial, but we couldn't find an easier proof of it than the one below.

Lemma 32. *Let x, y, z be elements of a group G, and suppose that $\langle x, y\rangle$, $\langle x, z\rangle$ and $\langle y, z\rangle$ are cyclic. Then $\langle x, y, z\rangle$ is cyclic.*

Proof. The result clearly holds if one of x, y, z is the identity; so suppose not. Now, a cyclic group cannot contain elements of both finite and infinite order, so either all three elements have finite order, or all three have infinite order.

Case 1: x, y, z have finite order. Then they generate a finite abelian group A. We first note that it suffices to do the case where the orders of x, y, z are powers of a prime p. For A is the direct sum of p-groups for various primes p; each p-group is generated by certain powers of x, y, z; and if each p-group is cyclic, then so is A. With this assumption, suppose that A is not cyclic. Since $\langle x, y\rangle$ is cyclic, A is the sum of two cyclic groups, and so contains a subgroup $Q \cong C_p \times C_p$. Each of $\langle x\rangle$, $\langle y\rangle$ and $\langle z\rangle$ intersects Q in a subgroup of order p; let these subgroups be X, Y, Z. Since $\langle x, y\rangle$ is cyclic, it meets Q in a subgroup of order p; so $X = Y$. Similarly $X = Z$. So $\langle x, y, z\rangle$ meets Q in a subgroup of order p, contradicting the assumption that $Q \leqslant \langle x, y, z\rangle$.

Case 2: x, y, z have infinite order. Then they generate a free abelian group; since $\langle x, y\rangle$ is cyclic, we see that $A = \langle x, y, z\rangle$ has rank at most 2. Consider the $\mathbb{Q}$-vector space $A \otimes_{\mathbb{Z}} \mathbb{Q}$, which has dimension at most 2. Since $\langle x, y\rangle$ is cyclic, the 1-dimensional subspaces $\langle x\rangle \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\langle y\rangle \otimes_{\mathbb{Z}} \mathbb{Q}$ have non-empty intersection, and so are equal. Similarly for $\langle z\rangle \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is 1-dimensional, so A is cyclic. □

Now, we have the following characterization of the maximal cliques in the enhanced power graph.

Lemma 33. *A maximal clique in the enhanced power graph is either a cyclic subgroup or a locally cyclic subgroup.*

Proof. Clearly, a cyclic or locally cyclic subgroup is a clique. Suppose that C is a maximal clique. If $x, y \in C$, then by Lemma 32, every element of $\langle x, y\rangle$ is joined to every element $z \in C$; so $C \cup \langle x, y\rangle$ is a clique. By maximality of C, we have $\langle x, y\rangle \subseteq C$; so C is a subgroup. Now, a simple induction shows that any finite subset of C generates a cyclic group, so that C is locally cyclic. □

Remark 34. Locally cyclic groups include the additive group of $\mathbb{Q}$ (or the subgroup consisting of rationals whose denominators only involve primes from a prescribed set), and direct sums of copies of the p-quasicyclic groups (the Prüfer groups) C_{p^∞} for distinct primes p.

Now, we have two immediate corollaries:

Corollary 35. *Let G be a group. Then $\omega(\mathcal{G}_e(G)) < \infty$ if and only if G is a group of finite exponent. If these conditions hold, then*

$$\omega(\mathcal{G}_e(G)) = \max\{o(g) \,:\, g \in G\}.$$

Remark 36. Note that this may be smaller than the exponent of G.

Proof. Clearly, if G is not a bounded exponent group, then $\mathcal{G}(G)$ as a subgraph of $\mathcal{G}_e(G)$ has infinite clique number by Lemma 7. Now, let G be a periodic group. Then the subsets $G_n = \{g \in G : o(g) = n\}$, for $n \in \mathbb{N}$, partition G into at most countably many subsets. On each of these subsets the power graph and the enhanced power graph coincide. In particular, if G is bounded exponent, then there are only finitely many classes. It is clear that, if x and y have the same order and generate a cyclic group, then each is a power of the other. □

Corollary 37. *A clique in the enhanced power graph of a group is at most countable.*

Proof. For a locally cyclic group is isomorphic to a subgroup of $\mathbb{Q}$ or $\mathbb{Q}/\mathbb{Z}$ and hence countable, see [19, Exercise 5, p.105]. □

Open problems

This paper concerns several graph theoretical parameters of the power graph of a group. In Section 2.1 we studied groups whose power graph has a finite independence number. In Theorem 6, we proved that if G is a nilpotent group and $\alpha(\mathcal{G}) < \infty$, then either G is a finite group or $G \cong C_{p^\infty} \times H$, for some prime number p, where H is a finite group and $p \nmid |H|$. This result motivates us to pose the following question.

Question 38. Let G be an infinite group. Is it true that $\alpha(\mathcal{G}(G)) < \infty$ if and only if $G \cong C_{p^\infty} \times H$, where H is a finite group and $p \nmid |H|$.

In Section 2.2, we showed that the chromatic number of the power graph of G is finite if and only if the clique number of the power graph of G is finite and this statement is also equivalent to the finiteness of the exponent of G. We proved that the clique number of the power graph of G is at most countable. The fact that the chromatic number is also at most countable was subsequently proved by Shitov [21].

It might be interesting to ask how much of Lemma 7 can be proved without the Axiom of Choice. Is there any way of showing that the chromatic number of a group of finite exponent is finite? A good test case for this question would be an abelian group of exponent 3. Colouring the non-identity elements with two colours requires choosing one of each pair $\{x, x^{-1}\}$, which requires AC (as Bertrand Russell famously pointed out).

In the study of the commuting graph, it is normal to delete vertices which lie in the centre of the group, since they would be joined to all other vertices. Similarly, in the study of the generating graph of a 2-generator group, the identity is an isolated vertex and is usually excluded. This convention is not used for the power graph. So any problem we raise will have two different forms, depending on which convention we use. For the power graph, the question of whether to include or exclude the identity is more interesting. Some of the results will be completely different in the two cases especially those dealing with connectedness. For example, if the identity is excluded, Theorem 18 fails, since indeed the power graph of the infinite cyclic group is connected when the identity is discarded. The next question seems interesting.

Question 39. Which groups do have the property that the power graph is connected when the identity is removed?

Or more generally:

Question 40. Which groups do have the property that the power graph is connected when the set of vertices which dominate the graph is removed?

The following question is the second version for Question 21.

Question 41. For which groups, are the induced subgraphs of the power graph and the commuting graph on $G \setminus \{e\}$ are equal. Note that free groups have this property.

Question 42. Consider the difference of the power graph and commuting graph, the graph in which x and y are joined if they commute but neither is a power of the other. What can be said about this difference graph? In particular, for which groups is it connected? Again this question can be asked with or without the identity. Note that in a periodic group the identity is isolated in the difference graph, but this is not true for arbitrary infinite groups.

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