# Tail positive words and generalized coinvariant algebras

Brendon Rhoades[*]
Department of Mathematics
University of California, San Diego
La Jolla, CA, 92093-0112, U.S.A.
bprhoades@math.ucsd.edu

Andrew Timothy Wilson[†]
Department of Mathematics
University of Pennsylvania
Philadelphia, PA, 19104-6395, U.S.A.
andwils@math.upenn.edu

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**Abstract**

Let $n, k$, and $r$ be nonnegative integers and let $S_n$ be the symmetric group. We introduce a quotient $R_{n,k,r}$ of the polynomial ring $\mathbb{Q}[x_1, \dots, x_n]$ in $n$ variables which carries the structure of a graded $S_n$-module. When $r \geqslant n$ or $k = 0$ the quotient $R_{n,k,r}$ reduces to the classical coinvariant algebra $R_n$ attached to the symmetric group. Just as algebraic properties of $R_n$ are controlled by combinatorial properties of permutations in $S_n$, the algebra of $R_{n,k,r}$ is controlled by the combinatorics of objects called *tail positive words*. We calculate the standard monomial basis of $R_{n,k,r}$ and its graded $S_n$-isomorphism type. We also view $R_{n,k,r}$ as a module over the 0-Hecke algebra $H_n(0)$, prove that $R_{n,k,r}$ is a projective 0-Hecke module, and calculate its quasisymmetric and noncommutative 0-Hecke characteristics. We conjecture a relationship between our quotient $R_{n,k,r}$ and the delta operators of the theory of Macdonald polynomials.

**Keywords:** symmetric function; coinvariant algebra; permutation

## 1 Introduction

Consider the action of the symmetric group $S_n$ on $n$ letters on the polynomial ring $\mathbb{Q}[\mathbf{x}_n] := \mathbb{Q}[x_1, \dots, x_n]$ given by variable permutation. The polynomials belonging to the invariant subring

$$\mathbb{Q}[\mathbf{x}_n]^{S_n} := \{f(\mathbf{x}_n) \in \mathbb{Q}[\mathbf{x}_n] \,:\, \pi.f(\mathbf{x}_n) = f(\mathbf{x}_n) \text{ for all } \pi \in S_n\} \tag{1.1}$$

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are the *symmetric polynomials* in the variable set $\mathbf{x}_n$. Let $e_d(\mathbf{x}_n)$ be the *elementary symmetric function* of degree $d$, that is $e_d(\mathbf{x}_n) = \sum_{1 \leqslant i_1 < \cdots < i_d \leqslant n} x_{i_1} \cdots x_{i_d}$. It is well known that the set $\{e_1(\mathbf{x}_n), \dots, e_n(\mathbf{x}_n)\}$ gives an algebraically independent homogeneous collection of generators for the ring $\mathbb{Q}[\mathbf{x}_n]^{S_n}$.

Let $\mathbb{Q}[\mathbf{x}_n]^{S_n}_+ \subset \mathbb{Q}[\mathbf{x}_n]^{S_n}$ be the subspace of symmetric polynomials with vanishing constant term. The *invariant ideal* $I_n \subseteq \mathbb{Q}[\mathbf{x}_n]$ is the ideal

$$I_n := \langle \mathbb{Q}[\mathbf{x}_n]^{S_n}_+ \rangle = \langle e_1(\mathbf{x}_n), \dots, e_n(\mathbf{x}_n) \rangle \tag{1.2}$$

generated by this subspace. The *coinvariant algebra* $R_n$ is the corresponding quotient:

$$R_n := \mathbb{Q}[\mathbf{x}_n]/I_n. \tag{1.3}$$

The algebra $R_n$ is a graded $S_n$-module.

The coinvariant algebra is among the most important representations in algebraic combinatorics; algebraic properties of $R_n$ are deeply tied to combinatorial properties of permutations in $S_n$. E. Artin proved [2] that the collection of 'sub-staircase' monomials $\{x_1^{i_1} \cdots x_n^{i_n} \,:\, 0 \leqslant i_j < j\}$ descends to a vector space basis for $R_n$, so that the Hilbert series of $R_n$ is given by

$$\mathrm{Hilb}(R_n; q) = (1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}) = [n]!_q, \tag{1.4}$$

the standard $q$-analog of $n!$. Chevalley [4] proved that as an *ungraded* $S_n$-module, we have $R_n \cong \mathbb{Q}[S_n]$, the regular representation of $S_n$. Lusztig (unpublished) and Stanley [18] refined this result to describe the *graded* isomorphism type of $R_n$ in terms of the major index statistic on standard Young tableaux.

In this paper we will study the following generalization of the coinvariant algebra $R_n$. Recall that the degree $d$ *homogeneous symmetric function* in $\mathbb{Q}[\mathbf{x}_n]$ is given by $h_d(\mathbf{x}_n) := \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_d \leqslant n} x_{i_1} \cdots x_{i_d}$.

**Definition 1.1.** *Let $n, k,$ and $r$ be nonnegative integers with $r \leqslant n$. Let $I_{n,k,r} \subseteq \mathbb{Q}[\mathbf{x}_n]$ be the ideal*

$$I_{n,k,r} := \langle h_{k+1}(\mathbf{x}_n), h_{k+2}(\mathbf{x}_n), \dots, h_{k+n}(\mathbf{x}_n), e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \dots, e_{n-r+1}(\mathbf{x}_n) \rangle$$

*and let*

$$R_{n,k,r} := \mathbb{Q}[\mathbf{x}_n]/I_{n,k,r}$$

*be the corresponding quotient ring.*

The ideal $I_{n,k,r}$ is homogeneous and stable under the action of the symmetric group, so that $R_{n,k,r}$ is a graded $S_n$-module. Since the generators of $I_{n,k,r}$ are symmetric polynomials, we have the containment of ideals $I_{n,k,r} \subseteq I_n$, so that $R_{n,k,r}$ projects onto the classical coinvariant algebra $R_n$. If $r = n$ or $k = 0$ we have the equality $I_{n,k,r} = I_n$, so that $R_{n,k,r} = R_n$.

Just as algebraic properties of $R_n$ are controlled by combinatorics of permutations $\pi_1 \dots \pi_n$ of the set $\{1, 2, \dots, n\}$, algebraic properties of $R_{n,k,r}$ will be controlled by the combinatorics of permutations $\pi_1 \dots \pi_{n+k}$ of the *multiset* $\{0^k, 1, 2, \dots, n\}$ whose last $r$ entries $\pi_{n+k-r+1} \dots \pi_{n+k-1}\pi_{n+k}$ are all nonzero. We will call such permutations *$r$-tail positive*. Let $S_{n,k,r}$ be the collection of all $r$-tail positive permutations of the multiset $\{0^k, 1, 2, \dots, n\}$. For example, we have

$$S_{2,2,1} = \{0012, 0021, 0102, 0201, 1002, 2001\}.$$

By considering the possible locations of the $k$ 0's in an element of $S_{n,k,r}$, it is immediate that

$$|S_{n,k,r}| = \binom{n+k-r}{k} \cdot |S_n| = \binom{n+k-r}{k} \cdot n!. \tag{1.5}$$

The basic enumeration of Equation 1.5 will manifest (see Theorem 3.6) in Hilbert series as

$$\mathrm{Hilb}(R_{n,k,r}; q) = \begin{bmatrix} n+k-r \\ k \end{bmatrix}_q \cdot \mathrm{Hilb}(R_n; q) = \begin{bmatrix} n+k-r \\ k \end{bmatrix}_q \cdot [n]!_q, \tag{1.6}$$

where ${\begin{bmatrix} m \\ i \end{bmatrix}}_q := \frac{[m]!_q}{[i]!_q [m-i]!_q}$ is the usual $q$-binomial coefficient. Going even further, we have (see Theorem 4.2) the following graded Frobenius image

$$\mathrm{grFrob}(R_{n,k,r}; q) = \begin{bmatrix} n+k-r \\ k \end{bmatrix}_q \cdot \mathrm{grFrob}(R_n; q) = \begin{bmatrix} n+k-r \\ k \end{bmatrix}_q \cdot \sum_{T \in \mathrm{SYT}(n)} s_{\mathrm{shape}(T)}, \tag{1.7}$$

which implies that the quotient $R_{n,k,r}$ consists of $\binom{n+k-r}{k}$ copies of the coinvariant algebra $R_n$, with grading shifts given by a $q$-binomial coefficient. The authors know of no direct way to see this from Definition 1.1.

The ideal $I_{n,k,r}$ defining the quotient $R_{n,k,r}$ is of 'mixed' type – its generators come in two flavors: the homogeneous symmetric functions $h_{k+1}(\mathbf{x}_n), h_{k+2}(\mathbf{x}_n), \dots, h_{k+n}(\mathbf{x}_n)$ and the elementary symmetric functions $e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \dots, e_{n-r+1}(\mathbf{x}_n)$. Several mixed ideals have recently been introduced to give combinatorial generalizations of the coinvariant algebra.

- Let $k \leqslant n$. Haglund, Rhoades, and Shimozono [11] studied the quotient of $\mathbb{Q}[\mathbf{x}_n]$ by the ideal
$$\langle x_1^k, x_2^k, \dots, x_n^k, e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \dots, e_{n-k+1}(\mathbf{x}_n) \rangle. \tag{1.8}$$
The generators are high degree $S_n$-invariants $e_n(\mathbf{x}_n), \dots, e_{n-k+1}(\mathbf{x}_n)$ together with a homogeneous system of parameters $x_1^k, \dots, x_n^k$ of degree $k$ carrying the defining representation of $S_n$. Algebraic properties of the corresponding quotient are controlled by combinatorial properties of $k$-block ordered set partitions of $\{1, 2, \dots, n\}$.

- Let $r \geqslant 2$ and let $\mathbb{Z}_r \wr S_n$ be the group of $n \times n$ monomial matrices whose nonzero entries are $r^{th}$ complex roots of unity (this is the group of '$r$-colored permutations' of $\{1, 2, \dots, n\}$). Let $k \leqslant n$ be non-negative integers. Chan and Rhoades [3] studied the quotient of $\mathbb{C}[\mathbf{x}_n]$ by the ideal
$$\langle x_1^{kr+1}, x_2^{kr+1}, \dots, x_n^{kr+1}, e_n(\mathbf{x}_n^r), e_{n-1}(\mathbf{x}_n^r), \dots, e_{n-k+1}(\mathbf{x}_n^r) \rangle, \tag{1.9}$$

where $f(\mathbf{x}_n^r) = f(x_1^r, \ldots, x_n^r)$ for any polynomial $f$. The generators here are high degree $\mathbb{Z}_r \wr S_n$-invariants $e_n(\mathbf{x}_n^r), \ldots, e_{n-k+1}(\mathbf{x}_n^r)$ together with a h.s.o.p. $x_1^{kr+1}, \ldots, x_n^{kr+1}$ of degree $kr+1$ carrying the dual of the defining representation of $\mathbb{Z}_r \wr S_n$. Algebraic properties of the corresponding quotient are controlled by $k$-dimensional faces in the Coxeter complex attached to $\mathbb{Z}_r \wr S_n$.

- Let $\mathbb{F}$ be any field and let $H_n(0)$ be the 0-Hecke algebra over $\mathbb{F}$ of rank $n$; the algebra $H_n(0)$ acts on the polynomial ring $\mathbb{F}[\mathbf{x}_n]$ by isobaric divided difference operators. Let $k \leqslant n$ be positive integers. Huang and Rhoades [14] studied the quotient of $\mathbb{F}[\mathbf{x}_n]$ by the ideal

$$\langle h_k(x_1), h_k(x_1, x_2), \ldots, h_k(x_1, x_2, \ldots, x_n), e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \ldots, e_{n-k+1}(\mathbf{x}_n) \rangle. \quad (1.10)$$

Once again, the generators consist of high degree $H_n(0)$-invariants

$$e_n(\mathbf{x}_n), \ldots, e_{n-k+1}(\mathbf{x}_n)$$

together with a h.s.o.p. of degree $k$ carrying the defining representation of $H_n(0)$. Algebraic properties of the quotient are controlled by 0-Hecke combinatorics of $k$-block ordered set partitions of $\{1, \ldots, n\}$.

The novelty of this paper is that our mixed ideals consist of high degree invariants of different kinds: elementary and homogeneous. It would be interesting to develop a more unified picture of the algebraic and combinatorial properties of mixed quotients of polynomial rings.

Our analysis of the rings $R_{n,k,r}$ will share many properties with the analyses of the previously mentioned mixed quotients. Since the generators $I_{n,k,r}$ do not form a regular sequence of homogeneous polynomials in $\mathbb{Q}[\mathbf{x}_n]$, the usual commutative algebra tools (e.g. the Koszul complex) used to study the coinvariant algebra $R_n$ are unavailable to us. These will be replaced by *combinatorial* commutative algebra tools (e.g. Gröbner theory). We will see that the ideal $I_{n,k,r}$ has an explicit minimal Gröbner basis (with respect to the lexicographic term order) in terms of Demazure characters. This Gröbner basis will yield the Hilbert series of $R_{n,k,r}$, as well as an identification of its standard monomial basis. The graded $S_n$-isomorphism type of $R_{n,k,r}$ will then be obtainable by constructing an appropriate short exact sequence to serve as a recursion.

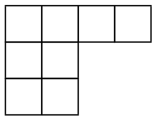
The rest of the paper is organized as follows. In **Section 2** we give background related to symmetric functions and Gröbner bases. In **Section 3** we determine the Hilbert series of $R_{n,k,r}$ and calculate the standard monomial basis for $R_{n,k,r}$ with respect to the lexicographic term order. In **Section 4** we determine the graded $S_n$-isomorphism type of $R_{n,k,r}$. We also view $R_{n,k,r}$ as a module over the 0-Hecke algebra $H_n(0)$ and calculate its graded noncommutative and bigraded quasisymmetric 0-Hecke characteristics. We close in **Section 5** with some open problems.

# 2 Background

## 2.1 Words, partitions, and tableaux

Let $w = w_1 \dots w_n$ be a word in the alphabet of nonnegative integers. An index $1 \leqslant i \leqslant n-1$ is a *descent* of $w$ if $w_i > w_{i+1}$. The *descent set* of $w$ is $\mathrm{Des}(w) := \{1 \leqslant i \leqslant n-1 : w_i > w_{i+1}\}$ and the *major index* of $w$ is $\mathrm{maj}(w) := \sum_{i \in \mathrm{Des}(w)} i$. A pair of indices $1 \leqslant i < j \leqslant n$ is called an *inversion* of $w$ if $w_i > w_j$; the *inversion number* $\mathrm{inv}(w)$ counts the inversions of $w$. The word $w$ is called *$r$-tail positive* if its last $r$ letters $w_{n-r+1} \dots w_{n-1} w_n$ are positive.

Let $n \in \mathbb{Z}_{\geqslant 0}$. A *partition* $\lambda$ of $n$ is a weakly decreasing sequence $\lambda = (\lambda_1 \geqslant \cdots \geqslant \lambda_k)$ of positive integers with $\lambda_1 + \cdots + \lambda_k = n$. We write $\lambda \vdash n$ to indicate that $\lambda$ is a partition of $n$ and use $|\lambda| = n$ to denote the sum of the parts of $\lambda$. The *Ferrers diagram* of $\lambda$ (in English notation) consists of $\lambda_i$ left-justified boxes in row $i$. The Ferrers diagram of $(4, 2, 2) \vdash 8$ is shown below on the left.



| | | | |
|---|---|---|---|
| 1 | 1 | 2 | 5 |
| 2 | 3 | | |
| 3 | 5 | | |

| | | | |
|---|---|---|---|
| 1 | 2 | 5 | 8 |
| 3 | 4 | | |
| 6 | 7 | | |

Let $\lambda \vdash n$. A *tableau* $T$ of shape $\lambda$ is a filling of the Ferrers diagram of $\lambda$ with positive integers. The tableau $T$ is called *semistandard* if its entires increase weakly across rows and strictly down columns. The tableau $T$ is a *standard Young tableau* if it is semistandard and its entries consist of $1, 2, \dots, n$. The tableau in the center above is semistandard and the tableau on the right above is standard. We let $\mathrm{shape}(T) = \lambda$ denote the *shape* of $T$ and let $\mathrm{SYT}(n)$ denote the collection of all standard Young tableaux with $n$ boxes.

Given a standard tableau $T \in \mathrm{SYT}(n)$, an index $1 \leqslant i \leqslant n-1$ is a *descent* of $T$ if $i+1$ appears in a lower row of $T$ than $i$. Let $\mathrm{Des}(T)$ denote the set of descents of $T$ and let $\mathrm{maj}(T) := \sum_{i \in \mathrm{Des}(T)} i$ be the *major index* of $T$. If $T$ is the standard tableau above we have $\mathrm{Des}(T) = \{2, 5\}$ so that $\mathrm{maj}(T) = 2 + 5 = 7$.

## 2.2 Symmetric functions

Let $\Lambda$ denote the ring of symmetric functions in an infinite variable set $\mathbf{x} = (x_1, x_2, \dots)$ over the ground field $\mathbb{Q}(q,t)$. The algebra $\Lambda$ is graded by degree: $\Lambda = \bigoplus_{n \geqslant 0} \Lambda_n$. The graded piece $\Lambda_n$ has dimension equal to the number of partitions $\lambda \vdash n$.

The vector space $\Lambda_n$ has many interesting bases, all indexed by partitions of $n$. Given $\lambda \vdash n$, let

$$m_\lambda, \;\; e_\lambda, \;\; h_\lambda, \;\; p_\lambda, \;\; s_\lambda, \;\; \widetilde{H}_\lambda$$

denote the associated *monomial, elementary, homogeneous, power sum, Schur,* and *modified Macdonald* symmetric function (respectively). As $\lambda$ ranges over the collection of partitions of $n$, all of these form bases for the vector space $\Lambda_n$.

Let $f = f(x_1, x_2, \dots) \in \Lambda$ be a symmetric function. We define an eigenoperator $\Delta_f : \Lambda \to \Lambda$ for the modified Macdonald basis of $\Lambda$ as follows. Given a partition $\lambda$, we set

$$\Delta_f(\widetilde{H}_\lambda) := f(\dots, q^{i-1} t^{j-1}, \dots) \cdot \widetilde{H}_\lambda, \tag{2.1}$$

where $(i, j)$ ranges over all matrix coordinates of cells in the Ferrers diagram of $\lambda$, and all other variables appearing in $f$ are set to 0. The symmetry of $f$ guarantees that the eigenvalue $f(\dots, q^{i-1}t^{j-1}, \dots)$ is independent of which variables $x_k$ are evaluated at which monomials $q^{i-1}t^{j-1}$, but we will replace variables by monomials by working along the rows of $\lambda$ from left to right, going from top to bottom. The reader familiar with plethysm will recognize this formula as

$$\Delta_f(\widetilde{H}_\lambda) := f[B_\lambda] \cdot \widetilde{H}_\lambda. \tag{2.2}$$

For example, if $\lambda = (3, 2) \vdash 5$, we fill the boxes of $\lambda$ with monomials

| | | |
|---|---|---|
| $1$ | $q$ | $q^2$ |
| $t$ | $qt$ | |

and see that

$$\Delta_f(\widetilde{H}_{(3,2)}) = f(1, q, q^2, t, qt) \cdot \widetilde{H}_{(3,2)}.$$

When $f = e_n$, the restriction of the delta operator $\Delta_{e_n}$ to the space $\Lambda_n$ of homogeneous degree $n$ symmetric functions is more commonly denoted $\nabla$:

$$\Delta_{e_n} |_{\Lambda_n} = \nabla. \tag{2.3}$$

In particular, we have $\Delta_{e_n} e_n = \nabla e_n$.

Given a partition $\lambda \vdash n$, let $S^\lambda$ denote the associated irreducible representation of the symmetric group $S_n$; for example, we have that $S^{(n)}$ is the trivial representation and $S^{(1^n)}$ is the sign representation. Given any finite-dimensional $S_n$-module $V$, there exist unique integers $c_\lambda$ such that $V \cong_{S_n} \bigoplus_{\lambda \vdash n} c_\lambda S^\lambda$. The *Frobenius character* of $V$ is the symmetric function

$$\mathrm{Frob}(V) := \sum_{\lambda \vdash n} c_\lambda \cdot s_\lambda \tag{2.4}$$

obtained by replacing the irreducible $S^\lambda$ with the Schur function $s_\lambda$.

If $V = \bigoplus_{d \geqslant 0} V_d$ is a graded vector space, the *Hilbert series* of $V$ is the power series

$$\mathrm{Hilb}(V; q) = \sum_{d \geqslant 0} \dim(V_d) \cdot q^d. \tag{2.5}$$

Similarly, if $V = \bigoplus_{d \geqslant 0} V_d$ is a graded $S_n$-module, the *graded Frobenius character* of $V$ is

$$\mathrm{grFrob}(V; q) = \sum_{d \geqslant 0} \mathrm{Frob}(V_d) \cdot q^d. \tag{2.6}$$

### 2.3 Quasisymmetric and nonsymmetric functions

The space $\Lambda$ of symmetric functions has many generalizations; in this paper we will also use the spaces QSym of quasisymmetric functions and **NSym** of noncommutative symmetric functions. We briefly review their definition below, as well as their relationship with the 0-Hecke algebra $H_n(0)$; for more details see [13, 14].

Let $n$ be a positive integer. A (strong) composition $\alpha$ of $n$ is a sequence $(\alpha_1, \ldots, \alpha_k)$ of positive integers with $\alpha_1 + \cdots + \alpha_k = n$. We write $\alpha \models n$ to indicate that $\alpha$ is a composition of $n$ and $|\alpha| = n$ to denote the sum of the parts of $\alpha$. The map $\alpha = (\alpha_1, \ldots, \alpha_k) \mapsto \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\}$ provides a bijection between compositions of $n$ and subsets of $[n-1]$; we will find it convenient to identify compositions with subsets.

Let $S \subseteq [n-1]$ be a subset. The *Gessel fundamental quasisymmetric function* $F_S = F_{S,n}$ attached to $S$ is the degree $n$ formal power series

$$F_S = F_{S,n} := \sum_{\substack{i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n \\ j \in S \Rightarrow i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}. \tag{2.7}$$

The space QSym of *quasisymmetric functions* is the $\mathbb{Q}(q,t)$-algebra of formal power series with basis given by $\{F_{S,n} : n \geqslant 0, S \subseteq [n-1]\}$. If a subset $S \subseteq [n-1]$ corresponds to a composition $\alpha$, we set $F_\alpha := F_{S,n}$.

For any composition $\alpha \models n$, define a symbol $\mathbf{s}_\alpha$ (the *noncommutative ribbon Schur function*), formally defined to have homogeneous degree $n$. Let $\mathbf{NSym}_n$ be the $2^{n-1}$-dimensional $\mathbb{Q}(q,t)$-vector space with basis $\{\mathbf{s}_\alpha : \alpha \models n\}$ and let $\mathbf{NSym}$ be the graded vector space $\mathbf{NSym} := \bigoplus_{n \geqslant 0} \mathbf{NSym}_n$. The space $\mathbf{NSym}$ is the space of *noncommutative symmetric functions*. Although there is more structure on $\mathbf{NSym}$ (and on QSym) than the graded vector space structure (namely, they are dual graded Hopf algebras), only the vector space structure will be relevant in this paper.

Let $\mathbb{F}$ be an arbitrary field. The *0-Hecke algebra* $H_n(0)$ of rank $n$ over $\mathbb{F}$ is the unital associative $\mathbb{F}$-algebra with generators $T_1, \ldots, T_{n-1}$ and relations

$$\begin{cases} T_i^2 = T_i & 1 \leqslant i \leqslant n-1 \\ T_i T_j = T_j T_i & |i-j| > 1 \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} & 1 \leqslant i \leqslant n-2. \end{cases} \tag{2.8}$$

For all $1 \leqslant i \leqslant n-1$, let $s_i := (i, i+1) \in S_n$ be the corresponding adjacent transposition. Given a permutation $\pi \in S_n$, we define $T_\pi := T_{i_1} \cdots T_{i_k} \in H_n(0)$, where $\pi = s_{i_1} \cdots s_{i_k}$ is a reduced (i.e., as short as possible) expression for $\pi$ as a product of adjacent transpositions. The element $T_\pi \in H_n(0)$ is independent of the choice of reduced expression $\pi = s_{i_1} \cdots s_{i_k}$ since any two such reduced expressions may be connected by 'braid moves' mirroring the defining relations of $H_n(0)$ (see, for example, [15]). It can be shown that $\{T_\pi : \pi \in S_n\}$ is a $\mathbb{F}$-basis for $H_n(0)$, so that $H_n(0)$ has dimension $n!$ as a $\mathbb{F}$-vector space and may be viewed as a deformation of the group algebra $\mathbb{F}[S_n]$. The algebra $H_n(0)$ is not semisimple, even when the field $\mathbb{F}$ has characteristic zero, so its representation theory has a different flavor from that of $S_n$.

The indecomposable projective representations of $H_n(0)$ are naturally labeled by compositions $\alpha \models n$ (see [13, 14]). For $\alpha \models n$, we let $P_\alpha$ denote the corresponding indecomposable projective and let

$$C_\alpha := \mathrm{top}(P_\alpha) = P_\alpha / \mathrm{rad}(P_\alpha) \tag{2.9}$$

be the corresponding irreducible $H_n(0)$-module. The set $\{C_\alpha : \alpha \models n\}$ forms a complete set of nonisomorphic irreducible $H_n(0)$-modules.

The Grothendieck group $G_0(H_n(0))$ is the $\mathbb{Z}$-module generated by all isomorphism classes $[V]$ of finite-dimensional $H_n(0)$-modules with a relation $[V]-[U]-[W]=0$ for every short exact sequence $0 \to U \to V \to W \to 0$ of $H_n(0)$-modules. The $\mathbb{Z}$-module $G_0(H_n(0))$ is free with basis given by (isomorphism classes of) the irreducibles $\{C_\alpha : \alpha \models n\}$. The *quasisymmetric characteristic map* Ch is defined on $G_0(H_n(0))$ by

$$\mathrm{Ch} : C_\alpha \mapsto F_\alpha. \tag{2.10}$$

If a $H_n(0)$-module $V$ has composition factors $C_{\alpha^{(1)}}, \ldots, C_{\alpha^{(k)}}$, then $\mathrm{Ch}(V) = F_{\alpha^{(1)}} + \cdots + F_{\alpha^{(k)}}$. Since $H_n(0)$ is not semisimple, the characteristic $\mathrm{Ch}(V)$ does *not* determine $V$ up to isomorphism.

Let $K_0(H_n(0))$ be the $\mathbb{Z}$-module generated by all isomorphism classes $[P]$ of finite-dimensional *projective* $H_n(0)$-modules with a relation $[P]-[Q]-[R]=0$ for every short exact sequence $0 \to Q \to P \to R \to 0$ of projective modules. The $\mathbb{Z}$-module $K_0(H_n(0))$ is free with basis given by (isomorphism classes of) the projective indecomposable $\{P_\alpha : \alpha \models n\}$. The *noncommutative characteristic map* **ch** is defined on $K_0(H_n(0))$ by

$$\mathbf{ch} : P_\alpha \mapsto \mathbf{s}_\alpha, \tag{2.11}$$

This extends to give a noncommutative symmetric function $\mathbf{ch}(P)$ for any projective $H_n(0)$-module $P$. Since any short exact sequence $0 \to U \to V \to W \to 0$ of *projective* $H_n(0)$-modules splits, a projective module $P$ is determined by $\mathbf{ch}(P)$ up to isomorphism.

There are graded refinements of the maps Ch and **ch**. Let $V = \bigoplus_{d \geqslant 0} V_d$ be a graded $H_n(0)$-module with each $V_d$ finite-dimensional. The *degree-graded quasisymmetric characteristic* is $\mathrm{Ch}_q(V) := \sum_{d \geqslant 0} \mathrm{Ch}(V_d) \cdot q^d$. If each $V_d$ is projective, the *degree-graded noncommutative characteristic* is $\mathbf{ch}_q(V) := \sum_{d \geqslant 0} \mathbf{ch}(V_d) \cdot q^d$.

The quasisymmetric characteristic Ch admits a bigraded refinement as follows. The 0-Hecke algebra $H_n(0)$ has a *length filtration*

$$H_n(0)^{(0)} \subseteq H_n(0)^{(1)} \subseteq H_n(0)^{(2)} \subseteq \cdots \tag{2.12}$$

where $H_n(0)^{(\ell)}$ is the subspace of $H_n(0)$ with $\mathbb{F}$-basis $\{T_\pi : \pi \in S_n, \mathrm{inv}(\pi) \geqslant \ell\}$. If $V = H_n(0)v$ is a cyclic $H_n(0)$-module with distinguished generator $v$, we get an induced length filtration of $V$ by

$$V^{(\ell)} := H_n(0)^{(\ell)} v. \tag{2.13}$$

The *length-graded quasisymmetric characteristic* is given by

$$\mathrm{Ch}_t(V) := \sum_{\ell \geqslant 0} \mathrm{Ch}(V^{(\ell)}/V^{(\ell-1)}) \cdot t^\ell. \tag{2.14}$$

Now suppose $V = \bigoplus_{d \geqslant 0} V_d$ is a graded $H_n(0)$-module which is also cyclic. We get a bifiltration of $V$ consisting of the modules $V^{(\ell,d)} := V^{(\ell)} \cap V_d$ for $\ell, d \geqslant 0$. The *length-degree-bigraded quasisymmetric characteristic* is

$$\mathrm{Ch}_{q,t}(V) := \sum_{\ell, d \geqslant 0} \mathrm{Ch}(V^{(\ell,d)}/V^{(\ell-1,d)}) \cdot q^d t^\ell. \tag{2.15}$$

More generally, if $V$ is a direct sum of graded cyclic $H_n(0)$-modules, we define $\mathrm{Ch}_{q,t}(V)$ by applying $\mathrm{Ch}_{q,t}$ to its cyclic summands. This may depend on the cyclic decomposition of the module $V$. [1]

### 2.4 Gröbner theory

A total order $<$ on the monomials in the polynomial ring $\mathbb{Q}[\mathbf{x}_n]$ is called a *term order* if

- we have $1 \leqslant m$ for all monomials $m$, and
- $m \leqslant m'$ implies $m \cdot m'' \leqslant m' \cdot m''$ for all monomials $m, m', m''$.

The term order used in this paper is the *lexicographic* term order given by $x_1^{a_1} \cdots x_n^{a_n} < x_1^{b_1} \cdots x_n^{b_n}$ if there exists an index $i$ with $a_1 = b_1, \ldots, a_{i-1} = b_{i-1}$ and $a_i < b_i$.

If $<$ is any term order any $f \in \mathbb{Q}[\mathbf{x}_n]$ is any nonzero polynomial, let $\mathrm{in}_<(f)$ be the leading (i.e., greatest) term of $f$ with respect to the order $<$. If $I \subseteq \mathbb{Q}[\mathbf{x}_n]$ is any ideal, the associated *initial ideal* is

$$\mathrm{in}_<(I) := \langle \mathrm{in}_<(f) \,:\, f \in I - \{0\} \rangle. \tag{2.16}$$

A finite collection $G = \{g_1, \ldots, g_r\}$ of nonzero polynomials in an ideal $I \subseteq \mathbb{Q}[\mathbf{x}_n]$ is called a *Gröbner basis* of $I$ if we have the equality of monomial ideals

$$\mathrm{in}_<(I) = \langle \mathrm{in}_<(g_1), \ldots, \mathrm{in}_<(g_r) \rangle. \tag{2.17}$$

If $G$ is a Gröbner basis of $I$ it follows that $I = \langle G \rangle$. A Gröbner basis $G = \{g_1, \ldots, g_r\}$ is called *minimal* if the $<$-leading coefficient of each $g_i$ is 1 and $\mathrm{in}(g_i) \nmid \mathrm{in}(g_j)$ for all $i \neq j$. A minimal Gröbner basis $G = \{g_1, \ldots, g_r\}$ is called *reduced* if in addition, for all $i \neq j$, no term of $g_j$ is divisible by $\mathrm{in}_<(g_i)$. After fixing a term order, every ideal $I \subseteq \mathbb{Q}[\mathbf{x}_n]$ has a unique reduced Gröbner basis (see [5, Prop. 6, p. 92]).

Let $I \subseteq \mathbb{Q}[\mathbf{x}_n]$ be an ideal and let $G$ be a Gröbner basis for $I$. The set of monomials in $\mathbb{Q}[\mathbf{x}_n]$

$$\{m \,:\, \mathrm{in}_<(f) \nmid m \text{ for all } f \in I - \{0\}\} = \{m \,:\, \mathrm{in}_<(g) \nmid m \text{ for all } g \in G\} \tag{2.18}$$

descends to a vector space basis for the quotient $\mathbb{Q}[\mathbf{x}_n]/I$ (see [5, Prop. 1, p. 230]). This is called the *standard monomial basis*; it is completely determined by the ideal $I$ and the term order $<$. If $I$ is a homogeneous ideal, the Hilbert series of $\mathbb{Q}[\mathbf{x}_n]/I$ is given by

$$\mathrm{Hilb}(\mathbb{Q}[\mathbf{x}_n]/I; q) = \sum_m q^{\deg(m)}, \tag{2.19}$$

where the sum is over all monomials in the standard monomial basis.

[1] Our conventions for $q$ and $t$ in the definitions of $\mathbf{ch}_q$ and $\mathrm{Ch}_{q,t}$ are reversed with respect to those in [13, 14] and elsewhere. We make these conventions so as to be consistent with the case of the graded Frobenius map on $S_n$-modules.

## 3 Hilbert series

In this section we will derive the Hilbert series and ungraded isomorphism type of the $S_n$-module $R_{n,k,r}$. The method that we use dates back to Garsia and Procesi in the context of Tanisaki ideals and quotients [7].

Let $Y \subseteq \mathbb{Q}^n$ be any finite set of points [2] and consider the ideal $\mathbf{I}(Y) \subseteq \mathbb{Q}[\mathbf{x}_n]$ of polynomials which vanish on $Y$. That is, we have

$$\mathbf{I}(Y) = \{f \in \mathbb{Q}[\mathbf{x}_n] \,:\, f(\mathbf{y}) = 0 \text{ for all } \mathbf{y} \in X\}. \tag{3.1}$$

We may identify ([7, p. 95]) the quotient $\mathbb{Q}[\mathbf{x}_n]/\mathbf{I}(Y)$ with the collection of all (polynomial) functions $Y \to \mathbb{Q}$; since $Y$ is finite we have

$$|Y| = \dim(\mathbb{Q}[\mathbf{x}_n]/\mathbf{I}(Y)). \tag{3.2}$$

If $Y$ is stable under the coordinate permutation action of $S_n$, we have the further identification of $S_n$-modules

$$\mathbb{Q}[Y] \cong_{S_n} \mathbb{Q}[\mathbf{x}_n]/\mathbf{I}(Y). \tag{3.3}$$

The ideal $\mathbf{I}(Y)$ is usually not homogeneous; we wish to replace it by a homogeneous ideal so that the associated quotient is graded. For any nonzero polynomial $f \in \mathbf{I}(X)$, write $f = f_d + \cdots + f_1 + f_0$ where $f_i$ is homogeneous of degree $i$ and $f_d \neq 0$. Let $\tau(f) = f_d$ be the top homogeneous component of $f$. The ideal $\mathbf{T}(Y) \subseteq \mathbb{Q}[\mathbf{x}_n]$ is given by

$$\mathbf{T}(Y) := \langle \tau(f) \,:\, f \in \mathbf{I}(Y) - \{0\}\rangle. \tag{3.4}$$

By construction the ideal $\mathbf{T}(Y)$ is homogeneous, so that the quotient $\mathbb{Q}[\mathbf{x}_n]/\mathbf{T}(Y)$ is graded. Furthermore, we still have the dimension equality

$$|Y| = \dim(\mathbb{Q}[\mathbf{x}_n]/\mathbf{I}(Y)) = \dim(\mathbb{Q}[\mathbf{x}_n]/\mathbf{T}(Y)) \tag{3.5}$$

and the $S_n$-module isomorphism

$$\mathbb{Q}[Y] \cong_{S_n} \mathbb{Q}[\mathbf{x}_n]/\mathbf{I}(Y) \cong_{S_n} \mathbb{Q}[\mathbf{x}_n]/\mathbf{T}(Y) \tag{3.6}$$

whenever the point set $Y$ is $S_n$-stable (see [7, Thm. 3.2]).

Recall that $S_{n,k,r}$ consists of all rearrangements $\pi = \pi_1 \dots \pi_{n+k}$ of the multiset

$$\{0^k, 1, 2, \dots, n\}$$

whose $r$-tail is positive. The symmetric group $S_n$ acts on $S_{n,k,r}$ by permuting the positive letters $1, 2, \dots, n$. We aim to prove that $R_{n,k,r} \cong \mathbb{Q}[S_{n,k,r}]$ as ungraded $S_n$-modules. To do this, our strategy is as follows.

1. Find a point set $Y_{n,k,r} \subseteq \mathbb{Q}^n$ which is stable under the action of $S_n$ such that there is a $S_n$-equivariant bijection from $Y_{n,k,r}$ to $S_{n,k,r}$.

[2] See [7, Sec. 3] for a more leisurely introduction when the point set $Y$ is a single $S_n$-orbit; the relevant facts and their proofs remain unchanged.

2. Prove that $I_{n,k,r} \subseteq \mathbf{T}(Y_{n,k,r})$ by showing that the generators of $I_{n,k,r}$ arise as top degree components of polynomials in $\mathbf{I}(Y_{n,k,r})$.

3. Prove that

$$\dim(R_{n,k,r}) = \dim(\mathbb{Q}[\mathbf{x}_n]/I_{n,k,r}) \leqslant |S_{n,k,r}| = \dim(\mathbb{Q}[\mathbf{x}_n]/\mathbf{T}(Y_{n,k,r}))$$

and use the relation $I_{n,k,r} \subseteq \mathbf{T}(Y_{n,k,r})$ to conclude that $I_{n,k,r} = \mathbf{T}(Y_{n,k,r})$.

The point set $Y_{n,k,r}$ which accomplishes Step 1 is the following.

**Definition 3.1.** *Fix $n+k$ distinct rational numbers $\alpha_1, \alpha_2, \ldots, \alpha_{n+k} \in \mathbb{Q}$. Let $Y_{n,k,r}$ be the set of points $(y_1, y_2, \ldots, y_n) \in \mathbb{Q}^n$ such that*

- *the coordinates $y_1, y_2, \ldots, y_n$ are distinct and lie in $\{\alpha_1, \alpha_2, \ldots, \alpha_{n+k}\}$, and*
- *the numbers $\alpha_{n+k-r+1}, \ldots, \alpha_{n+k-1}, \alpha_{n+k}$ all appear as coordinates of $(y_1, y_2 \ldots, y_n)$.*

It is clear that $Y_{n,k,r}$ is stable under the action of $S_n$. We have a natural identification of $Y_{n,k,r}$ with permutations in $S_{n,k,r}$ given by letting a copy of $\alpha_i$ in position $j$ of $(y_1, \ldots, y_n)$ correspond to the letter $j$ in position $i$ of the corresponding permutation in $S_{n,k,r}$. For example, if $(n,k,r) = (4,3,2)$ then

$$(\alpha_7, \alpha_2, \alpha_4, \alpha_6) \leftrightarrow 0203041.$$

This bijection $Y_{n,k,r} \leftrightarrow S_{n,k,r}$ is clearly $S_n$-equivariant, so Step 1 of our strategy is accomplished. Step 2 of our strategy is achieved in the following lemma.

**Lemma 3.2.** *We have $I_{n,k,r} \subseteq \mathbf{T}(Y_{n,k,r})$.*

*Proof.* We show that every generator of $I_{n,k,r}$ arises as the leading term of a polynomial in $\mathbf{I}(Y_{n,k,r})$. We begin with the elementary symmetric function generators

$$e_{n-r+1}(\mathbf{x}_n), \ldots, e_{n-1}(\mathbf{x}_n), e_n(\mathbf{x}_n).$$

Consider the rational function in $t$ given by

$$\frac{(1-x_1t)(1-x_2t)\cdots(1-x_nt)}{(1-\alpha_{n+k-r+1}t)\cdots(1-\alpha_{n+k-1}t)(1-\alpha_{n+k}t)} = \sum_{i,j \geqslant 0} (-1)^i e_i(\mathbf{x}_n) h_j(\alpha_{n+k-r+1}, \ldots, \alpha_{n+k}) \cdot t^{i+j}. \tag{3.7}$$

If $(x_1, \ldots, x_n) \in Y_{n,k,r}$, the $r$ factors in the denominator cancel with $r$ factors in the numerator, so that this rational expression is a polynomial in $t$ of degree $n-r$. In particular, for $n-r+1 \leqslant m \leqslant r$ taking the coefficient of $t^m$ on both sides gives

$$\sum_{i=0}^{m} (-1)^i e_i(\mathbf{x}_n) h_{m-i}(\alpha_{n+k-r+1}, \ldots, \alpha_{n+k}) \in \mathbf{I}(Y_{n,k,r}), \tag{3.8}$$

so that $e_m(\mathbf{x}_n) \in \mathbf{T}(Y_{n,k,r})$.

A similar trick shows that the homogeneous symmetric functions $h_{k+1}(\mathbf{x}_n), \ldots, h_{k+n}(\mathbf{x}_n)$ lie in $\mathbf{T}(Y_{n,k,r})$. Consider the rational function

$$\frac{(1-\alpha_1 t)(1-\alpha_2 t)\cdots(1-\alpha_{n+k} t)}{(1-x_1 t)(1-x_2 t)\cdots(1-x_n t)} = \sum_{i,j \geqslant 0} (-1)^j h_i(\mathbf{x}_n) e_j(\alpha_1, \ldots, \alpha_{n+k}) t^{i+j}. \tag{3.9}$$

If $(x_1, \ldots, x_n) \in Y_{n,k,r}$ the $n$ factors in the denominator cancel with $n$ factors in the numerator, giving a polynomial in $t$ of degree $k$. For $m \geqslant k+1$, taking the coefficient of $t^m$ on both sides gives

$$\sum_{i=0}^{m} (-1)^i h_{m-i}(\mathbf{x}_n) e_i(\alpha_1, \ldots, \alpha_{n+k}) \in \mathbf{I}(Y_{n,k,r}), \tag{3.10}$$

so that $h_m(\mathbf{x}_n) \in \mathbf{T}(Y_{n,k,r})$. □

Step 3 of our strategy will take more work. To begin, we identify a convenient collection of monomials in the initial ideal $\text{in}_<(I_{n,k,r})$ with respect to the lexicographic term order. Given a subset $S = \{s_1 < \cdots < s_m\} \subseteq [n]$ the corresponding *skip monomial* (see [11]) $\mathbf{x}(S)$ is given by

$$\mathbf{x}(S) := x_{s_1}^{s_1} x_{s_2}^{s_2-1} \cdots x_{s_m}^{s_m-m+1}. \tag{3.11}$$

In particular, if $n = 8$ we have $\mathbf{x}(2458) = x_2^2 x_4^3 x_5^3 x_8^5$. The adjective 'skip' refers to the fact that the exponent sequence of a skip monomial $\mathbf{x}(S)$ increases whenever the set $S$ skips an element of $[n]$.

**Lemma 3.3.** *Let $<$ be the lexicographic term order on $\mathbb{Q}[\mathbf{x}_n]$. If $S \subseteq [n]$ satisfies $|S| = n-r+1$ we have $\mathbf{x}(S) \in \text{in}_<(I_{n,k,r})$. Moreover, we have $x_1^{k+1}, x_2^{k+2}, \ldots, x_n^{k+n} \in \text{in}_<(I_{n,k,r})$.*

*Proof.* The assertion regarding skip monomials comes from combining [11, Lem. 3.4] (and in particular [11, Eqn. 3.5]) and [11, Lem. 3.5]. To prove the second assertion, the identities

$$h_{m+1}(x_i, x_{i+1}, \ldots, x_n) - x_i \cdot h_m(x_i, x_{i+1}, \ldots, x_n) = h_{m+1}(x_{i+1}, \ldots, x_n) \tag{3.12}$$

(for $1 \leqslant i \leqslant n$ and $m \geqslant 0$) imply that

$$h_{k+1}(x_1, \ldots, x_n), h_{k+2}(x_2, \ldots, x_n), \ldots, h_{k+n}(x_n) \in I_{n,k,r}, \tag{3.13}$$

so that $x_1^{k+1}, x_2^{k+2}, \ldots, x_n^{k+n} \in \text{in}_<(I_{n,k,r})$. □

The initial terms provided by Lemma 3.3 will be all we need. We name the monomials $m \in \mathbb{Q}[\mathbf{x}_n]$ which are not divisible by any of these initial terms as follows.

**Definition 3.4.** *A monomial $m \in \mathbb{Q}[\mathbf{x}_n]$ is $(n,k,r)$-*good *if*

- *we have $\mathbf{x}(S) \nmid m$ for all $S \subseteq [n]$ with $|S| = n-r+1$, and*
- *we have $x_i^{k+i} \nmid m$ for all $1 \leqslant i \leqslant n$.*

*Let $\mathcal{M}_{n,k,r}$ denote the set of all $(n,k,r)$-good monomials.*

By Lemma 3.3, the monomials in $\mathcal{M}_{n,k,r}$ contain the standard monomial basis of $R_{n,k,r}$, and so descend to a spanning set of $R_{n,k,r}$. We will see that $\mathcal{M}_{n,k,r}$ is in fact that standard monomial basis of $R_{n,k,r}$. We will do this using the following combinatorial result.

**Lemma 3.5.** *There is an bijection $\Psi : S_{n,k,r} \to \mathcal{M}_{n,k,r}$ with the property that $\deg(\Psi(\pi)) = \mathrm{inv}(\pi)$ for all $\pi \in S_{n,k,r}$.*

*Proof.* The map $\Psi$ will essentially be the inversion code. Let $\pi = \pi_1 \dots \pi_{n+k} \in S_{n,k,r}$ be a $r$-tail positive permutation of the multiset $\{0^k, 1, 2, \dots, n\}$. The *code* of $\pi$ is the sequence $(c_1, \dots, c_n)$ where

$$c_i = \text{the number of letters } 0, 1, 2, \dots, i-1 \text{ to the right of } i \text{ in } \pi. \tag{3.14}$$

For example, if $\pi = 40130052$ the code is $(c_1, c_2, c_3, c_4, c_5) = (2, 0, 3, 6, 1)$. It is clear that the sum of the code of $\pi$ gives the inversion number $\mathrm{inv}(\pi)$. If $\pi \in S_{n,k,r}$ has code $(c_1, \dots, c_n)$, we define $\Psi(\pi) := x_1^{c_1} \cdots x_n^{c_n}$.

We argue that $\Psi$ is a well defined function $S_{n,k,r} \to \mathcal{M}_{n,k,r}$, that is, we have $\Psi(\pi) \in \mathcal{M}_{n,k,r}$ for all $\pi \in S_{n,k,r}$. Let $\pi \in S_{n,k,r}$ have code $(c_1, \dots, c_n)$. Since $\pi$ contains $k$ copies of 0, it is clear that $c_i < k + i$ for all $1 \leqslant i \leqslant n$, so that $x_i^{k+i} \nmid \Psi(\pi)$ for all $1 \leqslant i \leqslant n$.

Now let $S = \{s_1 < \cdots < s_{n-r+1}\} \subseteq [n]$ and suppose $\mathbf{x}(S) \mid \Psi(\pi)$. This means that $c_{s_i} \geqslant s_i - i + 1$ for all $1 \leqslant i \leqslant n - r + 1$. Let $T = \{\pi_{n+k-r+1}, \dots, \pi_{n+k-1}, \pi_{n+k}\}$ be the $r$-tail of $\pi$; since $\pi \in S_{n,k,r}$ the set $T$ consists of $r$ positive numbers. We argue that $S \cap T = \varnothing$ as follows.

- If $s_1 \in T$ we would have $c_{s_1} \leqslant s_1 - 1$ (since $s_1$ could form inversions with only $1, 2, \dots, s_1 - 1$), contradicting the inequality $c_{s_1} \geqslant s_1$. We conclude that $s_1 \notin T$.
- If $s_1, \dots, s_{i-1} \notin T$ and $s_i \in T$, we would have $c_{s_i} \leqslant s_i - i$ (since $s_i$ can only form inversions with those letters in $1, 2, \dots, s_i - 1$ which lie in $T$), contradicting the inequality $c_{s_i} \geqslant s_i - i + 1$. We conclude that $s_i \notin T$.

Induction gives the result that $S \cap T = \varnothing$. However, this contradicts the facts that $|S| = n - r + 1$, $|T| = r$, and that there are a total of $n$ positive letters in $\pi$. This concludes the proof that the map $\Psi : S_{n,k,r} \to \mathcal{M}_{n,k,r}$ is well defined.

The relation $\deg(\Psi(\pi)) = \mathrm{inv}(\pi)$ is clear from construction. The fact that $\Psi$ is an injection is equivalent to the fact that a permutation $\pi = \pi_1 \dots \pi_{n+k} \in S_{n,k,r}$ is determined by its code $(c_1, \dots, c_n)$. This assertion is true more broadly for any permutation of the multiset $\{0^k, 1, 2, \dots, n\}$ (whether or not it is $r$-tail positive); we leave the verification to the reader.

It remains to show that $\Psi$ is surjective; we do this by induction on $n$. Let $m = x_1^{a_1} \cdots x_{n-1}^{a_{n-1}} x_n^{a_n}$ be an $(n,k,r)$-good monomial. Then $m' := x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}$ is an $(n-1, k, r-1)$-good monomial, so there exists a word $\pi' = \pi'_1 \pi'_2 \dots \pi'_{n+k-1} \in S_{n-1,k,r-1}$ with $\Psi(\pi') = m'$. Equivalently, the code $(c_1, \dots, c_{n-1})$ of $\pi'$ is $(a_1, \dots, a_{n-1})$. Let

$\pi = \pi_1 \dots \pi_{n+k}$ be the word obtained from $\pi'$ by inserting $n$ just before the length $a_n$ suffix $\pi'_{n+k-a_n} \cdots \pi'_{n+k-1}\pi'_{n+k-1}$ of $\pi'$. That is, we have (with dots inserted for legibility)

$$\pi = \pi'_1 \pi'_2 \dots \pi'_{n+k-a_n-1} \cdot n \cdot \pi'_{n+k-a_n} \cdots \pi'_{n+k-1}.$$

If we can show that $\pi \in S_{n,k,r}$, then $\Psi(\pi) = m$, completing the proof of the lemma. This amounts to showing that every letter in the $r$-tail of $\pi$ is positive. Since every letter in the $(r-1)$-tail of $\pi'$ is positive, this is true if $a_n < r$ or if $\pi'_{n+k-r}$ is positive, so we may assume $a_n \geqslant r$ and $\pi'_{n+k-r} = 0$.

Let $S = \{\pi'_1, \dots, \pi'_{n+k-r-1}\} \cap \{1, 2, \dots, n-1\}$ be the set of positive letters in $\pi'$ which do not lie in its $r$-tail, so that $|S| = n - r$ and $|S \cup \{n\}| = n - r + 1$. We claim that $\mathbf{x}(S \cup \{n\}) \mid m$, contradicting the assumption that $m$ is $(n, k, r)$-good. To do this, write $S = \{s_1 < \cdots < s_{n-r}\}$, so that $\mathbf{x}(S \cup \{n\}) = x_{s_1}^{s_1} x_{s_2}^{s_2 - 1} \cdots x_{s_{n-r}}^{s_{n-r}-n+r+1} x_n^r$. Since $m = x_1^{a_1} \cdots x_n^{a_n}$ and $a_n \geqslant r$, the exponent of $x_n$ in $m$ is $\geqslant$ the exponent of $x_n$ in $\mathbf{x}(S \cup \{n\})$. For $1 \leqslant i \leqslant n - r$, we have that

$$\begin{aligned}
(\text{exponent of } x_{s_i} \text{ in } m) &= (\text{exponent of } x_{s_i} \text{ in } m') \\
&= (\text{number of letters to the right of } s_i \text{ in } \pi' \text{ which are} < s_i) \\
&\leqslant |\{\pi'_{n+k-r+1}, \pi'_{n+k-r+2}, \dots, \pi'_{n+k-1}\} \cap \{1, 2, \dots, s_i - 1\}| + 1 \\
&= s_i - i + 1 \\
&= (\text{exponent of } x_{s_i} \text{ in } \mathbf{x}(S \cup \{n\}))
\end{aligned}$$

where the first equality uses $s_i < n$, the second is the definition of $\Psi$, the inequality considers only those inversions arising from the $r$-tail $0\pi'_{n+k-r+1} \cdots \pi'_{n+k-1}$ of $\pi'$, and the penultimate equality follows from the definition of $S$. We conclude that $\mathbf{x}(S \cup \{n\}) \mid m$, which was the desired contradiction. $\square$

We are ready to derive the Hilbert series of $R_{n,k,r}$.

**Theorem 3.6.** *Endow monomials in $\mathbb{Q}[\mathbf{x}_n]$ with the lexicographic term order. The standard monomial basis of $R_{n,k,r}$ is $\mathcal{M}_{n,k,r}$. The Hilbert series of $R_{n,k,r}$ is given by*

$$\mathrm{Hilb}(R_{n,k,r}; q) = \begin{bmatrix} n+k-r \\ k \end{bmatrix}_q \cdot [n]!_q. \tag{3.15}$$

*Proof.* Let $\mathcal{B}^{\mathbf{T}}$ be the standard monomial basis of $\mathbb{Q}[\mathbf{x}_n]/\mathbf{T}(Y_{n,k,r})$ and let $\mathcal{B}^J$ be the standard monomial basis of $R_{n,k,r} = \mathbb{Q}[\mathbf{x}_n]/I_{n,k,r}$. We know that $|S_{n,k,r}| = |\mathcal{B}^{\mathbf{T}}|$. Lemma 3.2 implies that $\mathcal{B}^{\mathbf{T}} \subseteq \mathcal{B}^J$. Lemma 3.3 further implies the containment $\mathcal{B}^J \subseteq \mathcal{M}_{n,k,r}$. Finally, Lemma 3.5 gives the relation $|\mathcal{M}_{n,k,r}| = |S_{n,k,r}|$. Putting these facts together gives

$$\mathcal{B}^{\mathbf{T}} = \mathcal{B}^J = \mathcal{M}_{n,k,r}, \tag{3.16}$$

and the fact that all of these sets have size $|S_{n,k,r}|$. In particular, the standard monomial basis of $R_{n,k,r}$ is $\mathcal{M}_{n,k,r}$.

By Lemma 3.5

$$\mathrm{Hilb}(R_{n,k,r}; q) = \sum_{m \in \mathcal{M}_{n,k,r}} q^{\deg(m)} = \sum_{\pi \in S_{n,k,r}} q^{\mathrm{inv}(\pi)}. \tag{3.17}$$

Given any fixed permutation $\pi \in S_n$, we can insert $k$ copies of 0 among the letters of $\pi$ while preserving a positive $r$-tail in $\binom{n+k-r}{k}$ ways. If we keep track of the effect of these 0's on inversion counts, we see $\sum_{\pi'} q^{\mathrm{inv}(\pi')} = {n+k-r \brack k}_q \cdot q^{\mathrm{inv}(\pi)}$, where $\pi'$ runs over all of the elements of $S_{n,k,r}$ so obtained from $\pi$. Applying the equality $\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} = [n]!_q$, we obtain

$$\sum_{\pi \in S_{n,k,r}} q^{\mathrm{inv}(\pi)} = {n+k-r \brack k}_q \cdot [n]!_q, \tag{3.18}$$

as desired. □

We can also derive the ungraded $S_n$-isomorphism type of the quotient $R_{n,k,r}$.

**Corollary 3.7.** *As an* ungraded *$S_n$-module we have $R_{n,k,r} \cong_{S_n} \mathbb{Q}[S_{n,k,r}]$.*

*Proof.* Lemma 3.2 and Theorem 3.6 give the isomorphisms

$$R_{n,k,r} \cong \mathbb{Q}[\mathbf{x}_n]/\mathbf{T}(Y_{n,k,r}) \cong \mathbb{Q}[\mathbf{x}_n]/\mathbf{I}(Y_{n,k,r}) \cong \mathbb{Q}[S_{n,k,r}] \tag{3.19}$$

of ungraded $S_n$-modules. □

We describe a minimal Gröbner basis for the ideal $I_{n,k,r}$. Given a subset $S = \{s_1 < \cdots < s_m\} \subseteq [n]$, let $\gamma(S) = (\gamma(S)_1, \dots, \gamma(S)_n)$ be the length $n$ *skip vector* of nonnegative integers given by

$$\gamma(S)_i = \begin{cases} s_j - j + 1 & i = s_j \\ 0 & i \notin S. \end{cases} \tag{3.20}$$

Let $\gamma(S)^* = (\gamma(S)_n, \dots, \gamma(S)_1)$ be the reversal of the vector $\gamma(S)$. If $\gamma = (\gamma_1, \dots, \gamma_n)$ is any length $n$ vector of nonnegative integers, let $\kappa_\gamma(\mathbf{x}_n) \in \mathbb{Q}[\mathbf{x}_n]$ be the associated *Demazure character* (see [11, Sec. 2] for its definition). Finally, if $f(\mathbf{x}_n) \in \mathbb{Q}[\mathbf{x}_n]$ is any polynomial, let $f(\mathbf{x}_n^*)$ be the polynomial obtained by reversing the variables in $f(\mathbf{x}_n)$ so that

$$f(\mathbf{x}_n^*) = f(x_n, x_{n-1}, \dots, x_1). \tag{3.21}$$

**Corollary 3.8.** *Endow monomials in $\mathbb{Q}[\mathbf{x}_n]$ with the lexicographic term order. A Gröbner basis for the ideal $I_{n,k,r}$ consists of the polynomials*

$$h_{k+1}(x_1, x_2, \dots, x_n), h_{k+2}(x_2, \dots, x_n), \dots, h_{k+n}(x_n)$$

*together with the polynomials*

$$\kappa_{\gamma(S)^*}(\mathbf{x}_n^*),$$

*where $S$ ranges over all $n-r+1$-element subsets of $[n]$. When $r < n$ and $k > 0$ this Gröbner basis is minimal.*

The Gröbner basis in Corollary 3.8 is typically not reduced.

*Proof.* The proof of Lemma 3.3 shows that the polynomial $h_{k+i}(x_i, x_{i+1}, \ldots, x_n)$ lies in the ideal $I_{n,k,r}$. By [11, Lem. 3.4] (and in particular [11, Eqn. 3.5]) the relevant variable reversed Demazure characters lie in $I_{n,k,r}$.

Let $<$ be the lexicographic term order on $\mathbb{Q}[\mathbf{x}_n]$. We have $\text{in}_<(h_{k+i}(x_i, x_{i+1}, \ldots, x_n)) = x_i^{k+i}$ and $\text{in}_<(\kappa_{\gamma(S)^*}(\mathbf{x}_n^*) = \mathbf{x}(S)$ (see [11, Lem. 3.5]). Since Theorem 3.6 tells us that $\mathcal{M}_{n,k,r}$ is the standard monomial basis of $R_{n,k,r}$, the definition of $\mathcal{M}_{n,k,r}$ shows that these initial terms generate $\text{in}_<(I_{n,k,r})$, proving the assertion about the claimed collection of polynomials being a Gröbner basis. When $r < n$ and $k > 0$, none of the relevant skip monomials $\mathbf{x}(S)$ are divisible by any of the variable powers $x_1^{k+1}, \ldots, x_n^{k+n}$. This proves the claim about minimality. $\square$

For example, consider the case $(n, k, r) = (5, 2, 3)$. A minimal Gröbner basis for $J_{5,2,3}$ is given by the polynomials

$$h_3(x_1, x_2, x_3, x_4, x_5), h_4(x_2, x_3, x_4, x_5), h_5(x_3, x_4, x_5), h_6(x_4, x_5), h_7(x_5)$$

together with the variable reversed Demazure characters

$$\begin{array}{ccccc} \kappa_{(0,0,1,1,1)}(\mathbf{x}_5^*), & \kappa_{(0,2,0,1,1)}(\mathbf{x}_5^*), & \kappa_{(3,0,0,1,1)}(\mathbf{x}_5^*), & \kappa_{(0,2,2,0,1)}(\mathbf{x}_5^*), & \kappa_{(3,0,2,0,1)}(\mathbf{x}_5^*), \\ \kappa_{(3,3,0,0,1)}(\mathbf{x}_5^*), & \kappa_{(0,2,2,2,0)}(\mathbf{x}_5^*), & \kappa_{(3,0,2,2,0)}(\mathbf{x}_5^*), & \kappa_{(3,3,0,2,0)}(\mathbf{x}_5^*), & \kappa_{(3,3,3,0,0)}(\mathbf{x}_5^*). \end{array}$$

Theorem 3.6 describes the standard monomial basis $\mathcal{M}_{n,k,r}$ of $R_{n,k,r}$ in terms of divisibility by skip monomials. However, a more direct characterization of this standard monomial basis is available. Let $k \geqslant 0$ and $r \leqslant n$. For any $(n-r)$-element subset $T \subseteq [n]$, define a length $n$ sequence $\delta(T) := (\delta(T)_1, \ldots, \delta(T)_n)$ by the formula

$$\delta(T)_i := \begin{cases} i + k - 1 & i \in T \\ j - 1 & i \notin T \text{ and } i = s_j, \end{cases} \tag{3.22}$$

where $[n] - T = \{s_1 < \cdots < s_r\}$. Any of the $\binom{n}{r}$ sequences which can be obtained in this way is an $(n, k, r)$-*staircase*. For example, the $(5, 2, 3)$-staircases are

$$\begin{array}{ccccc} (0,1,2,5,6), & (0,1,4,2,6), & (0,3,1,2,6), & (2,0,1,2,6), & (0,1,4,5,2), \\ (0,3,1,5,2), & (2,0,1,5,2), & (0,3,4,1,2), & (2,0,4,1,2), & (2,3,0,1,2). \end{array}$$

**Proposition 3.9.** *Endow monomials in $\mathbb{Q}[\mathbf{x}_n]$ with the lexicographic term order. The standard monomial basis $\mathcal{M}_{n,k,r}$ of $R_{n,k,r}$ consists of those monomials in $\mathbb{Q}[\mathbf{x}_n]$ whose exponent vectors are componentwise $\leqslant$ at least one $(n, k, r)$-staircase.*

*Proof.* Let $\mathcal{N}_{n,k,r}$ be the collection of monomials in $\mathbb{Q}[\mathbf{x}_n]$ whose exponent vectors are componentwise $\leqslant$ at least one $(n, k, r)$-staircase. Let $\delta_n(T) = (a_1, \ldots, a_n)$ be an $(n, k, r)$-staircase for some $(n-r)$-element set $T \subseteq [n]$ and let $m = x_1^{a_1} \cdots x_n^{a_n}$ be the corresponding monomial. We claim that $a_i < k + i$ for all $1 \leqslant i \leqslant n$, so that $x_i^{k+i} \nmid m$; indeed

- if $i \in T$ then $a_i = i + k - 1 < k + i$, and

- if $i \notin T$ then $a_i = |\{1, 2, \ldots, i-1\} - T| < i < k + i$.

If $S = \{s_1 < \cdots < s_{n-r+1}\} \subseteq [n]$ satisfies $|S| = n - r + 1$ then at least element $s_j \in S$ satisfies $s_j \notin T$; choose $j$ minimal such that $s_j \notin T$. Since the exponent of $x_{s_j}$ in $\mathbf{x}(S)$ is $s_j - j + 1$ and the exponent of $x_{s_j}$ in $m$ is $|\{1, 2, \ldots, s_j - 1\} - T| = (s_j - 1) - (j - 1) = s_j - j$ (by the minimality of $j$), we conclude $\mathbf{x}(S) \nmid m$. It follows that $\mathcal{N}_{n,k,r} \subseteq \mathcal{M}_{n,k,r}$.

On the other hand, we may construct a map

$$\Phi : S_{n,k,r} \to \mathcal{N}_{n,k,r} \tag{3.23}$$

by letting $\Phi(\pi) = (c_1, \ldots, c_n)$ be the code of any $r$-tail positive permutation $\pi \in S_{n,k,r}$. To see that $\Phi$ is well defined, let $\pi = \pi_1 \ldots \pi_{n+k} \in S_{n,k,r}$; we must show the code $(c_1, \ldots, c_n)$ of $\pi$ is componentwise $\leqslant$ some $(n, k, r)$-staircase. Indeed, let $T = [n] \cap \{\pi_1, \pi_2, \ldots, \pi_{n+k-r}\}$ be the set of positive letters in $\pi$ which are *not* contained in its $r$-tail, so that $|T| = n - r$ and write $\delta(T) = (a_1, \ldots, a_n)$. We claim that $c_i \leqslant a_i$ for all $i$; indeed

- if $i \in T$ then $a_i = k + i - 1$, but in $\pi$ the letter $i$ can only form inversions with $1, 2, \ldots, i-1$ and any of the $k$ copies of 0, forcing $c_i \leqslant k + i - 1$, and
- if $i \notin T$ then $i$ belongs to the $r$-tail of $\pi$ so that $i$ can only form inversions with the letters in $1, 2, \ldots, i-1$ which also lie in the $r$-tail of $\pi$, forcing $c_i \leqslant |\{1, 2, \ldots, i-1\} - T| = a_i$.

It is clear that $\Phi$ is injective, so that

$$|S_{n,k,r}| \leqslant |\mathcal{N}_{n,k,r}| \leqslant |\mathcal{M}_{n,k,r}| = |S_{n,k,r}| \tag{3.24}$$

and we have $\mathcal{N}_{n,k,r} = \mathcal{M}_{n,k,r}$, as desired. □

For example, if $(n, k, r) = (2, 2, 1)$ the $(2, 2, 1)$-staircases are $(0, 3)$ and $(2, 0)$ so that

$$\mathcal{M}_{2,2,1} = \{1, x_1, x_1^2, x_2, x_2^2, x_2^3\}.$$

## 4 Frobenius series

In this section we derive the Frobenius series of $R_{n,k,r}$. Our first lemma is a short exact sequence which establishes a Pascal-type recursion for $\mathrm{grFrob}(R_{n,k,r}; q)$.

**Lemma 4.1.** *Suppose $n, k, r \geqslant 0$ with $r < n$ and $k > 0$. There is a short exact sequence of $S_n$-modules*

$$0 \to R_{n,k-1,r} \to R_{n,k,r} \to R_{n,k,r+1} \to 0, \tag{4.1}$$

*where the first map is homogeneous of degree $n - r$ and the second map is homogeneous of degree 0. Equivalently, we have the equality of graded Frobenius characters*

$$\mathrm{grFrob}(R_{n,k,r}; q) = \mathrm{grFrob}(R_{n,k,r+1}; q) + q^{n-r} \cdot \mathrm{grFrob}(R_{n,k-1,r}; q). \tag{4.2}$$

*Proof.* We have the inclusion of ideals $I_{n,k,r} \subseteq I_{n,k,r+1}$; we let the second map be the canonical projection $\pi : R_{n,k,r} \twoheadrightarrow R_{n,k,r+1}$. We have a homogeneous map $\widetilde{\varphi} : \mathbb{Q}[\mathbf{x}_n] \to R_{n,k,r}$ of degree $n-r$ given by multiplication by $e_{n-r}(\mathbf{x}_n)$, and then projecting onto $R_{n,k,r}$.

We claim that $\widetilde{\varphi}(I_{n,k-1,r}) = 0$, so that $\widetilde{\varphi}$ induces a well defined map $\varphi : R_{n,k-1,r} \to R_{n,k,r}$. This is equivalent to showing that $h_k(\mathbf{x}_n) \cdot e_{n-r}(\mathbf{x}_n) \in I_{n,k,r}$. The Pieri Rule implies that

$$h_k(\mathbf{x}_n) \cdot e_{n-r}(\mathbf{x}_n) = s_{(k,1^{n-r})}(\mathbf{x}_n) + s_{(k+1,1^{n-r-1})}(\mathbf{x}_n); \tag{4.3}$$

we will show that both terms on the right hand side lie in $I_{n,k,r}$.

To see that $s_{(k,1^{n-r})}(\mathbf{x}_n) \in I_{n,k,r}$, observe that, for $1 \leqslant i \leqslant r$ we have

$$h_{k-r+i}(\mathbf{x}_n) \cdot e_{n-i+1}(\mathbf{x}_n) = s_{(k-r+i,1^{n-i+1})}(\mathbf{x}_n) + s_{(k-r+i+1,1^{n-i})}(\mathbf{x}_n) \in I_{n,k,r}. \tag{4.4}$$

It follows that modulo $I_{n,k,r}$ we have the congruences

$$s_{(k,1^{n-r})}(\mathbf{x}_n) \equiv -s_{(k+1,1^{n-r-1})}(\mathbf{x}_n) \equiv s_{(k+2,1^{n-r-2})}(\mathbf{x}_n) \equiv \cdots \equiv \pm s_{(k+n-r)}(\mathbf{x}_n) \equiv 0, \tag{4.5}$$

where the last congruence used the fact that $s_{(k+n-r)}(\mathbf{x}_n) = h_{k+n-r}(\mathbf{x}_n) \in I_{n,k,r}$ since $r < n$. This chain of congruences also shows that $s_{(k+1,1^{n-r-1})}(\mathbf{x}_n) \in I_{n,k,r}$.

By the last paragraph, we have a well defined induced map $\varphi : R_{n,k-1,r} \to R_{n,k,r}$. It is clear that $\mathrm{Im}(\varphi) = \mathrm{Ker}(\pi)$. Moreover, the Pascal relation implies that

$$|S_{n,k-1,r}| + |S_{n,k,r+1}| = |S_{n,k,r}|, \tag{4.6}$$

so that by Theorem 3.6 we have

$$\dim(R_{n,k-1,r}) + \dim(R_{n,k,r+1}) = \dim(R_{n,k,r}). \tag{4.7}$$

Since $\pi$ is a surjection, this forces the sequence

$$0 \to R_{n,k-1,r} \xrightarrow{\varphi} R_{n,k,r} \xrightarrow{\pi} R_{n,k,r+1} \to 0 \tag{4.8}$$

to be exact. To finish the proof, observe that the maps $\varphi$ and $\pi$ commute with the action of $S_n$. $\square$

We are ready to state the graded Frobenius image of $R_{n,k,r}$. We will give several formulas for this image. For any word $w$ over the nonnegative integers, define the monomial $\mathbf{x}^w$ to be

$$\mathbf{x}^w := x_1^{\#\text{ of 1's in } w} x_2^{\#\text{ of 2's in } w} \cdots ; \tag{4.9}$$

in particular, any copies of 0 in $w$ do not affect $\mathbf{x}^w$.

**Theorem 4.2.** *The graded Frobenius image of $R_{n,k,r}$ is given by*

$$\mathrm{grFrob}(R_{n,k,r}; q) = \begin{bmatrix} n+k-r \\ k \end{bmatrix}_q \cdot \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} \cdot s_{\mathrm{shape}(T)} \tag{4.10}$$

$$= \sum_{w} q^{\mathrm{inv}(w)} \mathbf{x}^w. \tag{4.11}$$

*The last sum ranges over all length $n+k$ words $w = w_1 \dots w_{n+k}$ in the alphabet of nonnegative integers which contain precisely $k$ copies of* 0 *and are $r$-tail positive.*

*Proof.* By considering the placement of the $k$ copies of 0 in a $r$-tail positive word $w$ appearing in the final sum, we see that

$$\sum_{w} q^{\mathrm{inv}(w)} \mathbf{x}^w = \begin{bmatrix} n+k-r \\ k \end{bmatrix}_q \cdot \sum_{\substack{v=v_1\dots v_n \\ v_i \in \mathbb{Z}_{>0}}} q^{\mathrm{inv}(v)} \mathbf{x}^v. \tag{4.12}$$

On the other hand, we have

$$\sum_{\substack{v=v_1\dots v_n \\ v_i \in \mathbb{Z}_{>0}}} q^{\mathrm{inv}(v)} \mathbf{x}^v = \sum_{\substack{v=v_1\dots v_n \\ v_i \in \mathbb{Z}_{>0}}} q^{\mathrm{maj}(v)} \mathbf{x}^v = \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} \cdot s_{\mathrm{shape}(T)} = \mathrm{grFrob}(R_n; q), \tag{4.13}$$

where the first equality uses the equidistribution of the statistics inv and maj on permutations of a fixed multiset of positive integer, the second follows from standard properties of the RSK correspondence, and the third is a consequence of the work of Lusztig-Stanley [18].

By the last paragraph, it suffices to prove the first equality asserted in the statement of the theorem. If $r \geqslant n$ or $k = 0$ then $R_{n,k,r} = R_n$ and this equality is trivial. Otherwise, we have the $q$-Pascal relation

$$\begin{bmatrix} n+k-r \\ k \end{bmatrix}_q = \begin{bmatrix} n+k-r-1 \\ k \end{bmatrix}_q + q^{n-r} \cdot \begin{bmatrix} n+k-r-1 \\ k-1 \end{bmatrix}_q, \tag{4.14}$$

so that the theorem follows from Lemma 4.1 and induction. □

The short exact sequence in Lemma 4.1 gives a recipe for constructing bases of $R_{n,k,r}$ from bases of the classical coinvariant algebra $R_n$. We switch from working over $\mathbb{Q}$ to working over an arbitrary field $\mathbb{F}$, so that the ideals $I_{n,k,r}, I_n$ are defined inside the ring $\mathbb{F}[\mathbf{x}_n] := \mathbb{F}[x_1, \dots, x_n]$ and we have $R_{n,k,r} := \mathbb{F}[\mathbf{x}_n]/I_{n,k,r}, R_n := \mathbb{F}[\mathbf{x}_n]/I_n$.

**Theorem 4.3.** *Let $\mathcal{C}_n = \{b_\pi(\mathbf{x}_n) : \pi \in S_n\}$ be a collection of polynomials in $\mathbb{F}[\mathbf{x}_n]$ indexed by permutations in $S_n$ which descends to a basis of $R_n$. The collection of polynomials*

$$\mathcal{C}_{n,k,r} := \{b_\pi(\mathbf{x}_n) \cdot e_\lambda(\mathbf{x}_n) \,:\, \pi \in S_n,\ \lambda_1 \leqslant n-r,\ \text{and } \lambda \text{ has} \leqslant k \text{ parts}\} \tag{4.15}$$

*in $\mathbb{F}[\mathbf{x}_n]$ descends to a basis of $R_{n,k,r}$.*

*Proof.* This is trivial when $k = 0$ or $r \geqslant n$, so we assume $k > 0$ and $r < n$.

The arguments of Section 3 apply to show that $\dim(R_{n,k,r}) = |S_{n,k,r}|$ when working over the arbitrary field $\mathbb{F}$. [3] The proof of Lemma 4.1 then applies over $\mathbb{F}$ to give a short exact sequence of graded $\mathbb{F}$-vector spaces

$$0 \to R_{n,k-1,r} \xrightarrow{\cdot e_{n-r}(\mathbf{x}_n)} R_{n,k,r} \xrightarrow{\pi} R_{n,k,r+1} \to 0, \tag{4.16}$$

[3] If $\mathbb{F}$ is a finite field, there might not be enough elements in $\mathbb{F}$ for the point set $Y_{n,k,r}$ of Definition 3.1 to make sense. To get around this, we may apply [14, Lem. 3.1] to harmlessly replace $\mathbb{F}$ by an extension field $\mathbb{K}$.

where $\pi$ is the canonical projection. We may inductively assume that $\mathcal{C}_{n,k-1,r}$ descends to a $\mathbb{F}$-basis of $R_{n,k-1,r}$ and that $\mathcal{C}_{n,k,r+1}$ descends to a $\mathbb{F}$-basis of $S_{n,k,r+1}$. Exactness implies that

$$\{f(\mathbf{x}_n) \,:\, f(\mathbf{x}_n) \in \mathcal{C}_{n,k,r+1}\} \cup \{g(\mathbf{x}_n) \cdot e_{n-r}(\mathbf{x}_n) \,:\, g(\mathbf{x}_n) \in \mathcal{C}_{n,k-1,r}\} = \mathcal{C}_{n,k,r} \tag{4.17}$$

descends to an $\mathbb{F}$-basis of $R_{n,k,r}$. □

Theorem 4.3 reinforces the fact that $R_{n,k,r}$ consists of $\binom{n+k-r}{k}$ copies of $R_n$, graded by the $q$-binomial coefficient $\begin{bmatrix} n+k-r \\ k \end{bmatrix}_q$. Interesting bases $\mathcal{C}_n$ to which Theorem 4.3 can be applied include

- the *Artin basis* [2]
$$\mathcal{C}_n = \{x_1^{i_1} \cdots x_n^{i_n} \,:\, 0 \leqslant i_j < j\} \tag{4.18}$$
(which is connected to the inv statistic on permutations in $S_n$) and
- the *Garsia-Stanton basis* (or the *descent monomial basis*) [6, 8] $\mathcal{C}_n = \{gs_\pi \,:\, \pi \in S_n\}$ where
$$gs_\pi = \prod_{\pi_i > \pi_{i+1}} x_{\pi_1} \cdots x_{\pi_i} \tag{4.19}$$
(which is connected to the maj statistic on permutations in $S_n$).

The GS basis above can be deformed somewhat to describe the isomorphism type of $R_{n,k,r}$ as a module over the 0-Hecke algebra. The algebra $H_n(0)$ acts on the polynomial ring $\mathbb{F}[\mathbf{x}_n]$ by letting the generator $T_i$ act by the *Demazure operator* $\sigma_i$, where

$$\sigma_i.f := \frac{x_i f - x_{i+1} s_i(f)}{x_i - x_{i+1}}. \tag{4.20}$$

Here $s_i(f)$ is the polynomial obtained by interchanging $x_i$ and $x_{i+1}$ in $f(\mathbf{x}_n)$. It can be shown that if $f \in \mathbb{F}[\mathbf{x}_n]^{S_n}$ is any symmetric polynomial and $g \in \mathbb{F}[\mathbf{x}_n]$ is an arbitrary polynomial then

$$\sigma_i(fg) = f\sigma_i(g). \tag{4.21}$$

Therefore, any ideal $I \subseteq \mathbb{F}[\mathbf{x}_n]$ generated by symmetric polynomials is stable under the action of $H_n(0)$. In particular, the ideal $I_{n,k,r}$ is stable under the action of $H_n(0)$, and the quotient $R_{n,k,r} = \mathbb{F}[\mathbf{x}_n]/I_{n,k,r}$ carries the structure of an $H_n(0)$-module.

Huang [13] studied the coinvariant ring $R_n$ as a graded module over the 0-Hecke algebra $H_n(0)$. We apply Theorem 4.3 to generalize Huang's results to the quotient $R_{n,k,r}$. If $V$ is any graded $H_n(0)$-module, we let $V(i)$ denote the graded $H_n(0)$-module with components $V(i)_j := V_{i+j}$, for all $j \geqslant -i$.

**Theorem 4.4.** *Let $n, k$, and $r$ be nonnegative integers with $r \leqslant n$. We have an isomorphism of graded $H_n(0)$-modules*

$$R_{n,k,r} \cong \bigoplus_{\lambda \subseteq (n-r) \times k} R_n(-|\lambda|). \tag{4.22}$$

*Here the direct sum is over all partitions $\lambda$ which satisfy $\lambda_1 \leqslant n - r$ and have at most $k$ parts. The module $R_n = \mathbb{F}[\mathbf{x}_n]/I_n$ is the coinvariant algebra viewed as a graded $H_n(0)$-module.*

*Proof.* Huang [13] introduced the following modified GS basis of $R_n$. For $1 \leqslant i \leqslant n-1$, define an operator $\bar{\sigma}_i$ on $\mathbb{F}[\mathbf{x}_n]$ by the rule $\bar{\sigma}_i := \sigma_i - 1$. For any permutation $\pi \in S_n$, define $\bar{\sigma}_\pi := \bar{\sigma}_{i_1} \cdots \bar{\sigma}_{i_k}$ where $s_{i_1} \cdots s_{i_k}$ is any reduced word for $\pi$. Finally, given $\pi \in S_n$, let $\mathbf{x}_{\mathrm{Des}(\pi)}$ be the monomial

$$\mathbf{x}_{\mathrm{Des}(\pi)} := \prod_{i \in \mathrm{Des}(\pi)} (x_1 x_2 \cdots x_i). \tag{4.23}$$

For example, we have $\mathbf{x}_{21543} = (x_1) \cdot (x_1 x_2 x_3) \cdot (x_1 x_2 x_3 x_4)$. Huang proved [13, Thm. 4.5] that the collection of polynomials

$$\mathcal{C}_n := \{\bar{\sigma}_\pi . \mathbf{x}_{\mathrm{Des}(\pi)} \,:\, \pi \in S_n\} \tag{4.24}$$

in $\mathbb{F}[\mathbf{x}_n]$ descends to a $\mathbb{F}$-basis for $R_n$.

Applying Theorem 4.3 to Huang's basis of $R_n$, we get a collection of polynomials $\mathcal{C}_{n,k,r}$ given by

$$\mathcal{C}_{n,k,r} := \{e_\lambda(\mathbf{x}_n) \cdot \bar{\sigma}_\pi . \mathbf{x}_{\mathrm{Des}(\pi)} \,:\, \pi \in S_n \text{ and } \lambda \subseteq (n-r) \times k\} \tag{4.25}$$

which descends to a basis of $R_{n,k,r}$. The symmetric polynomial $e_\lambda(\mathbf{x}_n)$ has degree $|\lambda|$ and the symmetry of $e_\lambda(\mathbf{x}_n)$ gives

$$\sigma_i . (e_\lambda(\mathbf{x}_n) \cdot \bar{\sigma}_\pi . \mathbf{x}_{\mathrm{Des}(\pi)}) = e_\lambda(\mathbf{x}_n) \sigma_i . (\bar{\sigma}_\pi . \mathbf{x}_{\mathrm{Des}(\pi)}). \tag{4.26}$$

It follows that, for $\lambda \subseteq (n-r) \times k$ fixed, the collection of polynomials

$$\mathcal{C}_{n,k,r}(\lambda) := \{e_\lambda(\mathbf{x}_n) \cdot \bar{\sigma}_\pi . \mathbf{x}_{\mathrm{Des}(\pi)} \,:\, \pi \in S_n\} \tag{4.27}$$

descends inside $R_{n,k,r}$ to a $\mathbb{F}$-basis of a copy of $R_n$ with degree shifted up by $|\lambda|$. □

Theorem 4.4 leads immediately to the fact that $R_{n,k,r}$ is a projective $H_n(0)$-module and formulas for the characteristics $\mathrm{Ch}_{q,t}(R_{n,k,r}), \mathrm{Ch}_q(R_{n,k,r})$, and $\mathbf{ch}_q(R_{n,k,r})$.

**Corollary 4.5.** *Let $n, k,$ and $r$ be nonnegative integers with $r \leqslant n$.*

1. *The length-degree bigraded quasisymmetric characteristic $\mathrm{Ch}_{q,t}(R_{n,k,r})$ is given by*

$$\mathrm{Ch}_{q,t}(R_{n,k,r}) = \begin{bmatrix} n+k-r \\ k \end{bmatrix}_q \cdot \mathrm{Ch}_{q,t}(R_n) = \begin{bmatrix} n+k-r \\ k \end{bmatrix}_q \cdot \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} t^{\mathrm{inv}(\pi)} F_{\mathrm{Des}(\pi^{-1}),n}, \tag{4.28}$$

*where $F_{\mathrm{Des}(\pi^{-1}),n}$ is the fundamental quasisymmetric function.*

2. *The degree graded quasisymmetric characteristic* $\mathrm{Ch}_q(R_{n,k,r})$ *is in fact symmetric and given by*

$$\mathrm{Ch}_q(R_{n,k,r}) = \begin{bmatrix} n+k-r \\ k \end{bmatrix}_q \cdot \mathrm{Ch}_q(R_n) = \begin{bmatrix} n+k-r \\ k \end{bmatrix}_q \cdot \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}(T)} s_{\mathrm{shape}(T)}. \tag{4.29}$$

*In particular, we have*

$$\mathrm{Ch}_q(R_{n,k,r}) = \mathrm{grFrob}(R_{n,k,r}; q). \tag{4.30}$$

3. *The* $H_n(0)$*-module* $R_{n,k,r}$ *is projective. Its degree graded noncommutative characteristic* $\mathbf{ch}_q(R_{n,k,r})$ *is*

$$\mathbf{ch}_q(R_{n,k,r}) = \begin{bmatrix} n+k-r \\ k \end{bmatrix}_q \cdot \mathbf{ch}_q(R_n) = \begin{bmatrix} n+k-r \\ k \end{bmatrix}_q \cdot \sum_{\alpha} q^{\mathrm{maj}(\alpha)} \mathbf{s}_\alpha, \tag{4.31}$$

*where* $\alpha$ *ranges over all strong compositions of* $n$*, the major index is*

$$\mathrm{maj}(\alpha) = \mathrm{maj}(\alpha_1, \ldots, \alpha_\ell) = \sum_{j=1}^{\ell-1} (\alpha_1 + \cdots + \alpha_j),$$

*and* $\mathbf{s}_\alpha$ *is the noncommutative ribbon Schur function.*

*Proof.* Parts 1 and 2 follow from the work of Huang [13, Cor. 4.9] and Theorem 4.4. Since $R_n$ is a projective $H_n(0)$-module (see [13, Thm. 4.5]) and direct sums of projective modules are projective, we can apply [13, Cor. 8.4, $\mu = (1^n)$] to get Part 3. □

We remark here (and thank an anonymous referee for pointing this out) that Lemma 4.1 may be used to give a more conceptual proof of Theorem 4.4 and Corollary 4.5 without direct reference to Theorem 4.3. If $W$ is any projective $H_n(0)$-module, then any short exact sequence

$$0 \to U \to V \to W \to 0$$

of $H_n(0)$-modules ending in $W$ splits, giving an isomorphism $V \cong U \oplus W$. Lemma 4.1 gives a short exact sequence

$$0 \to R_{n,k-1,r} \to R_{n,k,r} \to R_{n,k,r+1} \to 0 \tag{4.32}$$

for all $n, k, r \geqslant 0$ with $r < n$ and $k > 0$. Since $R_{n,0,r} = R_{n,k,n} = R_n$ and the classical coinvariant ring $R_n$ is known [13] to be a projective $H_n(0)$-module, we see inductively that

- the short exact sequences of (4.32) always split, and
- the $H_n(0)$-modules $R_{n,k,r} \cong R_{n,k-1,r} \oplus R_{n,k,r+1}$ are always projective.

In particular, the noncommutative characteristic $\mathbf{ch}_t(R_{n,k,r})$ makes sense. The splitting of (4.32), together with the degrees of the maps involved in Lemma 4.1, gives the graded $H_n(0)$-module decomposition $R_{n,k,r} \cong \bigoplus_{\lambda\subseteq(n-r)\times k} R_n(-|\lambda|)$ and the characteristics in Corollary 4.5 as before.

Although Theorem 4.3 gives a collection of *polynomials* in $\mathbb{F}[\mathbf{x}_n]$ generalizing the GS monomials which descend to a basis of $R_{n,k,r}$, the authors have been unable to find a collection of *monomials* in $\mathbb{F}[\mathbf{x}_n]$ which generalizes the GS monomials and descends to a basis of $R_{n,k,r}$ (such monomial bases were found for the quotients appearing in the work of Haglund-Rhoades-Shimozono and Huang-Rhoades [11, 14]). Judging from the construction in [11, Sec. 5] and the Hilbert series of $R_{n,k,r}$, one might expect that the set of monomials

$$\{gs_\pi \cdot x_{\pi_1}^{i_1} \cdots x_{\pi_{n-r}}^{i_{n-r}} : \pi \in S_n \text{ and } k \geqslant i_1 \geqslant \cdots \geqslant i_{n-r} \geqslant 0\} \tag{4.33}$$

would descend to a basis of $R_{n,k,r}$, but this set of monomials is linearly dependent in the quotient in general. A potential combinatorial obstruction to finding a GS monomial basis for $R_{n,k,r}$ is the fact that the statistics inv and maj do *not* share the same distribution on $S_{n,k,r}$.

# 5 Open problems

## 5.1 Bivariate generalization for $r = 1$

We propose a relationship between our quotient ring $R_{n,k,r}$ and the theory of Macdonald polynomials. In particular, consider the ideal $I'_{n,k,r} \subseteq \mathbb{Q}[\mathbf{x}_n]$ given by

$$I'_{n,k,r} := \langle p_{k+1}(\mathbf{x}_n), p_{k+2}(\mathbf{x}_n), \ldots, p_{k+n}(\mathbf{x}_n), e_n(\mathbf{x}_n), e_{n-1}(\mathbf{x}_n), \ldots, e_{n-r+1}(\mathbf{x}_n)\rangle \tag{5.1}$$

and let $R'_{n,k,r} := \mathbb{Q}[\mathbf{x}_n]/I_{n,k,r}$ be the corresponding quotient. The ideal $I'_{n,k,r}$ is obtained from the ideal $I_{n,k,r}$ by replacing the homogeneous symmetric functions with power sum symmetric functions.

As with the quotient $R_{n,k,r}$, the quotient $R'_{n,k,r}$ has the structure of a graded $S_n$-module. Although the ideals $I_{n,k,r}$ and $I'_{n,k,r}$ are not equal in general, we present

**Conjecture 5.1.** *There is an isomorphism of graded $S_n$-modules $R_{n,k,r} \cong R'_{n,k,r}$.*

The main reason for preferring the quotient rings $R'_{n,k,r}$ over the quotient rings $R_{n,k,r}$ is that they generalize more readily to two sets of variables. Let $\mathbf{x}_n = (x_1, \ldots, x_n)$ and $\mathbf{y}_n = (y_1, \ldots, y_n)$ be two sets of $n$ variables and let $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]$ be the polynomial ring in these variables. The symmetric group $S_n$ acts on $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]$ by the *diagonal action* $\pi.x_i = x_{\pi_i}, \pi.y_i = y_{\pi_i}$.

For any $a, b \geqslant 0$, let $p_{a,b}(\mathbf{x}_n, \mathbf{y}_n)$ be the *polarized power sum*

$$p_{a,b}(\mathbf{x}_n, \mathbf{y}_n) := \sum_{i=1}^{n} x_i^a y_i^b. \tag{5.2}$$

Moreover, let $\mathcal{M}_n$ be the set of the $2^n$ monomials $z_1 \dots z_n$ in $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]$ where $z_i \in \{x_i, y_i\}$ for all $1 \leqslant i \leqslant n$. For example, we have

$$\mathcal{M}_2 = \{x_1x_2, x_1y_2, y_1x_2, y_1y_2\}.$$

For a nonnegative integer $k$, let $DI_{n,k} \subseteq \mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]$ be the ideal generated by the polarized power sums $p_{a,b}(\mathbf{x}_n, \mathbf{y}_n)$ with $a + b \geqslant k + 1$ together with the monomials in $\mathcal{M}_n$. Let $DR_{n,k} := \mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]/DI_{n,k}$ be the corresponding quotient, which is a bigraded $S_n$-module.

**Conjecture 5.2.** *The bigraded Frobenius image of $DR_{n,k}$ is given by the delta operator image*

$$\mathrm{grFrob}(DR_{n,k}; q, t) = \Delta_{h_k e_n} e_n = \Delta_{s_{k+1,1^{n-1}}} e_n = \Delta_{h_k} \nabla e_n.$$

The latter three quantities in the conjecture are trivially equal by the definition of the delta operator. When $k = 0$, the ring $DR_{n,0}$ is the classical diagonal coinvariant ring $DR_n$, so that Conjecture 5.2 reduces to Haiman's celebrated result [12] that $\mathrm{grFrob}(DR_n) = \Delta_{e_n} e_n$. Setting the $\mathbf{y}_n$ variables equal to zero in the quotient $DR_{n,k}$ yields the ring $R'_{n,k,1}$, so that the ring $R'_{n,k,1}$ conjecturally gives the analog of the coinvariant ring (for one set of variables) attached to the operator $\Delta_{h_k e_n}$.

The following proposition states that our module $R_{n,k,1}$ has graded Frobenius series which agrees with any of the delta operator expressions in Conjecture 5.2 upon setting $q = 0$ and $t = q$.

**Proposition 5.3.** *We have*

$$\mathrm{grFrob}(R_{n,k,1}; t) = \Delta_{h_k e_n} e_n \mid_{q=0} = \Delta_{s_{k-1,1^{n-1}}} e_n \mid_{q=0} = \Delta_{h_k} \nabla e_n \mid_{q=0} .$$

*Proof.* In this proof we will use the notation of plethysm; we refer the reader to [9] for the relevant details on plethysm and symmetric functions.

Let $\mathrm{rev}_t$ be the operator which reverses the coefficient sequences of polynomials with respect to the variable $t$. For a partition $\lambda \vdash n$, let $Q'_\lambda = Q'_\lambda(\mathbf{x}; t)$ be the corresponding Hall-Littlewood symmetric function. It is well known that the modified Macdonald polynomial $\widetilde{H}_\lambda = \widetilde{H}_\lambda(\mathbf{x}; q, t)$ satisfies

$$\widetilde{H}_\lambda \mid_{q=0} = \mathrm{rev}_t(Q'_\lambda). \tag{5.3}$$

This means that, for any symmetric function $f$ and any partition $\lambda \vdash n$, we have

$$\Delta_f(\widetilde{H}_\lambda) \mid_{q=0} = f(1, t, t^2, \dots, t^{\ell(\lambda)-1}) \cdot \mathrm{rev}_t(Q'_\lambda), \tag{5.4}$$

where $\ell(\lambda)$ is the number of parts of $\lambda$.

In order to exploit Equation 5.4, we need to express $e_n$ in terms of the modified Macdonald basis. This expansion is found in [9, Eqn. 2.72]: we have

$$e_n = \sum_{\lambda \vdash n} \frac{MB_\lambda \Pi_\lambda \widetilde{H}_\lambda}{w_\lambda}, \tag{5.5}$$

where

- $M = (1-q)(1-t)$,
- $B_\lambda = \sum_{c=(i,j)\in\lambda} q^{i-1}t^{j-i}$, where the sum is over all cells $c$ with matrix coordinates $(i,j)$ in the Ferrers diagram of $\lambda$,
- $\Pi_\lambda = \prod_{c=(i,j)\neq(0,0)}(1-q^{i-1}t^{j-1})$, where $c=(i,j)$ is a cell in $\lambda$ other than the corner $(0,0)$,
- $w_\lambda = \prod_{c\in\lambda}(q^{a(c)}-t^{l(c)+1})(t^{l(c)}-q^{a(c)+1})$, where the product is over all cells $c$ in the Ferrers diagram of $\lambda$ and $a(c), l(c)$ denote the arm and leg lengths of $\lambda$ at $c$.

We apply the operator $\Delta_{h_k e_n} = \Delta_{h_k}\Delta_{e_n}$ to both sides of Equation 5.5 to get

$$\Delta_{h_k e_n} e_n = \sum_{\lambda\vdash n} h_k[B_\lambda]e_n[B_\lambda]\frac{MB_\lambda\Pi_\lambda\widetilde{H}_\lambda}{w_\lambda}. \tag{5.6}$$

Setting $q=0$ on both sides of Equation 5.6 gives

$$\Delta_{h_k e_n} e_n \mid_{q=0} = \left[\sum_{\lambda\vdash n} h_k[B_\lambda]e_n[B_\lambda]\frac{MB_\lambda\Pi_\lambda\widetilde{H}_\lambda}{w_\lambda}\right]_{q=0}. \tag{5.7}$$

For any $\lambda\vdash n$ and any symmetric function $f$, we have $f[B_\lambda]\mid_{q=0} = f(1,t,t^2,\ldots,t^{\ell(\lambda)-1})$. In particular, we have $e_n[B_\lambda]=0$ unless $\lambda=(1^n)$ and Equation 5.7 reduces to

$$\Delta_{h_k e_n} e_n \mid_{q=0} = h_k(1,t,\ldots,t^{n-1})\cdot e_n(1,t,\ldots,t^{n-1})\cdot\left[\frac{MB_{(1^n)}\Pi_{(1^n)}\widetilde{H}_{(1^n)}}{w_{(1^n)}}\right]_{q=0}. \tag{5.8}$$

The right hand side of Equation 5.8 simplifies to

$$\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_t \cdot \mathrm{rev}_t(Q'_{(1^n)}) = \mathrm{grFrob}(R_{n,k,1};t) \tag{5.9}$$

where we used Theorem 4.2 at $r=1$ and the well known fact that the graded Frobenius image of the classical coinvariant algebra $R_n$ is $\mathrm{grFrob}(R_n;t) = \mathrm{rev}_t(Q'_{(1^n)})$. □

### 5.2 Other bivariate generalizations

One may wonder if there is a bivariate generalization of the entire ring $R_{n,k,r}$, as we have only discussed the $r=1$ case so far. While we have not been able to find a full generalization, there is some progress in the Hilbert series case. The *skewing operator* acts on a symmetric function $f$ of degree $d$ uniquely so that

$$\langle \partial f, g\rangle = \langle f, p_1 g\rangle \tag{5.10}$$

for all symmetric functions $g$ of degree $d-1$, where the inner product is the usual Hall inner product on symmetric functions.

Given a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ of $n$ positive integers, an $\alpha$-*Tesler matrix* $U = (u_{i,j})_{1\leqslant i,j\leqslant n}$ is an $n \times n$ upper triangular matrix with nonnegative integer entries such that, for $i = 1$ to $n$,

$$u_{i,i} + u_{i,i+1} + \ldots + u_{i,n} - (u_{1,i} + u_{2,i} + \ldots + u_{i-1,i}) = \alpha_i. \tag{5.11}$$

We write $U \in \mathcal{T}(\alpha)$. For example, the matrix

$$U = \begin{pmatrix} 0 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

satisfies $U \in \mathcal{T}(3, 2, 2, 3)$. The *weight* of an $n \times n$ $\alpha$-Tesler matrix $U$ is equal to

$$\mathrm{wt}(U; q, t) = (-(1-q)(1-t))^{\mathrm{pos}(U)-n} \prod_{u_{i,j}>0} [u_{i,j}]_{q,t} \tag{5.12}$$

where $\mathrm{pos}(U)$ is the number of positive entries in $U$ and $[k]_{q,t}$ is the usual $q,t$-integer, i.e. $[k]_{q,t} = \frac{q^k - t^k}{q-t}$. For example, if $U$ is the Tesler matrix shown above, we have $\mathrm{pos}(U) = 7$ and

$$\mathrm{wt}(U; q, t) = (-(1-q)(1-t))^{7-4}[2]_{q,t}^2[3]_{q,t}[6]_{q,t}.$$

Finally, the $\alpha$-*Tesler polynomial* is

$$\mathrm{Tes}(\alpha; q, t) = \sum_{U \in \mathcal{T}(\alpha)} \mathrm{wt}(U; q, t). \tag{5.13}$$

This corollary follows from work in [1, 10, 16].

**Corollary 5.4.**

$$\mathrm{Hilb}(R_{n,k,r}; q) = {n+k-r \brack k}_q \cdot [n]!_q \tag{5.14}$$

$$= \partial^{n-r+1} \Delta_{h_k} \partial^{r-1} \nabla e_n \big|_{t=0} \tag{5.15}$$

$$= \sum_{\substack{\alpha \models n+k \\ \ell(\alpha)=n \\ \alpha_1=\ldots=\alpha_r=1}} \mathrm{Tes}(\alpha; q, 0) \tag{5.16}$$

It would be interesting to find an extension of this corollary to the entire graded Frobenius series of $R_{n,k,r}$ for general $r$.

### 5.3 A Schubert basis

There is also a basis for $R_{n,k,r}$ given by certain Schubert polynomials. We let $\Pi_{n,k,r}$ be all the permutations $\pi$ of $\{1, 2, \ldots, n+k\}$ that satisfy

- all descents in $\pi$ occur weakly left of position $n$, and
- $1, 2, \ldots, r$ all appear in $\pi_1\pi_2\ldots\pi_n$.

If $\pi \in \Pi_{n,k,r}$ is a permutation, let $\mathfrak{S}_\pi(\mathbf{x}_n)$ be the Schubert polynomial attached to $\pi$. Note that, since each $\pi$ has no descents after position $n$, there are at most $n$ variables that appear in the Schubert polynomial associated to $\pi$, so we have not truncated the variable set in any meaningful way. We will show that $\{\mathfrak{S}_\pi(\mathbf{x}_n^*) : \pi \in \Pi_{n,k,r}\}$ is a basis for $R_{n,k,r}$, where the asterisk represents the reversal of the vector of variables. This will follow from the fact that the leading terms are all $(n, k, r)$-good monomials.

**Proposition 5.5.** *Let $<$ be the lexicographic monomial order and let*

$$\mathcal{LT}_{n,k,r} = \{\mathrm{in}_<(\mathfrak{S}_\pi(\mathbf{x}_n^*)) : \pi \in \Pi_{n,k,r}\}.$$

*Then $\mathcal{LT}_{n,k,r} = \mathcal{M}_{n,k,r}$.*

*Proof.* We will construct a bijection $\Phi : \Pi_{n,k,r} \to \mathcal{M}_{n,k,r}$ that satisfies $\Phi(\pi) = \mathrm{in}_<(\mathfrak{S}_\pi(\mathbf{x}_n^*))$. The bijection itself is

$$\Phi(\pi) = \prod_{i=1}^{n} x_{n-i+1}^{d_i} \tag{5.17}$$

where $d_i$ counts the number of $j > i$ such that $\pi_i > \pi_j$. The fact that $\Phi(\pi) = \mathrm{in}_<(\mathfrak{S}_\pi(\mathbf{x}_n^*))$ follows directly from the definition of the Schubert polynomial. We need to show that $m = \Phi(\pi) \in \mathcal{M}_{n,k,r}$ and to construct its inverse. Our proof will be similar to that of Lemma 3.5. First, we check that

- we have $\mathbf{x}(S) \nmid m$ for all $S \subseteq [n]$ with $|S| = n - r + 1$, and
- we have $x_i^{k+i} \nmid m$ for all $1 \leqslant i \leqslant n$.

To check the first condition, we recall that $\mathbf{x}(S) = x_{s_1}^{s_1} x_{s_2}^{s_2-1} \ldots x_{s_{n-r+1}}^{s_{n-r+1}-n+r}$ if $S = \{s_1 < s_2 < \ldots < s_{n-r+1}\}$. Since $S \subseteq [n]$ and the entries 1 through $r$ all appear in $\pi_1$ through $\pi_n$, there is some $s_i$ such that $\pi_{s_i} \leqslant r$. Choose $i$ as large as possible such that $\pi_{s_i} \leqslant r$. Since $j > n$ implies $\pi_j > r$, $\pi_{s_i}$ can only be greater than at most $n - s_i$ entries to its right, i.e. $d_{s_i} \leqslant n - s_i$. Hence the power of $x_{n-s_i+1}$ in $m$ is at most $n - s_i$, which means $\mathbf{x}(S) \nmid m$. The second condition follows from the definition of $m$.

Given a monomial $m \in \mathcal{M}_{n,k,r}$, we would like to construct $\Phi^{-1}(m)$. This can be done using the usual bijection from codes $(d_1, d_2, \ldots, d_n)$ to permutations. For $i = 1$ to $n$, we choose $\pi_i$ such that it is greater than exactly $d_i$ of the entries in $[n + k]$ that have not already been placed to the left of position $i$ in $\pi$. The second condition for $(n, k, r)$-good monomials implies that the result is an honest permutation, and the first condition implies that $1, 2, \ldots, r$ all appear in the first $n$ entries. ☐

**Corollary 5.6.** $\{\mathfrak{S}_\pi(\mathbf{x}_n^*) : \pi \in \Pi_{n,k,r}\}$ *descends to a basis for $R_{n,k,r}$.*

It would be interesting to explore if this Schubert basis maintains many of the properties of the Schubert basis for the usual ring of coinvariants. For example, the following suggests that the structure constants of this Schubert basis are positive modulo $R_{n,k,r}$.

**Question 5.7.** *For two permutations $\pi, \pi' \in \Pi_{n,k,r}$, is it always true that the product*

$$\mathfrak{S}_\pi(\mathbf{x}_n^*) \cdot \mathfrak{S}_{\pi'}(\mathbf{x}_n^*) \tag{5.18}$$

*has positive integer coefficients when expanded in the basis $\{\mathfrak{S}_\pi(\mathbf{x}_n^*) : \pi \in \Pi_{n,k,r}\}$ modulo $I_{n,k,r}$? Using* Sage, *we have checked that this is true for $1 \leqslant n, k \leqslant 4$ and $0 \leqslant r \leqslant n$. If so, do these coefficients count intersections in some family of varieties?*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## References


[1] D. Armstrong, A. Garsia, J. Haglund, B. Rhoades, and B. Sagan. J. Haglund. Combinatorics of Tesler matrices in the theory of parking functions and diagonal harmonics. *J. Comb.*, **3** (2011), 451–494.

[2] E. Artin. *Galois Theory,* Second edition. Notre Dame Math Lectures, no. 2. Notre Dame: University of Notre Dame, 1944.

[3] J. Chan and B. Rhoades. Generalized coinvariant algebras for wreath products. Submitted, 2017. arXiv:1701.06256.

[4] C. Chevalley. Invariants of finite groups generated by reflections. *Amer. J. Math.*, **77 (4)** (1955), 778–782.

[5] D. Cox, J. Little and D. O'Shea, *Ideals, varieties, and algorithms.* Third Edition. Undergraduate Texts in Mathematics, Springer, New York, 1992.

[6] A. M. Garsia. Combinatorial methods in the theory of Cohen-Macaulay rings. *Adv. Math.*, **38** (1980), 229–266.

[7] A. M. Garsia and C. Procesi. On certain graded $S_n$-modules and the $q$-Kostka polynomials. *Adv. Math.*, **94 (1)** (1992), 82–138.

[8] A. M. Garsia and D. Stanton. Group actions on Stanley-Reisner rings and invariants of permutation groups. *Adv. Math.*, **51 (2)** (1984), 107–201.

[9] J. Haglund. *The q,t-Catalan numbers and the space of diagonal harmonics.* University Lecture Series, Vol. 41, American Mathematical Society, Providence, RI, (2008). With an appendix on the combinatorics of Macdonald polynomials.

[10] J. Haglund. A polynomial expression for the Hilbert series of the quotient ring of diagonal coinvariants. *Adv. Math.*, **227** (2011), 2092 – 2106.

[11] J. Haglund, B. Rhoades, and M. Shimozono. Ordered set partitions, generalized coinvariant algebras, and the Delta Conjecture. Submitted, 2016. arXiv:1609.07575

[12] M. Haiman. Vanishing theorems and character formulas for the Hilbert scheme of points in the plane. *Invent. Math.*, **149 (2)** (2002), 371–407.

[13] J. Huang. 0-Hecke actions on coinvariants and flags. *J. Algebraic Combin.*, **40** (2014), 245–278.

[14] J. Huang and B. Rhoades. Ordered set partitions and the 0-Hecke algebra. Submitted, 2016. arXiv:1611.01251

[15] J. Humphreys. *Reflection Groups and Coxeter Groups.* Cambridge studies in advanced mathematics, **29**. Cambridge University Press, 1990.

[16] R. Ini Liu, K. Mészáros, and A. H. Morales. Flow polytopes and the space of diagonal harmonics. Submitted, 2016. arXiv:1610.08370

[17] I. G. Macdonald. *Symmetric Functions and Hall Polynomials,* Second edition. Oxford Mathematican Monographs. New York: The Clarendon Press Oxford University Press, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.

[18] R. P. Stanley. Invariants of finite groups and their applications to combinatorics. *Bull. Amer. Math. Soc.*, **1** (1979), 475–511.