

Uniform Mixing and Association Schemes

Chris Godsil

Natalie Mullin

Aidan Roy

University of Waterloo, Canada

`cgodsil@uwaterloo.ca`

Submitted: Oct 15, 2014; Accepted: Jul 26, 2017; Published: Aug 11, 2017

Mathematics Subject Classifications: Primary 05E30; 81P40

Abstract

We consider continuous-time quantum walks on distance-regular graphs. Using results about the existence of complex Hadamard matrices in association schemes, we determine which of these graphs have quantum walks that admit uniform mixing.

First we apply a result due to Chan to show that the only strongly regular graphs that admit instantaneous uniform mixing are the Paley graph of order nine and certain graphs corresponding to regular symmetric Hadamard matrices with constant diagonal. Next we prove that if uniform mixing occurs on a bipartite graph X with n vertices, then n is divisible by four; if X is also regular, we show that n must be the sum of two integer squares. Our work on bipartite graphs implies that uniform mixing does not occur on C_{2m} for $m \geq 3$. Using a result of Haagerup, we show that uniform mixing does not occur on C_p for any prime p such that $p \geq 5$. In contrast to this result, we see that ϵ -uniform mixing occurs on C_p for all primes p .

1 Introduction

Quantum walks are a quantum analog of a random walk on a graph, and have recently been the subject of much investigation. For recent survey from a mathematician's viewpoint, see [13]. A quantum walk can behave quite differently from a classical walk: for example Childs et al. [10] found a graph in which the time to propagate from one node to another was sped up exponentially compared to any classical walk.

We describe the physical motivation for continuous-time quantum walks. Let X be a graph with Laplacian matrix L , and let $p(t) = (p_1(t), \dots, p_n(t))$ be a probability distribution over the vertices $\{1, \dots, n\}$ of X , for each time t . Then $p(t)$ is a continuous-time random walk on X if it satisfies the differential equation

$$\frac{dp(t)}{dt} = -Lp(t).$$

This is a continuous version of a discrete-time random walk, in which the Laplacian matrix is the transition matrix for a Markov process on the vertices. The solution to the

differential equation is

$$p(t) = e^{-Lt}p(0).$$

Since L has rows which sum to 0, e^{-Lt} has rows which sum to 1 and is therefore a stochastic process.

In a continuous-time quantum walk, instead of a probability distribution $p(t)$ we use a quantum state $|\psi(t)\rangle$ defined on a state space with orthonormal basis $\{|1\rangle, \dots, |n\rangle\}$. Instead of a stochastic process, the walk is governed by a unitary evolution. From Schrödinger's equation for the evolution of a quantum system

$$i\frac{d|\psi(t)\rangle}{dt} = L|\psi(t)\rangle,$$

we have the solution

$$|\psi(t)\rangle = e^{-itL}|\psi(0)\rangle.$$

Since the Laplacian L is Hermitian, e^{iLt} is unitary. Assume our quantum walk begins at some vertex j , so $|\psi(0)\rangle = |j\rangle$. If at some time t we measure the state $|\psi(t)\rangle$ in the vertex basis $\{|1\rangle, \dots, |n\rangle\}$, then the probability of outcome $|k\rangle$ is

$$|\langle k|\psi(t)\rangle|^2 = |\langle k|e^{iLt}|j\rangle|^2 = |(e^{iLt})_{kj}|^2.$$

If this probability distribution is uniform over all vertices k , and this is true for all initial states $|j\rangle$, then the graph is said to have uniform mixing.

In fact we will define the unitary evolution using the adjacency matrix rather than the Laplacian. When X is regular, the resulting states differ only by an overall phase factor, and this does not affect uniform mixing.

In this paper we focus on the question of which graphs have uniform mixing: can a quantum walk beginning at a single vertex result in a state whose probability distribution is uniform over all vertices? If this is possible at some small time t , then the behaviour is quite different from a classical walk, which typically approaches the uniform distribution as $t \rightarrow \infty$.

Moore and Russell [18] in 2001 gave the first example of uniform mixing, showing that the n -cube has uniform mixing at time $t = \pi/4$. Since then a large body of work has been produced to study this problem. We summarize the conclusions. The following graphs do admit uniform mixing:

- (a) The complete graphs K_2 , K_3 , and K_4 [2].
- (b) The Hamming graphs $H(n, q)$, $q \in \{2, 3, 4\}$ [18] [7].
- (c) The Cartesian product of any two graphs that admit uniform mixing at the same time.

Note that since the Hamming graph $H(n, q)$ is the n -th Cartesian power of K_q , the results in (b) follow from the results in (a). A number of Cayley graphs for \mathbb{Z}_2^d admit uniform mixing, but no classification is yet known. Some work in this direction appears in [3].

On the other hand, some graphs are known not to admit uniform mixing:

- (a) K_n for $n > 4$.
- (b) $H(n, q)$ for $q > 4$.
- (c) Cycles C_n for $n = 2^u q$, where $q = 1$ and $u \geq 3$ or $q \cong 3 \pmod{4}$ [1].
- (d) The cycle C_5 [7].
- (e) Complete multipartite graphs (except C_4) [2].
- (f) The transpositions Cayley graph $X(S_n, \{\text{transpositions}\})$ [12].

An obvious question raised by these results is which cycles admit uniform mixing. In this work we show that the cycle C_n does not admit uniform mixing if n is even and greater than four, or if n is a prime greater than three. We see that if a bipartite graph on n vertices admits uniform mixing then n must be divisible by four; if in addition the graph is regular then n must be the sum of two integer squares.

The graphs we have that do admit uniform mixing are highly regular, and this leads us to ask which strongly regular graphs admit uniform mixing. We use work of Chan [8] to show that the only strongly regular graphs that admit instantaneous uniform mixing are the Paley graph of order 9 and certain graphs corresponding to regular symmetric Hadamard matrices with constant diagonal.

Since uniform mixing seems to be rare, we consider a relaxed version, called ϵ -uniform mixing, and we demonstrate that this does take place on all cycles of prime length.

2 Uniform mixing

Let A denote the adjacency matrix of a graph X on n vertices. The transition operator $U(t)$ defined by

$$U(t) = e^{itA} = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} A^k$$

determines a continuous quantum walk. It is straightforward to observe that the transition matrix satisfies the following properties.

- (i) $U(t)^T = U(t)$.
- (ii) $\overline{U(t)} = U(-t)$.
- (iii) $U(t)$ is unitary.

Suppose that A and B are two $m \times n$ matrices. The *Schur product* $A \circ B$ is the $m \times n$ matrix given by

$$(A \circ B)_{j,k} = A_{j,k} B_{j,k}.$$

We say the graph X admits *uniform mixing* at time t if and only if

$$U(t) \circ U(-t) = \frac{1}{n}J$$

where J is the $n \times n$ all-ones matrix.

Let $\|\cdot\|$ denote the Frobenius norm. More explicitly, for two $n \times n$ matrices A and B we have

$$\|A - B\| = \sqrt{\sum_{j=1}^n \sum_{k=1}^n |A_{j,k} - B_{j,k}|^2}.$$

We say that a graph X admits ϵ -uniform mixing if and only if for every $\epsilon > 0$, there exists some time t such that the corresponding mixing matrix $U(t)$ satisfies

$$\|U(t) \circ U(t)^* - \frac{1}{n}J\| < \epsilon.$$

It is useful to note the following lemma, which relates uniform mixing on a graph and its complement at certain times.

2.1 Lemma. *Let X denote a regular graph on n vertices, and let t denote an integer multiple of $2\pi/n$. At time t , uniform mixing occurs on X if and only if it occurs on the complement \bar{X} .*

Proof. Note that

$$e^{it\bar{A}} = e^{it(J-I-A)} = e^{it(J-I)}e^{-itA} = \left(e^{i(n-1)t} \left(\frac{1}{n}J \right) + e^{-it} \left(I - \frac{1}{n}J \right) \right) e^{-itA}.$$

If t denotes an integer multiple of $2\pi/n$, then the above equation reduces to

$$e^{it\bar{A}} = e^{-it}e^{-itA},$$

which implies that the complement of X has uniform mixing if and only if X does. \square

In Section 4, we see that the previous lemma applies to all cases of strongly regular graphs that admit uniform mixing.

3 Type-II Matrices

We say that a real or complex matrix is *flat* if all its entries have the same absolute value.

Suppose that A is an $n \times m$ matrix with all non-zero entries. We say that the *Schur inverse* of A is the matrix given by

$$A_{j,k}^{(-)} = \frac{1}{A_{j,k}}.$$

We say that a complex $n \times n$ matrix H is a *type-II matrix* if and only if

$$HH^{(-)T} = nI.$$

3.1 Lemma. *The graph X admits uniform mixing at time t if and only if $\sqrt{n}U(t)$ is a type-II matrix.*

Proof. We note that a complex number x satisfies $|x| = 1$ if and only if $\bar{x} = x^{-1}$. Therefore the transition matrix satisfies

$$U(t) \circ U(t)^* = \frac{1}{n}J \iff U(t)^* = \frac{1}{n}U(t)^{(-)T} \iff U(t)U(t)^{(-)T} = nI.$$

Since $U(t)$ is a symmetric unitary matrix, it follows that $U(t)^{(-)T} = nU(t)^{-1}$. We conclude that $U(t)$ is flat if and only if $\sqrt{n}U(t)$ is type-II. \square

A flat type-II matrix is often referred to as a *complex Hadamard matrix*. Therefore, we have the following.

3.2 Corollary. *The graph X admits uniform mixing at time t if and only if $\sqrt{n}U(t)$ is a complex Hadamard matrix.* \square

For the known examples of graphs that admit uniform mixing, the entries of $\sqrt{n}U(t)$ at the corresponding time are roots of unity. Such matrices are called *Butson-type* complex Hadamard matrices. They live in a more general family of complex Hadamard matrices whose entries are all algebraic numbers.

We recall a useful formulation of the Gelfond-Schneider Theorem due to Michel Waldschmidt [6].

3.3 Theorem. *If x and y are two non-zero complex numbers with x irrational, then at least one of the numbers x , e^y , or e^{xy} is transcendental.* \square

The above theorem is crucial in the proof of the following result.

3.4 Theorem. *Let X denote a graph. If at some non-zero time t all entries of the transition matrix $U(t)$ are algebraic numbers, then the ratio of any two non-zero eigenvalues of X must be rational.*

Proof. Let A denote the adjacency matrix of X , and let $\{\theta_0, \dots, \theta_d\}$ denote the eigenvalues of A . We consider a fixed pair of eigenvalues θ_r and θ_s . Since A is an integer matrix, the characteristic polynomial of A is a monic polynomial with integer coefficients. Therefore each eigenvalue of A is an algebraic integer. Furthermore, since A is symmetric, we decompose A as

$$A = \sum_{k=0}^d \theta_k E_k,$$

where each E_k is orthogonal projection onto the k -th eigenspace. This directly implies that

$$U(t) = \sum_{k=0}^d e^{\theta_k it} E_k. \tag{3.1}$$

From this we see that the eigenvalues of $U(t)$ are $\{e^{it\theta_0}, \dots, e^{it\theta_a}\}$. If all entries of $U(t)$ are algebraic, then the eigenvalues of $U(t)$ must be algebraic. If we further suppose that θ_r/θ_s is irrational, then Theorem 3.3 implies that one of

$$\{\theta_r/\theta_s, e^{it\theta_s}, e^{it\theta_s(\theta_r/\theta_s)}\}$$

must be transcendental, which is a contradiction. Therefore θ_r/θ_s must be rational for every pair of distinct eigenvalues θ_r and θ_s of X . \square

Restricting our consideration to regular graphs yields the following corollary.

3.5 Corollary. *Suppose that X is a regular graph with transition matrix $U(t)$. Further suppose that H is a matrix with all algebraic entries such that*

$$U(t) = \gamma H,$$

for some non-zero time t and some γ in \mathbb{C} . Then X must have all integral eigenvalues.

Proof. By a well-known result from linear algebra, we know that for the complex matrix itA we have

$$\det(U(t)) = \det(e^{itA}) = e^{\text{tr}(itA)} = 1.$$

Under our assumption that $U(t) = \gamma H$, this result implies that

$$\gamma^n \det(H) = 1.$$

Since H has all algebraic entries, it follows that $\det(H)$ is an algebraic number. This implies that γ is algebraic, and hence all of the entries of $U(t)$ must be algebraic. Theorem 3.4 shows that the ratio of any two eigenvalues must be rational. Since X is regular, the largest eigenvalue is an integer, and so the rest of the eigenvalues of X must be rational. Recall that the eigenvalues of A are algebraic integers. The only rational algebraic integers are the integers themselves, and so we conclude that all of the eigenvalues of X must be integers. \square

4 Complex Hadamards and SRGs

Let X be a strongly regular graph with parameters (n, k, λ, μ) and let A be its adjacency matrix, with spectral decomposition

$$A = kE_k + \theta E_\theta + \tau E_\tau,$$

where $k > \theta \geq \tau$.

Recall that a strongly regular graph X is said to be *primitive* if both X and its complement are connected. The following result of Chan [8] is instrumental in our classification of strongly regular graphs that admit uniform mixing.

4.1 Theorem. Let A be the adjacency matrix of a primitive strongly regular graph X with eigenvalues k, θ, τ , and $W = I + xA + y(J - I - A)$. If W is a complex Hadamard matrix, then either X or its complement \bar{X} has one of the following parameter sets (n, k, λ, μ) :

(i) $(4\theta^2, 2\theta^2 - \theta, \theta^2 - \theta, \theta^2 - \theta)$

(ii) $(4\theta^2, 2\theta^2 + \theta, \theta^2 + \theta, \theta^2 + \theta)$

(iii) $(4\theta^2 - 1, 2\theta^2, \theta^2, \theta^2)$

(iv) $(4\theta^2 + 4\theta + 1, 2\theta^2 + 2\theta, \theta^2 + \theta - 1, \theta^2 + \theta)$

(v) $(4\theta^2 + 4\theta + 2, 2\theta^2 + \theta, \theta^2 - 1, \theta^2)$

Moreover, Chan determines the possible values of x and y that can occur in each of these cases.

5 Uniform mixing on SRGs

If we suppose a strongly regular graph X admits uniform mixing, then there are numbers c, x and y with $|c| = |x| = |y| = 1$ such that

$$c\sqrt{n}U(t) = I + xA + y\bar{A},$$

and $c\sqrt{n}U(t)$ is a complex Hadamard matrix. Applying Chan's results in the previous section, we obtain the following classification of primitive strongly regular graphs that admit uniform mixing. Our main result in this section follows.

5.1 Theorem. A primitive strongly regular graph X with adjacency matrix A has uniform mixing if and only if one of the following holds

(a) $J - 2A$ is a regular symmetric Hadamard matrix of order $4\theta^2$ with constant diagonal and positive row sum and θ is even.

(b) $J - 2A - 2I$ is a regular symmetric Hadamard matrix of order $4\theta^2$ with constant diagonal and positive row sum and θ is odd.

(c) The Paley graph of order 9, which has parameters $(9, 4, 1, 2)$.

To prove this theorem, we consider each possible case of Theorem 4.1 separately. First we lay some groundwork that applies to all cases. Recall that we have

$$e^{iAt} = e^{ikt} E_k + e^{i\theta t} E_\theta + e^{i\tau t} E_\tau,$$

and so we can expand both sides of this equation in terms of the spectral decomposition. Comparing the expressions for E_k , E_θ and E_τ on both sides, we get the following system of equations:

$$ce^{ikt}\sqrt{n} = 1 + xk + y(v - k - 1) \quad (5.1)$$

$$ce^{i\theta t}\sqrt{n} = 1 + x\theta + y(-\theta - 1) \quad (5.2)$$

$$ce^{i\tau t}\sqrt{n} = 1 + x\tau + y(-\tau - 1). \quad (5.3)$$

These equations are the *characterizing equations*. The graph X admits uniform mixing if and only if a solution to the characterizing equations exists in t , x , y and c with $|x| = |y| = |c| = 1$.

Chan shows [8, Lemma 2.2] that if X has at least 5 vertices, then $\theta + \tau \in \{0, -1, -2\}$. Moreover, for each value of $\theta + \tau$, Chan finds parameter sets (n, k, λ, μ) in Theorem 4.1 that can occur and also determines the values of x and y that can occur in the characterizing equations.

In each case, we must also consider if the complement has uniform mixing $ce^{it\bar{A}} = I + xA + y\bar{A}$. The eigenvalues of the complement are

$$\{n - k - 1, -\theta - 1, -\tau - 1\}.$$

The characterizing equations for the complement of X , written in terms of the parameters of X , are:

$$ce^{i(n-k-1)t}\sqrt{n} = 1 + xk + y(n - k - 1) \quad (5.4)$$

$$ce^{i(-\theta-1)t}\sqrt{n} = 1 + x\theta + y(-\theta - 1) \quad (5.5)$$

$$ce^{i(-\tau-1)t}\sqrt{n} = 1 + x\tau + y(-\tau - 1). \quad (5.6)$$

5.1 Regular symmetric Hadamard matrices

We first consider uniform mixing on strongly regular graphs with parameters given in (i) or (ii) of Theorem 4.1. If a graph X with adjacency matrix A has these parameters or the complementary parameters, then $J - 2A$ or $J - 2A - 2I$, respectively, is a regular symmetric Hadamard matrix [4].

5.2 Lemma. *Suppose X is a primitive strongly regular graph with eigenvalues $k > \theta > \tau$. Assume that X or \bar{X} has parameters $(4\theta^2, 2\theta^2 - \theta, \theta^2 - \theta, \theta^2 - \theta)$ or $(4\theta^2, 2\theta^2 + \theta, \theta^2 + \theta, \theta^2 + \theta)$ (as in (i) or (ii) of Theorem 4.1). Then X admits uniform mixing if and only if one of the following holds:*

- (a) *The parameters of X or \bar{X} are given in (i) and θ is even.*
- (b) *The parameters of X or \bar{X} are given in (ii) and θ is odd.*

Proof. First suppose that X is a graph with parameters $(4\theta^2, 2\theta^2 - \theta, \theta^2 - \theta, \theta^2 - \theta)$. Chan shows that $x = -1$ and $y = 1$. The characterizing equations above reduce to

$$\begin{aligned} ce^{i\theta(2\theta-1)t} &= 1 \\ ce^{i\theta t} &= -1 \\ ce^{-i\theta t} &= 1. \end{aligned}$$

Thus $c = e^{i\theta t} = \pm i$, so $\theta t = \pi(m + 1/2)$ for some integer m , and $t = \pi(m + 1/2)/\theta$ and $kt = (2\theta - 1)\pi(m + 1/2)$. When θ is even, any m is a valid solution and uniform mixing occurs. These are the Latin square graphs $L_\theta(2\theta)$. When θ is odd, no solutions occur.

For the complement, the characterizing equations for \overline{X} reduce to

$$ce^{i(2\theta^2+\theta-1)t} = 1 \tag{5.7}$$

$$ce^{i(\theta-1)t} = 1 \tag{5.8}$$

$$ce^{i(-\theta-1)t} = -1 \tag{5.9}$$

Hence $c = e^{-i(\theta-1)t}$ and the first and third equations are $e^{it2\theta^2} = 1$ and $e^{-i2\theta t} = -1$. Thus $e^{i\theta t} = \pm i$, so $\theta t = \pi(m + 1/2)$ for some m , and $t = \pi(m + 1/2)/\theta$. When θ is even, any m is a valid solution and uniform mixing occurs. When θ is odd, no solutions occur.

If X or \overline{X} has parameters $(4\theta^2, 2\theta^2 + \theta, \theta^2 + \theta, \theta^2 + \theta)$, then a similar results holds: there are no solutions for θ even, and for θ odd $t = \pi(m + 1/2)/\theta$ is a solution for every m . These are the negative Latin square graphs $NL_\theta(2\theta)$ (for these, see [17]). The same applies to the complement: uniform mixing occurs when θ is even, and no solutions occur when θ is odd. \square

There are infinite families of strongly regular graphs with these parameters [15].

5.2 Symplectic-type graphs

We turn the graphs in part (iii) of Theorem 4.1. These are the strongly regular graphs with parameters $(4\theta^2 - 1, 2\theta^2, \theta^2, \theta^2)$; their eigenvalues satisfy $\theta + \tau = 0$. Chan shows in this case that (x, y) is one of

$$\left\{ \left(-1, \frac{2\theta^2 - 3 \pm i\sqrt{4\theta^2 - 5}}{2(\theta^2 - 1)} \right), \left(\frac{-2\theta^2 + 1 \pm i\sqrt{4\theta^2 - 1}}{2\theta^2}, 1 \right) \right\}.$$

Case $(x, y) = \left(\frac{-2\theta^2 + 1 \pm i\sqrt{4\theta^2 - 1}}{2\theta^2}, 1 \right)$: The characterizing equations reduce to

$$ce^{i2\theta^2 t} \sqrt{n} = \pm i\sqrt{n} \tag{5.10}$$

$$ce^{i\theta t} \sqrt{n} = \theta(x - 1) \tag{5.11}$$

$$ce^{-i\theta t} \sqrt{n} = -\theta(x - 1). \tag{5.12}$$

Equations (5.11) and (5.12) imply that $e^{2i\theta t} = -1$ and so $e^{i\theta t} = \pm i$ and $e^{i2\theta^2 t} = (-1)^\theta$. Then Equation (5.10) implies that $c = \pm i$. But then the LHS of equation (5.11) is real while the RHS is not, a contradiction. So there are no solutions.

The characterizing equations for the complement of X are:

$$ce^{i(2\theta^2-2)t}\sqrt{n} = \pm i\sqrt{n} \quad (5.13)$$

$$ce^{i(-\theta-1)t}\sqrt{n} = \theta(x-1) \quad (5.14)$$

$$ce^{i(\theta-1)t}\sqrt{n} = -\theta(x-1). \quad (5.15)$$

Equations (5.14) and (5.15) imply that $e^{i2\theta t} = -1$ and so $e^{i2\theta^2 t} = (-1)^\theta$. Then equation (5.13) implies that $c = \pm ie^{i2t}$. Since e^{it} is a 4θ -th root of unity, so is $ce^{i(-\theta-1)t} = \pm ie^{it(1-\theta)}$. On the other hand, from equation (5.14),

$$e^{it(1-\theta)} = \frac{\theta}{\sqrt{n}}(x-1) = \frac{-\sqrt{n}\pm i}{2\theta},$$

which has algebraic degree exactly four since n is not a perfect square. This implies that $e^{it(1-\theta)}$ is a primitive m -th root of unity for some m such that $\phi(m) = 4$, namely $m \in \{5, 8, 10, 12\}$. (Here ϕ is the Euler totient function.) But then it is not difficult to check that none of these primitive m -th roots of unity have imaginary part $\pm 1/2\theta$, for any $\theta > 1$. Thus there are no solutions.

Case $(x, y) = (-1, \frac{2\theta^2-3\pm i\sqrt{4\theta^2-5}}{2(\theta^2-1)})$: The characterizing equations reduce to

$$ce^{i2\theta^2 t}\sqrt{n} = -2 \pm i\sqrt{4\theta^2-5} \quad (5.16)$$

$$ce^{i\theta t}\sqrt{n} = 1 - \theta + y(-\theta - 1) \quad (5.17)$$

$$ce^{-i\theta t}\sqrt{n} = 1 + \theta + y(\theta - 1). \quad (5.18)$$

Simplifying (5.17) plus (5.18) we have $\sqrt{nc}\operatorname{Re}(e^{i\theta t}) = 1 - y$. Taking the absolute value of y , we get $\operatorname{Re}(e^{i\theta t})^2 = 1/n(\theta^2 - 1)$. Since $e^{i\theta t}$ has norm 1, we get $\operatorname{Im}(e^{i\theta t})^2 = \theta^2(4\theta^2 - 5)/n(\theta^2 - 1)$. Thus we can solve:

$$e^{i\theta t} = \frac{1}{\sqrt{n(\theta^2 - 1)}}[\pm 1 \pm \theta i\sqrt{4\theta^2 - 5}], \quad (5.19)$$

$$c = \frac{1}{2\sqrt{\theta^2 - 1}}[\pm 1 \pm i\sqrt{4\theta^2 - 5}]. \quad (5.20)$$

We then check that there are no solutions to the characterizing equations for any choice of integer θ . For small θ we can check explicitly. When θ is large, we examine $\arg(ce^{i2\theta^2 t})$ in two different ways: once from combining (5.19) and (5.20) and once from (5.16). We use asymptotics with error bounds to show the values are not equal. We need one preliminary observation: when θ is large, $e^{i\theta t}$ is close to $\pm i$ and so the argument of $e^{i\theta t}$ is approximately $\pm(\pi/2 - \operatorname{Re}(e^{i\theta t}))$.

5.3 Lemma. For any x on the unit circle, $\arg(x) = \pm(\pi/2 - \operatorname{Re}(x)) + \epsilon$, where $|\epsilon| \leq \operatorname{Re}(x)^2$.

Proof. Assume $\operatorname{Im}(x) > 0$ and $\operatorname{Re}(x) < 0$: the other cases are similar. The part of $\arg(x)$ past $\pi/2$ is the portion of the unit circle arc in the upper left quadrant, which is contained

in a rectangle with horizontal width $\operatorname{Re}(x)$ and vertical height $1 - \sqrt{1 - \operatorname{Re}(x)^2}$. Since $\sqrt{z} \geq z$ for any $z \in [0, 1]$, we have

$$\begin{aligned} \arg(x) - \pi/2 &\leq -\operatorname{Re}(x) + 1 - \sqrt{1 - \operatorname{Re}(x)^2} \\ &\leq -\operatorname{Re}(x) + 1 - (1 - \operatorname{Re}(x)^2) \\ &= -\operatorname{Re}(x) + \operatorname{Re}(x)^2. \end{aligned}$$

On the the other hand $\arg(x) - \pi/2 \geq -\operatorname{Re}(x)$. □

Now consider the arguments of $e^{i\theta t}$ and c from Equations (5.19) and (5.20):

$$\begin{aligned} \theta t = \arg(e^{i\theta t}) &= \pm \frac{\pi}{2} \pm \frac{1}{2\theta^2} + O\left(\frac{1}{\theta^4}\right), \\ \arg(c) &= \pm \frac{\pi}{2} \pm \frac{1}{2\theta} + O\left(\frac{1}{\theta^2}\right), \\ \arg(ce^{i2\theta^2 t}) &= \arg(c) + 2\theta(\theta t) \\ &= \pm \frac{\pi}{2} \pm \pi\theta \pm \frac{1}{2\theta}[2 \pm 1] \pm \pi\theta + O\left(\frac{1}{\theta^2}\right). \end{aligned}$$

On the other hand from (5.16) we have $\operatorname{Re}(ce^{i2\theta^2 t}) = -2/\sqrt{n} \approx -\frac{1}{\theta}$, and so

$$\arg(ce^{i2\theta^2 t}) = \pm \frac{\pi}{2} - \frac{1}{\theta} + O\left(\frac{1}{\theta^2}\right),$$

a contradiction. Using Lemma 5.3 this asymptotic argument can be made exact. Details are left to the reader. (Small values of θ can be eliminated by direct computation.)

The characterizing equations of the complement reduce to

$$ce^{i(2\theta^2-2)t}\sqrt{n} = -2 \pm i\sqrt{4\theta^2 - 5} \tag{5.21}$$

$$ce^{i(-\theta-1)t}\sqrt{n} = 1 - \theta + y(-\theta - 1) \tag{5.22}$$

$$ce^{i(\theta-1)t}\sqrt{n} = 1 + \theta + y(\theta - 1). \tag{5.23}$$

Adding/subtracting the last two equations gives

$$\sqrt{n}ce^{-it}\operatorname{Re}(e^{i\theta t}) = 1 - y \quad \text{and} \quad \sqrt{n}ce^{-it}\operatorname{Im}(e^{i\theta t}) = \theta(1 + y).$$

Again we can solve:

$$e^{i\theta t} = \frac{1}{\sqrt{n(\theta^2 - 1)}}[\pm 1 \pm \theta i\sqrt{4\theta^2 - 5}] \tag{5.24}$$

$$ce^{-it} = \frac{1}{2\sqrt{\theta^2 - 1}}[\pm 1 \pm i\sqrt{4\theta^2 - 5}]. \tag{5.25}$$

Again we use an asymptotic description of the argument which can be made precise with Lemma 5.3. We get:

$$\begin{aligned} \theta t = \arg(e^{i\theta t}) &= \pm \frac{\pi}{2} \pm \frac{1}{2\theta^2} + O\left(\frac{1}{\theta^4}\right), \\ t &= \pm \frac{\pi}{2\theta} \pm \frac{1}{2\theta^3} + O\left(\frac{1}{\theta^5}\right), \\ \arg(c) &= t \pm \frac{\pi}{2} \pm \frac{1}{2\theta} + O\left(\frac{1}{\theta^2}\right), \\ \arg(ce^{i(2\theta^2-2)t}) &= \arg(c) + (2\theta^2 - 2)t \\ &= \pm \frac{\pi}{2} + \frac{1}{\theta}[\pm \frac{1}{2} \pm 1 \pm \pi/2] \pm \pi\theta + O\left(\frac{1}{\theta^2}\right). \end{aligned}$$

Meanwhile from equation (5.21),

$$\arg(ce^{i(2\theta^2-2)t}) = \pm\frac{\pi}{2} - \frac{1}{\theta} + O\left(\frac{1}{\theta^2}\right).$$

Since the two expressions are not equal, no solution exists. Thus there is no uniform mixing in the symplectic-type graphs or their complements.

5.3 Regular conference matrix type graphs

We treat the fourth family of graphs from Chan's theorem. These have parameters $(n, k, \lambda, \mu) = (4\theta^2 + 4\theta + 2, 2\theta^2 + \theta, \theta^2 - 1, \theta^2)$ and $\theta + \tau = -1$.

Chan found $x = \pm i$ or $x = \frac{-1 \pm i\sqrt{(4\theta^2(\theta+1)^2-1)}}{2\theta(\theta+1)}$ and in either case, $y = \bar{x}$. The characteristic equations reduce to:

$$ce^{ikt}\sqrt{n} = 1 + xk + y(n - k - 1) \tag{5.26}$$

$$ce^{i\theta t}\sqrt{n} = 1 + x\theta + \bar{x}(-\theta - 1) \tag{5.27}$$

$$ce^{i(-\theta-1)t}\sqrt{n} = 1 + x(-\theta - 1) + \bar{x}\theta. \tag{5.28}$$

From (5.27) plus (5.28) we have

$$\sqrt{nc}(e^{i\theta t} + e^{i(-\theta-1)t}) = 2 - 2\text{Re}(x),$$

so $ce^{i\theta t} + ce^{i(-\theta-1)t}$ is real. Thus either $ce^{i\theta t} = -ce^{i(-\theta-1)t}$ or $ce^{i\theta t} = \overline{ce^{i(-\theta-1)t}}$. But in the first case $x = y = 1$, which is not a solution. We conclude that $ce^{i\theta t} = \overline{ce^{i(-\theta-1)t}} = \bar{c}e^{i(\theta+1)t}$ and therefore $c = \pm e^{it/2}$. The same analysis shows that $c = \pm e^{it/2}$ for the complement of X as well.

Case $x = \pm i$: The characteristic equations reduce to:

$$\pm\sqrt{n}e^{it(k+1/2)} = 1 - i(2\theta + 1), \tag{5.29}$$

$$\pm\sqrt{n}e^{it(\theta+1/2)} = 1 + i(2\theta + 1), \tag{5.30}$$

(where the signs on the left side agree). It follows that $e^{it(k+\theta+1)} = 1$, so e^{it} is a $(k+\theta+1) = (2\theta^2 + 2\theta + 1)$ -th root of unity and $e^{it(\theta+1/2)}$ is a $(4\theta^2 + 4\theta + 2)$ -th root of unity. But by (5.30), noting that \sqrt{n} is not an integer, $e^{it(\theta+1/2)}$ has algebraic degree exactly 4. Since none of the primitive m -th roots of unity of algebraic degree 4 have real part $1/\sqrt{n}$ for any θ , there are no solutions.

For the complement of X , the characteristic equations reduce to

$$\pm\sqrt{n}e^{it((n-k-1)+1/2)} = 1 - i(2\theta + 1) \tag{5.31}$$

$$\pm\sqrt{n}e^{it(\theta+1/2)} = 1 - i(2\theta + 1) \tag{5.32}$$

So $e^{it(n-k-1-\theta)} = 1$ and e^{it} is a $n - k - 1 - \theta = (2\theta^2 + 2\theta + 1)$ -th root of unity. Again there are no solutions.

Case $x = \frac{-1 \pm i\sqrt{4\theta^2(\theta+1)^2 - 1}}{2\theta(\theta+1)}$: This case is similar. The characteristic equations reduce to:

$$\begin{aligned}\sqrt{n}e^{it(k+1/2)} &= \frac{-2\theta^2 - 2\theta - 1 \pm i(2\theta + 1)\sqrt{4\theta^2(\theta + 1)^2 - 1}}{2\theta(\theta + 1)} \\ \sqrt{n}e^{it(\theta+1/2)} &= \frac{2\theta^2 + 2\theta + 1 \mp i(2\theta + 1)\sqrt{4\theta^2(\theta + 1)^2 - 1}}{2\theta(\theta + 1)}\end{aligned}$$

Thus $-e^{it(\theta+1/2)} = e^{it(k+1/2)}$ and so $e^{it(k-\theta)} = -1$ and e^{it} is a $2(k - \theta) = 4\theta^2$ -th root of unity while $e^{it(k+1/2)}$ is an $8\theta^2$ -th root of unity. Since none of the primitive m -th roots of unity of algebraic degree 4 have real part $\frac{2\theta^2+2\theta+1}{2\sqrt{n}\theta(\theta+1)}$ for any θ , there are no solutions.

For the complement of X , the characteristic equations are:

$$\begin{aligned}\sqrt{n}e^{it((n-k-1)+1/2)} &= \frac{-2\theta^2 - 2\theta - 1 \pm i(2\theta + 1)\sqrt{4\theta^2(\theta + 1)^2 - 1}}{2\theta(\theta + 1)} \\ \sqrt{n}e^{it(\theta+1/2)} &= \frac{2\theta^2 + 2\theta + 1 \pm i(2\theta + 1)\sqrt{4\theta^2(\theta + 1)^2 - 1}}{2\theta(\theta + 1)}\end{aligned}$$

Then $e^{it(n-k+\theta)} = -1$, so e^{it} is a $(4\theta^2 + 8\theta + 4)$ -th root of unity. Again no solutions exist.

5.4 Conference graphs

The final family of graphs that arise in Chan's theorem are known as conference graphs. A conference graph has parameters $((2\theta + 1)^2, 2\theta^2 + 2\theta, \theta^2 + \theta - 1, \theta^2 + \theta)$ with $\theta + \tau = -1$. Chan shows that $y = \bar{x}$. As in the previous case, the characteristic equations reduce to (5.26), (5.27), and (5.28) with $c = e^{it/2}$. Then equations (5.27) minus (5.28) reduce to $\text{Im}(x) = \text{Im}(e^{it(\theta+1/2)})$ and therefore $x = e^{it(\theta+1/2)}$ or $x = -e^{-it(\theta+1/2)}$. There are two possible values of (x, y) , namely

$$x \in \left\{ \frac{-1 \pm \sqrt{(2\theta + 1)(2\theta - 1)}i}{2\theta}, \frac{1 \pm \sqrt{(2\theta + 1)(2\theta + 3)}i}{2(\theta + 1)} \right\}, y = \bar{x}.$$

Case $x = \frac{-1 \pm \sqrt{(2\theta+1)(2\theta-1)}i}{2\theta}$: In this case (5.26) reduces to $ce^{ikt} = -1$. Thus $t = \frac{\pi(2m+1)}{k+1/2} = \frac{2\pi(2m+1)}{n}$, for some integer m .

Suppose θ is an integer. Now $x = e^{it(\theta+1/2)} = e^{i\pi(2m+1)/(2\theta+1)}$ or $-e^{-it(\theta+1/2)} = -e^{-i\pi(2m+1)/(2\theta+1)}$, so x is a $2(2\theta+1)$ -th root of unity. But we claim that x is a $2(2\theta+1)$ -th root of unity only when $\theta = 1$. To see this, note that $|\text{Re}(x)| = 1/2\theta$, while any $2(2\theta+1)$ -th root of unity has a real part of larger absolute value. (The root smallest real part in absolute value is $\omega = e^{i\pi(\theta+1)/(2\theta+1)}$, which has $|\text{Re}(\omega)| = \sin(\frac{\pi}{2(2\theta+1)}) > 1/2\theta$ for $\theta > 1$.) In the case $\theta = 1$, we get the 3×3 Latin square graph, for which uniform mixing does occur.

If θ is not an integer, then by Corollary 3.5 uniform mixing cannot occur on the corresponding conference graph. So there are no solutions if θ is not an integer.

Case $x = \frac{1 \pm \sqrt{(2\theta+1)(2\theta+3)}i}{2(\theta+1)}$: This case is similar: we find $ce^{ikt} = 1$. Thus $t = \frac{\pi(2m+1)}{k+1/2} = \frac{2\pi(2m+1)}{2k+1}$, for some integer m . So $t(\theta + 1/2) = (2\pi m)/(2\theta + 1)$, and $x = e^{it(\theta+1/2)}$ or $-e^{-it(\theta+1/2)}$ is a $(2\theta + 1)$ -th root of unity. Since $|\operatorname{Re}(x)|$ is too small, this never occurs. Thus the only conference graph having uniform mixing is the 3×3 Latin square.

The combination of the work above proves Theorem 5.1.

6 Bipartite graphs

Next we consider bipartite graphs. To begin, suppose that X is a bipartite graph on n vertices with adjacency matrix A . We assume the rows and columns of A are ordered such that

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}.$$

Since A is a block matrix, we easily compute the even and odd powers of A .

$$\begin{aligned} A^{2k} &= \begin{pmatrix} (BB^T)^k & 0 \\ 0 & (B^T B)^k \end{pmatrix} \\ A^{2k+1} &= \begin{pmatrix} 0 & (BB^T)^k B \\ (B^T B)^k B^T & 0 \end{pmatrix}. \end{aligned}$$

Now we derive an expression for the transition matrix of X in terms of these blocks.

$$\begin{aligned} U(t) &= e^{itA} = \sum_{k \geq 0} \frac{1}{k!} (itA)^k \\ &= \sum_{k \geq 0} \frac{(-1)^k t^{2k}}{(2k)!} \begin{pmatrix} (BB^T)^k & 0 \\ 0 & (B^T B)^k \end{pmatrix} \\ &\quad + i \sum_{k \geq 0} \frac{(-1)^k t^{2k+1}}{(2k+1)!} \begin{pmatrix} 0 & (BB^T)^k B \\ (B^T B)^k B^T & 0 \end{pmatrix}. \end{aligned}$$

To simplify notation we write $U(t)$ in block form as

$$U(t) = \begin{pmatrix} F_1(t) & iK(t) \\ iK^T(t) & F_2(t) \end{pmatrix}.$$

We pay particular attention to the fact that $F_1(t)$, $F_2(t)$, and $K(t)$ are all real matrices for all times t . This observation leads to the following result.

6.1 Lemma. *Suppose X is a bipartite graph on $n > 2$ vertices with transition matrix $U(t)$. If X admits uniform mixing at time t , then each entry of $\sqrt{n}U(t)$ is a fourth root of unity.*

Proof. If X admits uniform mixing at time t , then

$$|U(t)_{j,k}| = \frac{1}{\sqrt{n}}$$

for all $1 \leq j, k \leq n$. In terms of the blocks that comprise $U(t)$, this means that all of the entries of the real matrices $F_1(t)$, $F_2(t)$ and $K(t)$ are equal to $\pm 1/\sqrt{n}$. Hence all entries of $\sqrt{n}U(t)$ are fourth roots of unity. \square

Lemma 6.1 can be used to rederive Kay's result [16] concerning the phase factors for perfect state transfer in bipartite graphs. In addition, combining Lemma 6.1 with Theorem 3.4 yields the following results.

6.2 Lemma. *If X is a bipartite graph on $n > 2$ vertices that admits uniform mixing, then the ratio of any two non-zero eigenvalues must be rational.*

Proof. Let $U(t)$ denote the transition matrix of X , and suppose that X admits uniform mixing at non-zero time t . By Lemma 6.1, we know that each entry of $U(t)$ must be a fourth root of unity. Therefore, all the entries of $U(t)$ are algebraic numbers. By Theorem 3.4, the ratio of any two non-zero eigenvalues of X must be rational. \square

6.3 Theorem. *If X is a bipartite graph on $n > 2$ vertices that admits uniform mixing, then n is divisible by four.*

Proof. Let $U(t)$ denote the transition matrix of X , and suppose that X admits uniform mixing at time t . By Lemma 6.1, we know that each entry of $U(t)$ must be a fourth root of unity.

Let Γ_1 and Γ_2 denote the two colours classes on X . For convenience, we assume that the vertices in Γ_1 correspond to the first $|\Gamma_1|$ rows of $U(t)$. Let D denote the diagonal matrix of order $n \times n$ such that

$$D_{u,u} = \begin{cases} 1 & \text{if } u \in \Gamma_1 \\ i & \text{if } u \in \Gamma_2. \end{cases}$$

Let H denote the matrix given by

$$H = \sqrt{n}DU(t)D.$$

In terms of the blocks of $U(t)$, we see that

$$H = \sqrt{n} \begin{pmatrix} F_1 & -K \\ -K^T & -F_2 \end{pmatrix}.$$

This implies that H is a real matrix with entries equal to 1 or -1 . A straightforward computation also reveals that

$$HH^* = nDU(t)DD^*U(t)^*D^* = nI.$$

Therefore H is a real Hadamard matrix. It is well known that if H is a real Hadamard matrix of order n such that $n > 2$, then n is divisible by four. \square

The work done by Adamczak et al. [1] implies that if uniform mixing occurs on C_{2m} , then m must be a sum of two squares. Here we show that this is true for all regular bipartite graphs.

6.4 Theorem. *If X is a regular, bipartite graph with n vertices that admits uniform mixing, then n is the sum of two integer squares.*

Proof. Suppose X is a regular bipartite graph with transition matrix $U(t)$. Further suppose that X admits uniform mixing at time t . By Lemma 6.1, each of the entries of $\sqrt{n}U(t)$ is a fourth root of unity. In particular, this implies that

$$\sqrt{n}U(t)\mathbf{1} = (a + ib)\mathbf{1},$$

and

$$\sqrt{n}U(t)^*\mathbf{1} = \sqrt{n\overline{U(t)}}\mathbf{1} = (a - ib)\mathbf{1}$$

for some integers a and b where $\mathbf{1}$ is the all-ones vector. Taking the product of both of these expressions yields

$$nU(t)U(t)^*\mathbf{1} = (a - ib)(a + ib)\mathbf{1} = (a^2 + b^2)\mathbf{1}.$$

Since $U(t)$ is unitary, we know that $U(t)U(t)^* = I$. We conclude that $n = a^2 + b^2$. \square

7 Cycles

It was conjectured by Ahmadi et al. [2] that no cycle C_n , except for C_3 and C_4 , admits uniform mixing. Adamczak et al. [1] show that C_n does not admit uniform mixing if $n = 2^u$ for $u \geq 3$ or if $n = 2^u m$ where m is not the sum of two integer squares and $u \geq 1$. Carlson et al. [7] show that uniform mixing does not occur on C_5 . (This result also follows from the earlier result in this paper that uniform mixing does not occur on conference graphs.)

Using Corollary 3.5, we show that uniform mixing cannot occur on any even cycle. Likewise, we see that uniform mixing cannot occur on C_p where p is a prime such that $p \geq 5$. For both of these results, we use the irrationality of the eigenvalues of the underlying graph.

First we recall that the eigenvalues of a general cycle C_n have the form

$$\theta_r = \omega^r + \omega^{-r},$$

for $0 \leq r \leq n-1$ where $\omega = e^{2\pi i/n}$. If n is even, then 2 and -2 are both simple eigenvalues of C_n , and the remaining eigenvalues each have multiplicity 2. On the other hand, if n is odd, then $\theta_0 = 2$ is the only simple eigenvalue, and the nontrivial eigenvalues each have multiplicity two.

7.1 Lemma. *If $n = 5$ or $n \geq 7$, then C_n has an irrational eigenvalue.*

Proof. Using the notation above, we note that $\theta_1 = 2 \cos(2\pi/n)$. Since n is a positive integer, the eigenvalue $\theta_1 = 2 \cos(2\pi/n)$ is rational if and only if $n \in \{1, 2, 3, 4, 6\}$ (see [20].) If we assume $n = 5$ and $n \geq 7$, then θ_1 is an irrational eigenvalue of C_n . \square

Applying Corollary 3.5, we obtain the following result.

7.2 Theorem. *The cycle C_4 is the unique even cycle that admits uniform mixing.*

Proof. Let $U(t)$ denote the transition matrix of C_{2m} . If $m = 2$, it is straightforward to observe that $U(t)$ is flat at time $t = \pi/4$. Suppose for a contradiction that $m > 2$ and $U(t)$ is flat at some time t . By Theorem 6.3, it follows that m must be even, and so $m > 3$. Furthermore, by Lemma 6.1 all of the entries of $U(t)$ are $\pm 1/\sqrt{2m}$ or $\pm i/\sqrt{2m}$, and hence they are all algebraic. Thus by Corollary 3.5, the eigenvalues of C_{2m} must be integers, which contradicts Lemma 7.1. \square

Next we turn our attention to cycles of odd prime order. It is known that C_3 admits uniform mixing at time $t = 2\pi/9$. On the other hand, it is known that C_5 does not admit uniform mixing [7]. (This is also a consequence of our earlier results about conference graphs.) We extend these results to show that C_p does not admit uniform mixing for any prime p such that $p \geq 5$. To obtain this result, we consider the following important result due to Haagerup [14].

7.3 Theorem. *Let p denote a prime number. There are only a finite number of circulant type-II matrices of order p with constant diagonal 1.* \square

Haagerup's work refers to so-called cyclic p -roots, which are equivalent to circulant $p \times p$ type-II matrices with constant diagonal 1. Using Haagerup's Theorem, we obtain the following new result.

7.4 Theorem. *The cycle C_3 is the unique cycle of odd prime order that admits uniform mixing.*

Proof. Let p denote an odd prime such that $p \geq 5$, and let $U(t)$ denote the transition matrix of C_p . Suppose that C_p admits uniform mixing at time t . This implies that $\sqrt{p}U(t)$ is a scalar multiple of a type-II matrix of order p with constant diagonal 1. By Haagerup's work we know that there are a finite such number of such type-II matrices. The set of all cyclic type-II matrices with constant diagonal 1 of a given order form an algebraic variety defined over \mathbb{Q} . It is well known that if such an algebraic variety has only a finite number of points, then the coordinates of each point are algebraic numbers. (A proof is given as Theorem 4.6.1 in [19], it is an easy consequence of Theorem 6 on page 243 of [11].) Therefore $U(t)$ must be a scalar multiple of a matrix with all algebraic entries. In turn, Theorem 3.5 implies that C_p must have all integral eigenvalues. This contradicts the fact that C_p has an irrational eigenvalue, which we proved in Lemma 7.1. \square

8 Cyclotomic number theory

We recall some information about number fields related to the cyclic schemes. The following results are well-known. See [21], for example.

Suppose that p is an odd prime. Let $d = (p - 1)/2$, and let $\theta_1, \dots, \theta_d$ denote the nontrivial eigenvalues of C_p . Also let ϕ denote the Euler phi function. As before, ω is the primitive p -root of unity given by

$$\omega = e^{2\pi i/p}.$$

8.1 Lemma. *Each of the eigenvalues $\theta_0, \dots, \theta_d$ can be expressed as a polynomial in θ_1 with integral coefficients.*

Proof. First recall that $\theta_0 = 2$ is trivially a polynomial in θ_1 . Next note that

$$(\omega + \omega^{-1})^2 = \omega^2 + \omega^{-2} + 2.$$

Rearranging this equation yields $\theta_2 = \theta_1^2 - 2$. More generally, for $2 \leq k \leq d$, we see that $(\omega + \omega^{-1})^k$ can be expressed as an integral linear combination of $1, \theta_1, \dots, \theta_{k-1}$. By induction we conclude that each eigenvalue $\theta_0, \dots, \theta_d$ can be expressed as a polynomial in θ_1 with integral coefficients. \square

Using this lemma, we obtain useful information about the smallest number field containing all of the eigenvalues of the cycle.

8.2 Lemma. *The extension field $\mathbb{Q}(\theta_1, \dots, \theta_d)$ is isomorphic to $\mathbb{Q}(\theta_1)$, and*

$$[\mathbb{Q}(\theta_1) : \mathbb{Q}] = (p - 1)/2.$$

Proof. By Lemma 8.1, we immediately see that $\mathbb{Q}(\theta_1, \dots, \theta_d)$ is isomorphic to $\mathbb{Q}(\theta_1)$. Recall that the cyclotomic field $\mathbb{Q}(\omega)$ is an algebraic extension of \mathbb{Q} , and

$$[\mathbb{Q}(\omega) : \mathbb{Q}] = \phi(p) = p - 1,$$

where ϕ denotes the Euler phi function. Since ω is the root of a quadratic polynomial over $\mathbb{Q}(\theta_1)$ and θ_1 is real, we see that $[\mathbb{Q}(\theta_1) : \mathbb{Q}] = (p - 1)/2$. \square

Finally, we obtain a very useful theorem about the linear independence of a subset of the eigenvalues of the cycle.

8.3 Theorem. *The set $\{1, \theta_1, \theta_2, \dots, \theta_{(p-3)/2}\}$ is linearly independent over \mathbb{Q} .*

Proof. Suppose, for a contradiction, that there exists some rational coefficients α_k for $0 \leq k \leq (p - 3)/2$ such that

$$\alpha_0 + \alpha_1\theta_1 + \alpha_2\theta_2 + \dots + \alpha_{(p-3)/2}\theta_{(p-3)/2} = 0,$$

with at least one α_k non-zero. Since we have $\theta_k = \omega^k + \omega^{-k}$, we see can re-express this equation in terms of ω as

$$\alpha_0 + \alpha_1(\omega + \omega^{-1}) + \alpha_2(\omega^2 + \omega^{-2}) + \dots + \alpha_{(p-3)/2}(\omega^{(p-3)/2} + \omega^{-(p-3)/2}) = 0.$$

Multiplying both sides by $\omega^{(p-3)/2}$ yields a polynomial equation in terms of ω with rational coefficients and degree at most $p - 3$. This contradicts the fact that $[\mathbb{Q}(\omega) : \mathbb{Q}] = p - 1$. \square

In our following work with ϵ -uniform mixing on C_p , we consider the scaled exponents of the eigenvalues of $U(t)$ as elements of the additive group \mathbb{R}/\mathbb{Z} . Recall that the direct product of d copies of \mathbb{R}/\mathbb{Z} is a *compact torus*. Suppose that t is an element of a compact torus T . We say that t is a *generator* if the smallest closed subgroup of T containing t is T itself. The following version of Kronecker's theorem appears as Theorem 15.1 in [5].

8.4 Theorem (Kronecker). *Let (t_1, \dots, t_r) denote an element of \mathbb{R}^r , and let t be the image of this point in $T = (\mathbb{R}/\mathbb{Z})^r$. Then t is a generator of T if and only if $\{1, t_1, \dots, t_r\}$ are linearly independent over \mathbb{Q} . \square*

9 ϵ -uniform mixing on C_p

In this section we show that ϵ -uniform mixing occurs on cycles of prime length. In particular, we show in these cases that $U(t)$ gets arbitrarily close to the scalar multiple of a complex Hadamard matrix.

We rely on the viewpoint of cyclic association schemes. Let p denote an odd prime and consider the cycle C_p . Let $d = \lfloor \frac{p}{2} \rfloor$. For $0 \leq r \leq d$ we define the following adjacency matrices.

$$[A_r]_{j,k} = \begin{cases} 1 & \text{if } j - k \in \{r, -r\} \pmod{p} \\ 0 & \text{otherwise.} \end{cases}$$

Note that A_0 is the $p \times p$ identity matrix, and A_1 is the adjacency matrix of the cycle C_p . Let $\mathcal{A} = \{A_0, \dots, A_d\}$. The set of matrices $\mathcal{A} = \{A_0, \dots, A_d\}$ form the *cyclic association scheme* of order p .

It is also convenient to consider the underlying permutation matrix C that is the adjacency matrix of a directed cycle. We index the rows and columns of C with elements of \mathbb{Z}_p such that

$$C_{j,k} = \begin{cases} 1 & \text{if } j - k \equiv 1 \pmod{p} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{A} = \{A_0, \dots, A_d\}$ denote the cyclic association scheme on p vertices. Since p is an odd prime, we see that \mathcal{A} has $d = (p - 1)/2$ classes. We further let $\{E_0, \dots, E_d\}$ denote the spectral idempotents of \mathcal{A} . We assume that these idempotents have been ordered such that $E_0 = \frac{1}{p}J$ and

$$E_r = \sum_{j=0}^{p-1} (\omega^{jr} + \omega^{-jr}) C^j.$$

for $1 \leq r < \lfloor n/2 \rfloor$.

To begin our new work, we define

$$F = \sum_{r=0}^d \omega^{r^2} E_r, \tag{9.1}$$

where $\omega = e^{i2\pi/p}$.

9.1 Lemma. *The matrix F is a flat unitary matrix.* □

Proof. First we verify that F is unitary. We do this by a direct computation. It is convenient to recall that

$$E_r^2 = E_r \quad \text{and} \quad E_r E_j = 0 \quad \text{if } r \neq j.$$

Using these observations we see that

$$FF^* = \left(\sum_{r=0}^d \omega^{r^2} E_r \right) \left(\sum_{r=0}^d \omega^{-r^2} E_r \right) = \sum_{r=0}^d E_r = I.$$

Next we use the discrete Fourier transform Θ to show that F is flat. Note that for an arbitrary matrix M in $\mathbb{C}[\mathcal{A}]$ there exists a unique polynomial $p(x)$ of degree at most $p-1$ in $\mathbb{C}[x]$ such that $M = p(C)$. Further note that Θ is defined such that

$$\Theta(M) = \sum_{j=0}^{p-1} p(\omega^j) C^j.$$

Since Θ is linear, we have

$$\begin{aligned} \Theta(F)\Theta(F^*) &= \left(\sum_{r=0}^d \omega^{r^2} \Theta(E_r) \right) \left(\sum_{r=0}^d \omega^{-r^2} \Theta(E_r) \right) \\ &= \left(\sum_{r=0}^d \omega^{r^2} A_r \right) \left(\sum_{r=0}^d \omega^{-r^2} A_r \right) \\ &= \left(\sum_{j=0}^{p-1} \omega^{j^2} C^j \right) \left(\sum_{j=0}^{p-1} \omega^{-j^2} C^j \right) \\ &= \sum_{k=0}^{p-1} \left(\sum_{j=0}^{p-1} \omega^{j^2 - (k-j)^2} \right) C^k \\ &= \sum_{k=0}^{p-1} \omega^{-k^2} \left(\sum_{j=0}^{p-1} \omega^{jk} \right) C^k \\ &= pC^0 = pI. \end{aligned}$$

By a well-known property of the discrete Fourier transform, $\Theta(F)\Theta(F^*) = pI$ implies that

$$F \circ F^* = \frac{1}{p} J.$$

Therefore F is flat. □

Our goal is to show that $U(t)$ gets arbitrarily close to a complex scalar multiple of F as t ranges over all real numbers. Since F is a flat matrix, achieving this goal implies that C_p admits ϵ -uniform mixing. The proof of this result relies heavily on Kronecker's Theorem.

9.2 Theorem. *The odd prime cycle C_p admits ϵ -uniform mixing.*

Proof. Let $U^-(t)$ denote the scaled transition matrix given by

$$U^-(t) = e^{-2it}U(t) = E_0 + e^{(\theta_1-2)it}E_1 + \dots + e^{(\theta_{d-2})it}E_{d-1}. \quad (9.2)$$

Note that $U^-(t)$ is a unitary matrix, and $U^-(t)$ is flat if and only if $U(t)$ is flat. Let ϵ denote a positive real number. We proceed by showing that there exists some time t such that

$$\|U^-(t) - F\| < \frac{\epsilon}{2}.$$

We consider $U^-(t)$ at times that are an integer multiple of $2\pi/p$. For any s in \mathbb{Z} , we see that Equation 9.2 becomes

$$U^-(2s\pi/p) = \sum_{r=0}^{d-1} e^{2s(\theta_r-2)\pi i/p} E_r.$$

In terms of e , we express Equation 9.1 as

$$F = \sum_{r=0}^{d-1} e^{2r^2\pi i/p} E_r.$$

Our goal is to find a time t such that the coordinates of F and $U^-(t)$ are close to the same value. In terms of the exponents of these coefficients, this is equivalent to finding some integer s such that $\frac{1}{p}r^2 \approx \frac{1}{p}(\theta_r - 2)s$ in (\mathbb{R}/\mathbb{Z}) for $0 \leq r \leq d-1$. For two elements x and y in \mathbb{R}/\mathbb{Z} , we define the distance $|x - y|_{\mathbb{R}/\mathbb{Z}}$ to be

$$|x - y|_{\mathbb{R}/\mathbb{Z}} = \inf_{k \in \mathbb{Z}} |x - y - k|,$$

where the norm on the right hand side of the definition is the absolute value of $x - y - k$ considered as a real number.

From Theorem 8.3, we know that $1, \theta_1, \dots, \theta_{d-1}$ are linearly independent over \mathbb{Q} , and consequently $\left\{1, \frac{1}{p}(\theta_1 - 2), \dots, \frac{1}{p}(\theta_{d-1} - 2)\right\}$ is linearly independent over the rationals.

By Kronecker's Theorem (Theorem 8.4), we see that

$$D = \left\{ \left(\frac{1}{p}(\theta_1 - 2)s, \dots, \frac{1}{p}(\theta_{d-1} - 2)s \right) : s \in \mathbb{Z} \right\}$$

is dense in $(\mathbb{R}/\mathbb{Z})^{d-1}$.

Therefore for any $\delta > 0$, we can find some s in \mathbb{Z} such that

$$\left| \frac{1}{p}(\theta_r - 2)s - \frac{r^2}{p} \right|_{\mathbb{R}/\mathbb{Z}} < \delta \quad (9.3)$$

in (\mathbb{R}/\mathbb{Z}) for all $1 \leq r \leq d-1$. It remains to consider the coordinates of $U^-(t)$ and F with respect to E_d . Recall that for a cyclic association scheme we have

$$\theta_d = -1 - \theta_1 - \theta_2 \cdots - \theta_{d-1}.$$

We can use this to derive an expression for the d -th coordinate of $U^-(t)$ in terms of the first $d-1$ coordinates, as follows:

$$\begin{aligned} \frac{1}{p}(\theta_d - 2)s &= \frac{1}{p} \left(-3 - \sum_{r=1}^{d-1} \theta_r \right) s \\ &= -\frac{1}{p}(2(d-1) + 3)s - \sum_{r=1}^{d-1} \frac{1}{p}(\theta_r - 2)s \\ &= -s - \sum_{r=1}^{d-1} \frac{1}{p}(\theta_r - 2)s. \end{aligned}$$

Now working in \mathbb{R}/\mathbb{Z} , we see that the exponent of the d -th coordinate of $U^-(t) - F$ is

$$\frac{1}{p}(\theta_d - 2)s - \frac{1}{p}d^2 = -s - \sum_{r=1}^{d-1} \frac{1}{p}(\theta_r - 2)s - \frac{1}{p}d^2 \quad (9.4)$$

$$= -\sum_{r=1}^{d-1} \left(\frac{1}{p}(\theta_r - 2)s - \frac{1}{p}r^2 \right) - \frac{1}{p} \sum_{r=0}^d r^2. \quad (9.5)$$

Note that

$$\frac{1}{p} \sum_{r=1}^d r^2 = \frac{d(d+1)(2d+1)}{6p} = \frac{(p-1)(p+1)}{24}$$

Since p is an odd prime, we know that both $p-1$ and $p+1$ are even, and exactly one of those values is divisible by 4. Therefore $(p-1)(p+1)$ is divisible by 8. Since we are assuming that $p \neq 3$, we also know that $p-1$ or $p+1$ is divisible by 3. It follows that

$$\frac{(p-1)(p+1)}{24} \in \mathbb{Z}.$$

must be an integer. We simplify Equation 9.4 in \mathbb{R}/\mathbb{Z} to

$$\frac{1}{p}(\theta_d - 2)s - \frac{1}{p}d^2 = -\sum_{r=1}^{d-1} \left(\frac{1}{p}(\theta_r - 2)s - \frac{1}{p}r^2 \right).$$

Now we use this expression and Inequality 9.3 to bound the coefficient d -th coordinate of $U^-(t) - F$ in terms of δ as follows:

$$\left| \frac{1}{p}(\theta_d - 2)s - \frac{1}{p}d^2 \right|_{\mathbb{R}/\mathbb{Z}} \leq \sum_{r=1}^{d-1} \left| \frac{1}{p}(\theta_r - 2)s - \frac{1}{p}r^2 \right|_{\mathbb{R}/\mathbb{Z}} < (d-1)\delta.$$

This implies that for any $\epsilon > 0$, we can find a sufficiently small δ such that

$$\|U^-(2s\pi/p) - F\| < \frac{\epsilon}{2}.$$

It can be shown that if A and B are symmetric $n \times n$ complex matrices, such that

$$\|A - B\| \leq \gamma,$$

for some positive real number γ . Then

$$\|A \circ A^* - B \circ B^*\| \leq 2\gamma.$$

Therefore it follows that

$$\|U^-(2s\pi/p) \circ U^-(2s\pi/p)^* - \frac{1}{n}J\| < \epsilon.$$

Finally, we note that $U^-(2s\pi/p) \circ U^-(2s\pi/p)^* = U(2s\pi/p) \circ U(2s\pi/p)^*$, which proves that ϵ -uniform mixing occurs on C_p . \square

10 Future work

Our results strongly supports the conjecture of Ahmadi et al. [2] that no cycle C_n , except for C_3 and C_4 , admits uniform mixing. In this paper we proved that no prime cycles, except for C_3 , admit uniform mixing. Thus it would be desirable to relate the mixing properties of C_{pm} to mixing properties of C_m for any prime p and integer m . This could lead to a complete proof of Ahmadi et al.'s conjecture.

Another area for future work is finding additional examples of graphs in association schemes that admit uniform mixing. Recent work by Chan shows that such graphs exist in certain schemes derived from n -cubes [9].

References

- [1] William Adamcak, Kevin Andrew, Leon Bergen, Dillon Ethier, Peter Hernberg, Jennifer Lin, and Christino Tamon. Non-uniform mixing of quantum walk on cycles. *International Journal of Quantum Information*, 05(06):781–793, 2007.
- [2] Amir Ahmadi, Ryan Belk, Christino Tamon, and Carolyn Wendler. On mixing in continuous-time quantum walks on some circulant graphs. *Quantum Inf. Comput.*, 3(6):611–618, 2003.

- [3] Ana Best, Markus Kliegl, Shawn Mead-Gluchacki, and Christino Tamon. Mixing of quantum walks on generalized hypercubes. *International Journal of Quantum Information*, 06(06):1135–1148, 2008.
- [4] A. E. Brouwer and J. H. van Lint. Strongly regular graphs and partial geometries. In *Enumeration and Design (Waterloo, Ont., 1982)*, pages 85–122. Academic Press, Toronto, ON, 1984.
- [5] Daniel Bump. *Lie Groups*. Springer, New York, second edition, 2013.
- [6] Edward B. Burger and Robert Tubbs. *Making Transcendence Transparent*. Springer-Verlag, New York, 2004.
- [7] William Carlson, Allison Ford, Elizabeth Harris, Julian Rosen, Christino Tamon, and Kathleen Wrobel. Universal mixing of quantum walk on graphs. *Quantum Inf. Comput.*, 7(8), 2007.
- [8] Ada Chan. Complex Hadamard matrices and strongly regular graphs. [arXiv:1102.5601](https://arxiv.org/abs/1102.5601), 2011.
- [9] Ada Chan. Complex Hadamard matrices, instantaneous uniform mixing and cubes. [arXiv:1305.5811](https://arxiv.org/abs/1305.5811), 2013.
- [10] Andrew M. Childs, Richard Cleve, Enrico Deotto, Edward Farhi, Sam Gutmann, and Daniel A. Spielman. Exponential algorithmic speedup by a quantum walk. In *Proceedings of the Thirty-Fifth Annual ACM Symposium on Theory of Computing, STOC '03*, pages 59–68, New York, NY, USA, 2003. ACM.
- [11] David Cox, John Little, and Donal O’Shea. *Ideals, Varieties, and Algorithms*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1997.
- [12] Heath Gerhardt and John Watrous. Continuous-time quantum walks on the symmetric group. In *Approximation, Randomization, and Combinatorial Optimization*, volume 2764 of *Lecture Notes in Comput. Sci.*, pages 290–301. Springer, Berlin, 2003.
- [13] Chris Godsil. State transfer on graphs. *Discrete Math.*, 312(1):129–147, 2012.
- [14] Uffe Haagerup. Cyclic p -roots of prime lengths p and related complex Hadamard matrices. [arXiv:0803.2629](https://arxiv.org/abs/0803.2629), pages 1–29, 2008.
- [15] Willem H. Haemers and Qing Xiang. Strongly regular graphs with parameters $(4m^4, 2m^4 + m^2, m^4 + m^2, m^4 + m^2)$ exist for all $m > 1$. *European J. Combin.*, 31(6):1553–1559, 2010.
- [16] Alastair Kay. Basics of perfect communication through quantum networks. *Phys. Rev. A*, 84:022337, Aug 2011.
- [17] Dale M. Mesner. A new family of partially balanced incomplete block designs with some Latin square design properties. *Ann. Math. Statist.*, 38:571–581, 1967.
- [18] Cristopher Moore and Alexander Russell. Quantum walks on the hypercube. In *Randomization and Approximation Techniques in Computer Science*, volume 2483 of *Lecture Notes in Comput. Sci.*, pages 164–178. Springer, Berlin, 2002.
- [19] Natalie Ellen Mullin. *Uniform Mixing of Quantum Walks and Association Schemes*. PhD thesis, University of Waterloo, 2013.

- [20] J. M. H. Olmsted. Discussions and notes: rational values of trigonometric functions. *Amer. Math. Monthly*, 52(9):507–508, 1945.
- [21] Pierre Samuel. *Algebraic Theory of Numbers*. Translated from the French by Allan J. Silberger. Houghton Mifflin Co., Boston, Mass., 1970.