Frankl's Conjecture for subgroup lattices

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Abstract

We show that the subgroup lattice of any finite group satisfies Frankl's Union-Closed Conjecture. We show the same for all lattices with a modular coatom, a family which includes all supersolvable and dually semimodular lattices. A common technical result used to prove both may be of some independent interest.

1 Introduction

1.1 Frankl's Conjecture

All groups and lattices considered in this paper will be finite. An element of a lattice is a *join-irreducible* if it cannot be written as the join of strictly lesser elements. We will examine the following conjecture, attributed to Frankl from 1979.

Conjecture 1.1 (Frankl's Union-Closed Conjecture)**.** *If L is a lattice with at least 2 elements, then there is a join-irreducible a with $\left|[a, \hat{1}]\right| \leqslant \frac{1}{2}\left|L\right|$.*

There are a number of different equivalent forms of this conjecture. The original form that Frankl considered involved a related condition for families of sets that are closed under intersection. The first appearance in print was in the conference proceedings [26], arising from its mention by Duffus in a problem session. Three forms of the problem are given in [26]: a statement about families of sets closed under union, Frankl's original form, and the lattice statement as we have here. Conjecture 1.1 appears as a 5-difficulty problem in [30], where it is called a "diabolical" problem. See [6] for further information and history. The conjecture has been the subject of a Polymath project [4].

We will henceforth refer to Conjecture 1.1 as Frankl's Conjecture. We will focus on the lattice form. If we wish to refer to the join-irreducible a satisfying the required condition, we will say that L satisfies Frankl's Conjecture with a.

Frankl's Conjecture, while open in general, is known to hold for many families of lattices. Poonen in [24] proved and generalized remarks of Duffus from [26]: namely, that the conjecture holds for distributive lattices, and for relatively complemented (including geometric) lattices. Reinhold [25] showed the conjecture to hold for dually semimodular lattices (see also [1, 29]). Whether the conjecture holds for semimodular lattices is in general unknown, but Czédli and Schmidt in [10] verified it for semimodular lattices that have a high ratio of elements to join-irreducibles. Joshi, Waphare, and Kavishwar have recently in [14] shown the conjecture to hold for dismantlable lattices.

We remark that Blinovsky has an arXiv preprint which claims to settle the Frankl Conjecture. However: his argument is difficult to follow, and has gone through a large number of arXiv versions in a short time. Moreover, he has also claimed to solve several other difficult conjectures in a short period, using the same technique. There does not seem at this time to be a consensus that his proof is correct.

1.2 Subgroup lattices

Recall that for a group G, the *subgroup lattice* of G is the set $L(G)$ of all subgroups of G, ordered by inclusion.

Our first main theorem verifies that Frankl's Conjecture holds for subgroup lattices.

Theorem 1.2. *If G is a group and $L(G)$ is the subgroup lattice of G, then $L(G)$ satisfies Conjecture 1.1.*

Subgroup lattices of groups form a large family of lattices. Indeed, it is an important open question (first asked by Pálfy and Pudlák [22]) as to whether every finite lattice occurs as an interval in the subgroup lattice of some finite group. Although most experts on the topics appear to believe the answer to the Pálfy-Pudlák question to be negative, progress has been somewhat limited. Indeed, the problem is difficult [5] even for lattices of height 2! See [2] and its references for further discussion of the Pálfy-Pudlák question and attempts to disprove it.

In light of the question of Pálfy and Pudlák, it would be highly interesting to settle Frankl's Conjecture in intervals of the form $[H, G]$ of $L(G)$. We cannot do this in general, but give group-theoretic sufficient conditions. We will state these conditions carefully in Corollary 1.5. We also verify that Frankl's Conjecture holds for every interval in a solvable group in Corollary 1.10.

1.3 Modular elements, subgroup lattices, and Frankl's Conjecture

An essential tool in the proof of Theorem 1.2 also has applications to many other lattices. For this reason, we give it in a quite general form.

An element m of a lattice L is *left-modular* if for every $a < b$ in L, the expression $a \vee m \wedge b$ can be written without parentheses. That is, if $a \vee (m \wedge b) = (a \vee m) \wedge b$ for every $a < b$. We show:

Theorem 1.3 (Main Technical Theorem). *Let L be a lattice, let $m \in L \setminus \{\hat{1}\}$ be left-modular, and let $x, y \in L$ be (not necessarily distinct) join-irreducibles. If $m \vee x \vee y = \hat{1}$, then L satisfies Frankl's Conjecture with either x or y.*

It follows from the well-known Dedekind Identity (see Section 2.1 below) that any normal subgroup N of G is left-modular in $L(G)$. It is straightforward to see that a subgroup X is a join-irreducible in $L(G)$ if and only if X is cyclic of prime-power order. Thus, we obtain the following as an easy consequence of Theorem 1.3.

Corollary 1.4. *If G is a group with a normal subgroup $N \lhd G$, such that G/N is generated by at most two elements of prime-power order, then $L(G)$ satisfies Frankl's Conjecture.*

The proof of Theorem 1.2 will be obtained by combining Corollary 1.4 with results on finite simple groups.

We similarly obtain a relative version for upper intervals in groups. The statement is somewhat harder to work with, as we are not aware of any short description for join-irreducibles in intervals of subgroup lattices.

Corollary 1.5. *Let G be a group and H be a subgroup. If X and Y are join-irreducibles of the interval $[H, G]$, and $N \lhd G$ is such that $HN < G$ but $HN \vee X \vee Y = G$, then the interval $[H, G]$ satisfies Frankl's Conjecture.*

1.4 The Averaged Frankl's Condition

A related question to Frankl's Conjecture asks for which lattices the average size over a join-irreducible element (other than $\hat{0}$) is at most $\frac{1}{2}|L|$. We call this condition the *Averaged Frankl's Condition*. The Averaged Frankl's Condition does not hold for all lattices, but is known to hold for lattices with a large ratio of elements to join-irreducibles [9]. The condition obviously holds for uncomplicated subgroup lattices such as $L(\mathbb{Z}_{p^n})$ or $L(\mathbb{Z}_p^n)$. Indeed, our techniques allow us to show a stronger condition for a restrictive class of groups.

Proposition 1.6. *If G is a supersolvable group in which all Sylow subgroups are elementary abelian, then G satisfies Frankl's Conjecture with any join-irreducible X.*

Supersolvable groups with elementary abelian subgroups are also known as *complemented groups*, and were first studied by Hall [12]. We don't know whether the subgroup lattices of arbitrary groups satisfy the Averaged Frankl's Condition.

1.5 Other lattices

Left-modular elements also occur in lattices from elsewhere in combinatorics. A situation that is both easy and useful is:

Corollary 1.7. *If a lattice L has a left-modular coatom m, then L satisfies Frankl's Conjecture.*

Proof. If $\hat{1}$ is a join-irreducible, then the result is trivial. Otherwise, there is some join-irreducible x such that $m \vee x = \hat{1}$, and we apply Theorem 1.3. $\square$

Remark 1.8. Shewale, Joshi, and Kharat prove [29, Theorem 2] that if every coatom of a lattice L is left-modular, then L satisfies Frankl's Conjecture. Indeed, a similar technique is already applied by Reinhold in [25]. After submission of this paper, we learned that the Shewale, Joshi, and Kharat go on to remark [29, Remark 2] that the same argument yields (a more general result than) Corollary 1.7.

There has been much study of classes of lattices that have a left-modular coatom. Dually semimodular lattices have every coatom left-modular, so we recovery the earlier-mentioned result [25] that such lattices satisfy Frankl's Conjecture. We also obtain the new result that *supersolvable* and *left-modular* lattices (those with a maximal chain consisting of left-modular elements) satisfy Frankl's Conjecture. See e.g. [20] for background on supersolvable lattices.

Still more generally, the *comodernistic* lattices recently examined by the second author and Schweig [28] are those lattices with a left-modular coatom on every interval. This class of lattices includes all supersolvable, left-modular, and dually semimodular lattices. It also includes other large classes of examples, including subgroup lattices of solvable groups and k-equal partition lattices.

Theorem 1.9. *Comodernistic lattices (including supersolvable, left-modular, and dually semimodular lattices) satisfy Frankl's Conjecture.*

Subgroup lattices of solvable groups are one family of examples of comodernistic lattices [28, Theorem 1.7]. That is, every interval in the subgroup lattice of a solvable group has a left-modular coatom. It follows immediately that:

Corollary 1.10. *If G is a solvable group, then every interval in $L(G)$ satisfies Frankl's Conjecture.*

Since the $\hat{0}$ element of any lattice is left-modular, Theorem 1.3 also yields the following:

Corollary 1.11. *If L is a lattice such that $\hat{1} = x \vee y$ for join-irreducibles x, y, then L satisfies Frankl's Conjecture.*

1.6 Organization

In Section 2 we will discuss the group-theoretic aspects of the problem. We will complete the proof of Corollary 1.4 and Theorem 1.2, pending only on the proof of Theorem 1.3. In Section 3, we will prove Theorem 1.3 and generalizations, as well as Proposition 1.6.

Acknowledgements

2 Groups, generation, and subgroup lattices

The main purpose of this section is to prove Theorem 1.2, as we do in Section 2.3. We first begin with some basic background on the combinatorics of subgroup lattices.

2.1 Modular elements in subgroup lattices

We first recall the well-known Dedekind Identity (see [27, 1.3.14] or [13, Exercise 2.9]):

Lemma 2.1 (Dedekind Modular Identity)**.** *If H, K, N are subgroups of a group G such that $H \leqslant K$, then $H(N \cap K) = HN \cap K$.*

It is also well known that HN is a subgroup of G if and only if $HN = NH = H \vee N$. These conditions are obviously satisfied when N is a normal subgroup, and are sometimes otherwise satisfied.

It is thus immediate from the Dedekind Identity that whenever HN is a subgroup, we also have that N satisfies the modular relation with H and any $K > H$. In particular, we recover our earlier claim that normal subgroups are left-modular in $L(G)$.

2.2 Proof of Corollary 1.4

Corollary 1.4 follows from the left-modularity of a normal subgroup N, together with another routine exercise: If $\overline{x}$ and $\overline{y}$ are elements of prime-power order in G/N, then there are $x, y \in G$ of prime-power order such that $\overline{x} = Nx$, $\overline{y} = Ny$ [13, Exercise 3.12]. In particular, the modular subgroup N and the join-irreducibles $\langle x \rangle$ and $\langle y \rangle$ satisfy the conditions of Theorem 1.3.

Corollary 1.5 follows by a similar argument.

2.3 Proof of Theorem 1.2

In order to prove Theorem 1.2, we will need a result on finite simple groups. King recently proved the following in [15].

Theorem 2.2 (Prime Generation Theorem [15]). *If G is any nonabelian finite simple group, then G is generated by an involution and an element of prime order.*

We now complete the proof of Theorem 1.2. Whenever N is a maximal normal subgroup of G, the quotient G/N is simple. Of course, abelian simple groups are generated by a single element of prime order. Nonabelian simple groups are handled by Theorem 2.2. Theorem 1.2 now follows from Corollary 1.4.

2.4 Overview of generation of simple groups by elements of prime order

The substantive work of King [15] in proving Theorem 2.2 builds on a large body of preceding work. We will briefly survey some history and mathematical details. We assume basic knowledge of the Classification of Finite Simple Groups in this discussion, but will not assume any such elsewhere in the paper.

Definition 2.3. A group G is said to be (p, q)-*generated* if G is generated by an element of order p and one of order q.

The case of $(2, 3)$-generation is particularly well-studied in the literature. Such groups are exactly the quotients (having order at least 6) of the infinite group $PSL_2(\mathbb{Z})$. There is also a connection with automorphism groups of compact Riemann surfaces [8]. In addition to the references below, see e.g. [23, 31] for more background on $(2, 3)$-generation.

The following has been known to hold for some time.

Proposition 2.4. *With at most finitely many exceptions, every nonabelian finite simple group is either $(2, 3)$- or $(2, 5)$-generated.*

We summarize the history behind Proposition 2.4. The alternating group A_n was shown to be $(2, 3)$-generated by Miller [21] for $n \neq 6, 7, 8$; while A_6, A_7 and A_8 are easily seen to be $(2, 5)$-generated. Excluding the groups $PSp_4(q)$, all but finitely many of the classical groups are $(2, 3)$-generated by work of Liebeck and Shalev [16]. In the same paper [16], the authors showed that, excluding finitely many exceptions, in characteristic 2 or 3 the groups PSp_4 are $(2, 5)$-generated. Cazzola and Di Martino in [7] showed PSp_4 to be $(2, 3)$-generated in all other characteristics. Lübeck and Malle [17] (building on earlier work by Malle [18, 19]) showed all simple exceptional groups excluding the Suzuki groups to be $(2, 3)$-generated. Evans [11] showed the Suzuki groups to be $(2, p)$-generated for any odd prime p dividing the group order, and in particular to be $(2, 5)$-generated. Proposition 2.4 now follows by combining the results enumerated here with the Classification of Finite Simple Groups.

We caution that $PSU_3(3^2)$ is known not to be $(2, 3)$-generated [32], and since it has order $|PSU_3(3^2)| = 2^5 \cdot 3^3 \cdot 7$, the group is certainly not $(2, 5)$-generated either.

King's proof of Theorem 2.2 proceeds by showing that every classical simple group G is either $(2, 3)$-, $(2, 5)$-, or $(2, r)$-generated, where r is a so-called Zsigmondy prime for G.

3 Proof of Theorem 1.3

Since $m \neq \hat{1}$, we see that $x \vee y \not\leqslant m$. If $x \leqslant m$, then we may replace the triple m, x, y with m, y, y while still meeting the conditions of the theorem. Thus, we may suppose without loss of generality that neither x nor y is on the interval $[\hat{0}, m]$.

Suppose without loss of generality that $[x, \hat{1}]$ has at most as many elements as $[y, \hat{1}]$. We will show that $|[x, \hat{1}]| \leqslant \frac{1}{2} |L|$ by constructing an injection from $[x, \hat{1}]$ to its complement in L.

We construct this injection in two stages. First, since $|[x, \hat{1}]| \leqslant |[y, \hat{1}]|$, there is an injection φ_1 that maps

$$\varphi_1 : [x, \hat{1}] \setminus [x \vee y, \hat{1}] \to [y, \hat{1}] \setminus [x \vee y, \hat{1}].$$

(We notice that if $x = y$, then $x \vee y = x = y$, and this will cause no trouble in our argument.) We then consider the map

$$\varphi_2 : [x \vee y, \hat{1}] \to [\hat{0}, m]$$
$$\alpha \mapsto m \wedge \alpha.$$

As $x \vee y \vee (m \wedge \alpha) = (x \vee y \vee m) \wedge \alpha = \hat{1} \wedge \alpha = \alpha$ by left-modularity, the map φ_2 is an injection. Since $x \not\leqslant m$, the image of φ_2 is contained in the complement of $[x, \hat{1}]$.

The two maps φ_1, φ_2 have disjoint domains. Combining them yields the desired injection.

3.1 Generalizations

Examining our proof of Theorem 1.3, we observe that we do not use the full power of left-modularity, but only that m satisfies the left-modular relation for any $\alpha > x \vee y$. Thus, we have actually proved the following generalization:

Proposition 3.1. *Let L be a lattice, and let $x, y \in L$ be (not necessarily distinct) join-irreducibles. If $m \in L \setminus \{\hat{1}\}$ satisfies $(x \vee y \vee m) \wedge \alpha = (x \vee y) \vee (m \wedge \alpha)$ for any $\alpha > x \vee y$, and $m \vee x \vee y = \hat{1}$, then L satisfies Frankl's Conjecture with either x or y.*

While the statement of Proposition 3.1 appears notably more complicated than that of Theorem 1.3, it yields a reasonably uncomplicated corollary for intervals in subgroup lattices.

Corollary 3.2. *Let G be a group, let $H < G$, and let X, Y be join-irreducibles of $[H, G]$. If there is a subgroup K with $H < K < G$ such that $K(X \vee Y) = G$, then the interval $[H, G]$ satisfies Frankl's Conjecture.*

We in particular are now able to prove Proposition 1.6.

Proof (of Proposition 1.6). It follows by a theorem of Hall [12] that for every subgroup H in G, there is some subgroup K such that $KH = G$ and $H \cap K = 1$. The result follows by combining the theorem of Hall with Corollary 3.2. $\square$

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