On some Euler-Mahonian distributions

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Submitted: Dec 22, 2016; Accepted: Jul 26, 2017; Published: Aug 11, 2017

Abstract

We prove that the pair of statistics (des, maj) on multiset permutations is equidistributed with the pair (stc, inv) on certain quotients of the symmetric groups. We define an analogue of the statistic stc on multiset permutations whose joint distribution with the inversions equals that of (des, maj). We extend the definition of the statistic stc to hyperoctahedral and even hyperoctahedral groups. These functions, together with the Coxeter length, are equidistributed with (ndes, nmaj) and (ddes, dmaj), respectively.

1 Introduction

The first result about the enumeration of multiset permutations with respect to the statistics now called *descent number* and *major index* is due to MacMahon. Let $\rho = (\rho_1, \dots, \rho_m)$ be a composition of $N \in \mathbb{N}$. We denote by S_ρ the set of all permutations of the multiset $\{1^{\rho_1}, \dots, m^{\rho_m}\}$. The *descent set* $\mathrm{Des}(w)$ of the multiset permutation $w = w_1 \cdots w_N \in S_\rho$ is $\mathrm{Des}(w) = \{i \in [N-1] \mid w_i > w_{i+1}\}$. The descent and major index statistics on S_ρ are given by

$$\mathrm{des}(w) = |\,\mathrm{Des}(w)| \quad \text{and} \quad \mathrm{maj}(w) = \sum_{i \in \mathrm{Des}(w)} i.$$

Then ([10, §462, Vol. 2, Ch. IV, Sect. IX])

$$\sum_{k \geqslant 0} \left(\prod_{j=1}^{m} \binom{\rho_j + k}{k}_q \right) x^k = \frac{\sum_{w \in S_\rho} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)}}{\prod_{i=0}^{N} (1 - xq^i)} \in \mathbb{Z}[q][\![x]\!], \tag{1}$$

*Supported by the German-Israeli Foundation for Scientific Research and Development, grant no. 1246.

where, for $n, k \in \mathbb{N}$, we set

$$\binom{n}{k}_p = \frac{[n]_p!}{[n-k]_p![k]_p!}, \qquad [n]_p! = \prod_{i=1}^{n} [i]_p, \qquad [n]_p = \sum_{i=0}^{n-1} p^i.$$

The well-known result about the equidistribution on multiset permutations of the *inversion number* with the major index also goes back to MacMahon. In [7] Foata and Schützenberger proved that this equidistribution refines, in the case of the symmetric group, to inverse descent classes. A pair of statistics that is equidistributed with $(\mathrm{des}, \mathrm{maj})$ is called *Euler-Mahonian*. In [12] Skandera introduced an Eulerian statistic, which he called stc, on the symmetric group and proved that the pair $(\mathrm{stc}, \mathrm{inv})$ is Euler-Mahonian. He pointed out that the word statistic obtained by generalising stc in the natural way is not Eulerian. He also left open the question of finding a suitable generalisation with this property.

In this note we prove that the joint distribution of $(\mathrm{stc}, \mathrm{inv})$ on certain quotients of the symmetric group is indeed the same as the distribution of $(\mathrm{des}, \mathrm{maj})$ on multiset permutations; we use this result to define a statistic mstc that is Eulerian on multiset permutations and that, together with inv, is equidistributed with the pair $(\mathrm{des}, \mathrm{maj})$.

To the author's knowledge, not much is known about factorisations of the bivariate polynomials defined by Euler-Mahonian distributions. More is known (or conjectured) about the Eulerian polynomial $\sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)}$ and its generalisations to multiset permutations. Frobenius proved (see [8]) that the Eulerian polynomial has real, simple, negative roots, and that -1 features as a root if and only if n is even. Later, Simion proved that its analogues on permutations of any multiset are also real rooted with simple, negative roots (see [11]).

We use our first result of equidistribution to show that the polynomial of the joint distribution of des and maj admits, on the set of permutations of words in the alphabet $\{1^r, 2^r\}$, for odd r, a unique *unitary* factor (i.e. a factor which arises from a univariate polynomial, all of whose roots lie on the unit circle, by means of a monomial substitution; cf. Section 2 for a precise definition). Together with the factorisation of Carlitz's q-Eulerian polynomial showed in [4, Lemma 2.7], this result may be considered a refinement of Frobenius' and Simion's results. The previous results support a conjecture we made in [4, Conjecture B] and which we translate in Section 2 in terms of the joint distribution of $(\mathrm{stc}, \mathrm{inv})$ on quotients of the symmetric group.

Our interest in detecting factorisations of this type is motivated by questions regarding analytic properties of certain Dirichlet series studied by the author and Voll in [4]. These series are closely related to Hadamard products of subgroup zeta functions of free abelian groups of finite rank. Their numerators are exactly the polynomials of the joint distributions of $(\mathrm{des}, \mathrm{maj})$ on appropriate sets of multiset permutations. This combinatorial description allows us to easily read off analytic properties such as abscissa of convergence, meromorphic continuation or the existence of a natural boundary.

Generalisations of MacMahon's result (1) to signed permutations were first obtained by Adin, Brenti and Roichman in [1] and to even-signed permutations by Biagioli in [2]. In the last section of this note we define Eulerian statistics nstc and dstc which, together with the length, are equidistributed with the Euler-Mahonian pairs (ndes, nmaj) on the hyperoctahedral group and (ddes, dmaj) on the even hyperoctahedral group, respectively.

2 Stc on quotients of the symmetric group and multiset permutations

We start with some definitions and notation, for further notation and basic facts about Coxeter groups we refer the reader to [3].

For $n, m \in \mathbb{N}$, $m \leqslant n$ we denote with $[n] = \{1, \ldots, n\}$ and $[m, n] = \{m, m+1, \ldots, n\}$. For a permutation $\sigma \in S_n$ we use the one-line notation or the disjoint cycle notation. For a (signed) permutation $\sigma \in S_n$ (respectively, B_n), we let

$$\operatorname{Inv}(\sigma) = \{(i, j) \in [n] \times [n] \mid i < j,\, \sigma(i) > \sigma(j)\} \text{ and } \operatorname{inv}(\sigma) = |\operatorname{Inv}(\sigma)|.$$

The symmetric group S_n is a Coxeter group with respect to the set of generators $S = \{s_1, \ldots, s_{n-1}\}$, where s_i is the simple transposition $(i,\, i+1)$. It is well-known that the Coxeter length coincides with the inversion number. For $J \subseteq [n-1]$ the corresponding quotient is defined as

$$S_n^J = \{w \in S_n \mid \operatorname{Des}(w) \subseteq [n-1] \setminus J\}.$$

Moreover we denote with

$$IS_n^J = \{w \in S_n \mid \operatorname{Des}(w^{-1}) \subseteq [n-1] \setminus J\}$$

the inverse descent class corresponding to $J \subseteq [n-1]$.

It is well-known that the symmetric group S_n is in bijection with the set of words $w = w_1 \cdots w_n \in E_n$ where

$$E_n = \{w_1 \cdots w_n \mid w_i \in [0, n-i] \text{ for } i = 1, \ldots, n\}.$$

One of such bijections is the Lehmer code, defined as follows. For $\sigma \in S_n$, $\operatorname{code}(\sigma) = c_1 \cdots c_n \in E_n$ where $c_i = |\{j \in [i+1, n] \mid \sigma(i) > \sigma(j)\}|$. The sum of the c_is gives, for each permutation, the inversion number. The Eulerian statistic stc is defined as follows (cf. [12, Definition 3.1]): $\operatorname{stc}(\sigma) = \operatorname{st}(\operatorname{code}(\sigma))$, where for a word $w \in E_n$,

$$\operatorname{st}(w) = \max\{r \in [n] \mid \exists\, 1 \leqslant i_1, \cdots < i_r \leqslant n \mid w_{i_1} \cdots w_{i_r} > (r-1)(r-2) \cdots 1\, 0\}.$$

Informally, $\operatorname{st}(w)$ is the maximum r for which there exists a subword of w of length r elementwise strictly greater than the *r-staircase word* $(r-1)(r-2) \cdots 1\, 0$.

For example let $\sigma = 452361 \in S_6$. Then $\operatorname{code}(\sigma) = 331110$, $\operatorname{inv}(\sigma) = \sum_i c_i = 9$, $\operatorname{stc}(\sigma) = \operatorname{st}(\operatorname{code}(\sigma)) = 3$. The statistic stc constitutes an Eulerian partner for the inversions on S_n.

Theorem 1 ([12, Theorem 3.1]). *Let $n \in \mathbb{N}$. Then*

$$\sum_{w \in S_n} x^{\operatorname{des}(w)} q^{\operatorname{maj}(w)} = \sum_{w \in S_n} x^{\operatorname{stc}(w)} q^{\ell(w)}$$

Recall that given a composition ρ of N, we denote by S_ρ the set of permutations of the multiset $\{1^{\rho_1}, \ldots, m^{\rho_m}\}$, i.e. rearrangements of the word $w_\rho = \underbrace{1\ldots1}_{\rho_1}\underbrace{2\ldots2}_{\rho_2}\cdots\underbrace{m\ldots m}_{\rho_m}$.

A natural way to associate a permutation to a multiset permutation is the standardisation, which we now describe; cf. also [13, §1.7]. Given ρ a composition of N and a word w in the alphabet $\{1^{\rho_1}, \ldots, m^{\rho_m}\}$, $\mathrm{std}(w)$ is the element of S_N obtained by w substituting, in the order of appearance in w from left to right, the ρ_1 1s with the sequence $1\,2\ldots\rho_1$, the ρ_2 2s with the sequence $\rho_1+1\ldots\rho_1+\rho_2$ and so on. So for example if $\rho=(2,3,2)$ and $w = 1223132 \in S_\rho$, then $\mathrm{std}(w) = 1346275 \in S_7$. Clearly the map $\mathrm{std} : S_\rho \to S_N$ is not surjective. The set S_ρ is in bijection with certain quotients and inverse descent classes of S_N depending on ρ. In particular, for $\rho = (\rho_1, \ldots, \rho_m)$ a composition of N, for $i = 1\ldots, m-1$ we let

$$r_i = \sum_{k=1}^{i} \rho_k, \qquad \mathbf{r} = \{r_i \mid i \in [m-1]\} \subseteq [N-1] \qquad \text{and} \qquad R = [N-1] \setminus \mathbf{r}. \tag{2}$$

The quotient and the inverse descent class that will play a role in our results of equidistribution are S_N^R and IS_N^R, respectively.

We will need the following result due to Foata and Han.

Proposition 2 ([6, Proprieté 2.2])**.** *Let $n \in \mathbb{N}$, $J \subseteq [n-1]$. Then*

$$\sum_{\substack{\{w\in S_n \mid \\ \mathrm{Des}(w)=J\}}} x^{\mathrm{des}(w^{-1})} q^{\mathrm{maj}(w^{-1})} = \sum_{\substack{\{w\in S_n \mid \\ \mathrm{Des}(w)=J\}}} x^{\mathrm{stc}(w)} q^{\ell(w)}. \tag{3}$$

Our first result is the following.

Proposition 3. *Let $N \in \mathbb{N}$, ρ a composition of N and $R \subseteq [N-1]$ as in* (2)*. The pair* (stc, ℓ) *on S_N^R is equidistributed with* $(\mathrm{des}, \mathrm{maj})$ *on S_ρ:*

$$C_\rho(x,q) = \sum_{w\in S_\rho} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)} = \sum_{w\in S_N^R} x^{\mathrm{stc}(w)} q^{\ell(w)}. \tag{4}$$

Proof. The standardisation std is a bijection between S_ρ and IS_N^R, and preserves des and maj, so

$$\sum_{w\in S_\rho} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)} = \sum_{w\in S_\rho} x^{\mathrm{des}(\mathrm{std}(w))} q^{\mathrm{maj}(\mathrm{std}(w))} = \sum_{w\in IS_N^R} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)} = \sum_{w\in S_N^R} x^{\mathrm{stc}(w)} q^{\ell(w)},$$

where the last equality follows from Proposition 2. □

As an application, we prove a result about the bivariate factorisation of the polynomial $C_\rho(x,q)$, that in [4] is used to deduce analytic properties of some orbit Dirichlet series. We will need the following definition (see also [4, Remark 2.9]).

Definition 4. We say that a bivariate polynomial $f(x,y) \in \mathbb{Z}[x,y]$ is unitary if there exist integers $\alpha, \beta \geqslant 0$ and $g \in \mathbb{Z}[t]$ so that $f(x,y) = g(x^\alpha y^\beta)$ and all the complex roots of g lie on the unit circle.

Proposition 5. *Let $\rho = (r,r)$ where $r \equiv 1 \pmod 2$. Then*

$$C_\rho(x,q) = (1 + xq^r)\widetilde{C}_\rho(x,q), \tag{5}$$

where $\widetilde{C}_\rho(x,q)$ has no unitary factor.

Before we prove Proposition 5, we give a characterisation of the stc for permutations with at most one descent.

Lemma 6. *Let $\rho = (\rho_1, \rho_2)$, $N = \rho_1 + \rho_2$. Let $w \in S_N^{\{\rho_1\}^c}$. Then*

$$\mathrm{stc}(w) = |\{i \in [\rho_1] \mid w(i) > \rho_1\}|.$$

Proof. A permutation $w \in S_N^{\{\rho_1\}^c}$ has at most a descent at ρ_1, so its code is of the form $\mathrm{code}(w) = c_1 \cdots c_{\rho_1} 0 \cdots 0$, with $0 \leqslant c_1 \leqslant \ldots \leqslant c_{\rho_1}$. The first (possibly) non-zero element of the code is exactly the number of elements of the second block for which the image is in the first block. This number coincides with the length of the longest possible subword of the code which is elementwise greater than a staircase word. □

Proof of Proposition 5. The polynomial $C_\rho(x,1)$, descent polynomial of S_ρ, has all real, simple, negative roots (cf. [11, Corollary 2]). Thus a factorisation of the form (5) implies that $\widetilde{C}_\rho(x,q)$ has no unitary factor. To prove (5) we define an involution φ on S_N^R such that, for all $w \in S_N^R$, $|\ell(\varphi(w)) - \ell(w)| = r$ and $|\mathrm{stc}(\varphi(w)) - \mathrm{stc}(w)| = 1$.

For $w \in S_N^R$ we define

$$M_w = \{i \in [r] \mid w^{-1}(i) \leqslant r \text{ and } w^{-1}(i+r) > r \text{ or } w^{-1}(i) > r \text{ and } w^{-1}(i+r) \leqslant r\},$$

the set of $i \in [r]$ for which i and $i + r$ are not in the same ascending block. Since r is odd, M_w is non-empty for all $w \in S_\rho$. We then define $\varphi(w) = ((\iota, \iota + r)w)^R$, where $\iota = \min\{i \in M_w\}$ and, for $\sigma \in S_N$, σ^R denotes the unique minimal coset representative in the quotient S_N^R. By Lemma 6 clearly $\mathrm{stc}(\varphi(w)) = \mathrm{stc}(w) \pm 1$. Suppose now that $w^{-1}(\iota) \leqslant r$ and $w^{-1}(\iota) > r$ (the other case is analogous). Then

$$\begin{aligned}\ell(\varphi(w)) &= \ell(w) + |\{i \in [r] \mid w(i) > \iota\}| + |\{i \in [r+1, 2r] \mid w(i) < \iota + r\}| \\ &= \ell(w) + r - i + i\end{aligned}$$

as desired. □

We now reformulate [4, Conjecture B] in terms of the bivariate distribution of (stc, ℓ) on quotients of the symmetric group.

Conjecture A. *Let ρ be a composition of N and $R \subseteq [N-1]$ constructed as in (2). Then $C_\rho(x,q) = \sum_{w \in S_N^R} x^{\mathrm{stc}(w)} q^{\ell(w)}$ has a unitary factor if and only if $\rho = (\rho_1, \ldots, \rho_m)$ where $\rho_1 = \ldots = \rho_m = r$ for some odd r and even m. In this case*

$$\sum_{w \in S_N^R} x^{\mathrm{stc}(w)} q^{\ell(w)} = (1 + xq^{\frac{rm}{2}}) \widetilde{C}_\rho(x,q)$$

for some $\widetilde{C}_\rho(x,q) \in \mathbb{Z}[x,q]$ with no unitary factors.

Proposition 3 suggests a natural extension of the definition of the statistic stc to multiset permutations, thus answering a question raised by Skandera, see [12, Question 5.1].

For $w \in S_\rho$, $\mathrm{std}(w) \in IS_N^R$. This yields a bijection between multiset permutations S_ρ and the quotient S_N^R

$$\mathrm{istd} : S_\rho \to S_N^R, \quad \mathrm{istd}(w) = (\mathrm{std}(w))^{-1}$$

which is inversion preserving: $\mathrm{inv}(w) = \mathrm{inv}(\mathrm{istd}(w))$.

Definition 7. Let ρ be a composition of N. For a multiset permutation $w \in S_\rho$ the *multistc* is

$$\mathrm{mstc}(w) = \mathrm{stc}(\mathrm{istd}(w)).$$

The pair $(\mathrm{mstc}, \mathrm{inv})$ is equidistributed with $(\mathrm{des}, \mathrm{maj})$ on S_ρ, as

$$\sum_{w \in S_\rho} x^{\mathrm{mstc}(w)} q^{\mathrm{inv}(w)} = \sum_{w \in S_N^R} x^{\mathrm{stc}(w)} q^{\mathrm{inv}(w)} = \sum_{w \in S_\rho} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)},$$

which together with (1) proves the following theorem.

Theorem 8. *Let ρ be a composition of $N \in \mathbb{N}$. Then*

$$\sum_{k \geqslant 0} \left(\prod_{j=1}^{m} \binom{\rho_j + k}{k}_q \right) x^k = \frac{\sum_{w \in S_\rho} x^{\mathrm{mstc}(w)} q^{\mathrm{inv}(w)}}{\prod_{i=0}^{N} (1 - xq^i)} \in \mathbb{Z}[q][\![x]\!].$$

3 Signed and even-signed permutations

MacMahon's result (1) for the symmetric group (i.e. for $\rho_1 = \ldots \rho_m = 1$) is often present in the literature as Carlitz's identity, satisfied by Carlitz's q-Eulerian polynomial $A_n(x,q) = \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} q^{\mathrm{maj}(\sigma)}$.

This result was extended, for suitable statistics, to the groups of signed and even-signed permutations. The major indices defined in such extensions are in both cases equidistributed with the Coxeter length ℓ. In this section we define type B and type D analogues of the statistic stc, that together with the length satisfy these generalised Carlitz's identities.

3.1 Eulerian companion for the length in type B

Let $n \in \mathbb{N}$. The hyperoctahedral group B_n is the group of permutations $\sigma = \sigma_1 \cdots \sigma_n$ of $\{\pm 1, \ldots, \pm n\}$ for which $|\sigma| = |\sigma_1| \ldots |\sigma_n| \in S_n$. For $\sigma \in B_n$, the negative set and negative statistic are

$$\operatorname{Neg}(\sigma) = \{i \in [n] \mid \sigma(i) < 0\} \quad \operatorname{neg}(\sigma) = |\operatorname{Neg}(\sigma)|.$$

The Coxeter length ℓ for σ in B_n has the following combinatorial interpretation (see, for instance [3]):

$$\ell(\sigma) = \operatorname{inv}(\sigma) + \operatorname{neg}(\sigma) + \operatorname{nsp}(\sigma),$$

where inv is the usual inversion number and $\operatorname{nsp}(\sigma) = |\{(i,j) \in [n] \times [n] \mid i < j, \sigma(i) + \sigma(j) < 0\}|$ is the number of negative sum pairs.

In [1] an Euler-Mahonian pair of the negative type was defined as follows. The negative descent and negative major index are, respectively,

$$\operatorname{ndes}(\sigma) = \operatorname{des}(\sigma) + \operatorname{neg}(\sigma), \qquad \operatorname{nmaj}(\sigma) = \operatorname{maj}(\sigma) - \sum_{i \in \operatorname{Neg}(\sigma)} \sigma(i). \tag{6}$$

The pair (ndes, nmaj) satisfies the following generalised Carlitz's identity.

Theorem 9 ([1, Theorem 3.2]). *Let $n \in \mathbb{N}$. Then*

$$\sum_{r \geqslant 0} [r+1]_q^n x^r = \frac{\sum_{\sigma \in B_n} x^{\operatorname{ndes}(\sigma)} q^{\operatorname{nmaj}(\sigma)}}{(1-x) \prod_{i=1}^{n} (1 - x^2 q^{2i})} \quad \text{in } \mathbb{Z}[q][\![x]\!]. \tag{7}$$

Motivated by (6) and the well-known fact that the length in type B may be also written as

$$\ell(\sigma) = \operatorname{inv}(\sigma) - \sum_{i \in \operatorname{Neg}(\sigma)} \sigma(i), \tag{8}$$

we define the analogue of the statistic stc for signed permutations as follows.

Definition 10. Let $\sigma \in B_n$. Then

$$\operatorname{nstc}(\sigma) = \operatorname{stc}(\sigma) + \operatorname{neg}(\sigma).$$

Theorem 11. *Let $n \in \mathbb{N}$. Then*

$$\sum_{\sigma \in B_n} x^{\operatorname{nstc}(\sigma)} q^{\ell(\sigma)} = \sum_{\sigma \in B_n} x^{\operatorname{ndes}(\sigma)} q^{\operatorname{nmaj}(\sigma)}.$$

Proof. We use essentially the same argument as in the proof of [9, Theorem 3]. There, the following decomposition of B_n is used. Every permutation $\tau \in S_n$ is associated with 2^n elements of B_n, via the choice of the n signs. More precisely, given a signed permutation

$\sigma \in B_n$ one can consider the ordinary permutation in which the elements are in the same relative positions as in σ. We write $\pi(\sigma) = \tau$. Then

$$B_n = \bigcup_{\tau \in S_n} B(\tau),$$

where $B(\tau) = \{\sigma \in B_n \mid \pi(\sigma) = \tau\}$. So every $\sigma \in B_n$ is uniquely identified by the permutation $\tau = \pi(\sigma)$ and the choice of signs $J(\sigma) = \{\sigma(j) \mid j \in \mathrm{Neg}(\sigma)\}$.

Clearly, for $\sigma \in B_n$ we have $\mathrm{Inv}(\sigma) = \mathrm{Inv}(\pi(\sigma))$, and thus $\mathrm{stc}(\sigma) = \mathrm{stc}(\pi(\sigma))$. So, for $\tau = \pi(\sigma)$

$$x^{\mathrm{nstc}(\sigma)} q^{\ell(\sigma)} = x^{\mathrm{stc}(\tau)} q^{\mathrm{inv}(\tau)} \prod_{j \in J(\sigma)} x q^j.$$

The claim follows, as

$$\sum_{\sigma \in B_n} x^{\mathrm{nstc}(\sigma)} q^{\ell(\sigma)} = \sum_{\sigma \in B(\tau)} \sum_{\tau \in S_n} x^{\mathrm{stc}(\tau)} q^{\mathrm{inv}(\tau)} \sum_{J \subseteq [n]} \prod_{j \in J} x q^j = A_n(x,q) \prod_{i=1}^{n} (1 + x q^i).$$

□

Corollary 12. *Let $n \in \mathbb{N}$. Then*

$$\sum_{r \geqslant 0} [r+1]_q^n x^r = \frac{\sum_{\sigma \in B_n} x^{\mathrm{nstc}(\sigma)} q^{\ell(\sigma)}}{(1-x) \prod_{i=1}^{n} (1 - x^2 q^{2i})} \quad \textit{in } \mathbb{Z}[q][\![x]\!]. \tag{9}$$

3.2 Eulerian companion for the length in type D

The even hyperoctahedral group D_n is the subgroup of B_n of signed permutations for which the negative statistic is even:

$$D_n = \{\sigma \in B_n \mid \mathrm{neg}(\sigma) \equiv 0 \pmod 2\}.$$

Also for σ in D_n the Coxeter length can be computed in terms of the following statistics:

$$\ell(\sigma) = \mathrm{inv}(\sigma) + \mathrm{nsp}(\sigma). \tag{10}$$

The problem of finding an analogue, on the group D_n of even signed permutations, was solved in [2], where type D statistics des and maj were defined, as follows. For $\sigma \in D_n$

$$\mathrm{ddes}(\sigma) = \mathrm{des}(\sigma) + |\,\mathrm{DNeg}(\sigma)|, \qquad \mathrm{dmaj}(\sigma) = \mathrm{maj}(\sigma) - \sum_{i \in \mathrm{DNeg}(\sigma)} (\sigma(i) + 1), \tag{11}$$

where $\mathrm{DNeg}(\sigma) = \{i \in [n] | \sigma(i) < -1\}$. The following holds.

Theorem 13 ([2, Theorem 3.4])**.** *Let $n \in \mathbb{N}$. Then*

$$\sum_{r\geqslant 0}[r+1]_q^n x^r = \frac{\sum\limits_{\sigma\in D_n} x^{\mathrm{ddes}(\sigma)}q^{\mathrm{dmaj}(\sigma)}}{(1-x)(1-xq^n)\prod\limits_{i=1}^{n-1}(1-x^2q^{2i})} \text{ in } \mathbb{Z}[q][\![x]\!]. \tag{12}$$

Definition 14. Let $\sigma \in D_n$. We set

$$\mathrm{dstc}(\sigma) = \mathrm{stc}(\sigma) + |\,\mathrm{DNeg}(\sigma)| = \mathrm{stc}(\sigma) + \mathrm{neg}(\sigma) + \varepsilon(\sigma),$$

where

$$\varepsilon(\sigma) = \begin{cases} -1 \text{ if } \sigma^{-1}(1) < 0, \\ 0 \quad \text{otherwise} \;. \end{cases}$$

We now show that the statistic just defined constitutes an Eulerian partner for the length on D_n, that is, the following holds.

Theorem 15. *Let $n \in \mathbb{N}$. Then*

$$\sum_{\sigma\in D_n} x^{\mathrm{dstc}(\sigma)}q^{\ell(\sigma)} = \sum_{\sigma\in D_n} x^{\mathrm{ddes}(\sigma)}q^{\mathrm{dmaj}(\sigma)}.$$

Proof. We use, as in [2] the following decomposition of D_n. Let

$$T_n = \{\alpha \in D_n \mid \mathrm{des}(\alpha) = 0\} = \{\alpha \in D_n \mid \mathrm{Inv}(\alpha) = \varnothing\} \tag{13}$$

then D_n can be rewritten as the following disjoint union:

$$D_n = \bigcup_{\tau\in S_n}\{\alpha\tau \mid \alpha \in T_n\}. \tag{14}$$

For $\alpha \in T_n$ and $\tau \in S_n$ the following hold:

$$\begin{aligned} \ell(\alpha\tau) &= \ell(\alpha) + \ell(\tau) = \mathrm{nsp}(\alpha) + \mathrm{inv}(\tau), \\ \mathrm{nsp}(\alpha\tau) &= \mathrm{nsp}(\alpha), \\ \mathrm{dstc}(\alpha\tau) &= \mathrm{stc}(\tau) + \mathrm{neg}(\alpha) + \varepsilon(\sigma). \end{aligned}$$

The last equation follows from the second equality in (13). Thus

$$\begin{aligned} \sum_{\sigma\in D_n} x^{\mathrm{dstc}(\sigma)}q^{\ell(\sigma)} &= \sum_{\alpha\in T_n}\sum_{\tau\in S_n} x^{\mathrm{stc}(\tau)+\mathrm{neg}(\alpha)+\varepsilon(\alpha)}q^{\ell(\alpha)+\ell(\tau)} \\ &= \sum_{\alpha\in T_n} x^{\mathrm{neg}(\alpha)+\varepsilon(\alpha)}q^{\mathrm{nsp}(\alpha)}\sum_{\tau\in S_n} x^{\mathrm{stc}(\tau)}q^{\mathrm{inv}(\tau)} \\ &= \prod_{i=1}^{n-1}(1+xq^i)A_n(x,q) \end{aligned}$$

for the last equality see [2, Lemma 3.3]. The result follows, as

$$\sum_{\sigma \in D_n} x^{\mathrm{ddes}(\sigma)} q^{\mathrm{dmaj}(\sigma)} = \prod_{i=1}^{n-1} (1 + xq^i) A_n(x, q).$$

□

Corollary 16. *Let $n \in \mathbb{N}$. Then*

$$\sum_{r \geqslant 0} [r+1]_q^n x^r = \frac{\sum_{\sigma \in D_n} x^{\mathrm{dstc}(\sigma)} q^{\ell(\sigma)}}{(1-x)(1-xq^n) \prod_{i=1}^{n-1} (1 - x^2 q^{2i})} \quad \text{in } \mathbb{Z}[q][\![x]\!]. \tag{15}$$

Acknowledgements

The author would like to thank Christopher Voll for his comments.

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