Beyond Degree Choosability

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Abstract

Let G be a connected graph with maximum degree Δ. Brooks' theorem states that G has a Δ-coloring unless G is a complete graph or an odd cycle. A graph G is *degree-choosable* if G can be properly colored from its lists whenever each vertex v gets a list of $d(v)$ colors. In the context of list coloring, Brooks' theorem can be strengthened to the following. Every connected graph G is degree-choosable unless each block of G is a complete graph or an odd cycle; such a graph G is a *Gallai tree*.

This degree-choosability result was further strengthened to Alon–Tarsi orientations; these are orientations of G in which the number of spanning Eulerian subgraphs with an even number of edges differs from the number with an odd number of edges. A graph G is *degree-AT* if G has an Alon–Tarsi orientation in which each vertex has indegree at least 1. Alon and Tarsi showed that if G is degree-AT, then G is also degree-choosable. Hladký, Kráľ, and Schauz showed that a connected graph is degree-AT if and only if it is not a Gallai tree. In this paper, we consider pairs (G, x) where G is a connected graph and x is some specified vertex in $V(G)$. We characterize pairs such that G has no Alon–Tarsi orientation in which each vertex has indegree at least 1 and x has indegree at least 2. When G is 2-connected, the characterization is simple to state.

Keywords: list-coloring, choosability, degree-choosable, Alon–Tarsi orientation, Gallai tree

1 Introduction

Brooks' theorem is one of the fundamental results in graph coloring. For every connected graph G, it says that G has a Δ-coloring unless G is a complete graph $K_{\Delta+1}$ or an odd cycle. When we seek to prove coloring results by induction, we often want to color a

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subgraph H where different vertices have different lists of allowable colors (those not already used on their neighbors in the coloring of $G-H$). This gives rise to list coloring. Vizing [13] and, independently, Erdős, Rubin, and Taylor [5] extended Brooks' theorem to list coloring. They proved an analogue of Brooks' theorem when each vertex v has Δ allowable colors (possibly different colors for different vertices). Borodin [3] and Erdős, Rubin, and Taylor [5] strengthened this Brooks' analogue to the following result, where a *Gallai tree* is a connected graph in which each block is a complete graph or an odd cycle. Gallai tree

Theorem A. *If G is connected and not a Gallai tree, then for any list assignment L with $|L(v)| = d(v)$ for all $v \in V(G)$, graph G has a proper coloring φ with $\varphi(v) \in L(v)$ for all v.*

The graphs in Theorem A are *degree-choosable*. It is easy to check that every Gallai tree is not degree-choosable. So the set of all connected graphs that are not degree-choosable are precisely the Gallai trees. Hladký, Král', and Schauz [7] extended this characterization to the setting of Alon–Tarsi orientations. degree-choosable

For any digraph D, a *spanning Eulerian subgraph* is one in which each vertex has indegree equal to outdegree. The *parity* of a spanning Eulerian subgraph is the parity of its number of edges. For an orientation of a graph G, let EE (resp. EO) denote the number of even (resp. odd) spanning Eulerian subgraphs. An orientation is *Alon–Tarsi* (or AT) if EE and EO differ. A graph G is *f-AT* if it has an Alon–Tarsi orientation D such that $d^+(v) \leqslant f(v) - 1$ for each vertex v. In particular, G is *degree-AT* (resp. *k-AT*) if it is f-AT, where $f(v) = d(v)$ (resp. $f(v) = k$) for all v. Similarly, a graph G is *f-choosable* if G has a proper coloring φ from any list assignment L such that $|L(v)| = f(v)$ for all $v \in V(G)$. Alon and Tarsi [1] used algebraic methods to prove the following theorem for choosability. Later, Schauz [11] strengthened the result to paintability, which we discuss briefly in Section 4. Alon–Tarsi orientation f-AT, k-AT degree-AT f-choosable

Theorem B. *For a graph G and $f\colon V(G) \to \mathbb{N}$, if G is f-AT, then G is also f-choosable.*

In this paper we characterize those graphs G with a specified vertex x that are not f-AT, where $f(x) = d(x) - 1$ and $f(v) = d(v)$ for all other $v \in V(G)$. All such graphs are formed from a few 2-connected building blocks, by repeatedly applying a small number of operations. Most of the work in the proof is spent on the case when G is 2-connected. This result is easy to state, so we include it a bit later in the introduction, as our Main Lemma. Near the end of Section 3, with a little more work we extend our Main Lemma, by removing the hypothesis of 2-connectedness, to characterize all pairs (G, h_x) that are not AT. This result is Theorem 3.6.

This line of research began in the fifties with Dirac, who studied the minimum number of edges in an n-vertex k-critical graph G. Since G has minimum degree at least $k-1$, clearly $|E(G)| \geqslant \frac{k-1}{2}n$. Gallai [6] improved this bound by classifying all connected subgraphs that can be induced by vertices of degree $k-1$ in a k-critical graph. By Theorem A, all such graphs are Gallai trees. Here we consider graphs G that are critical with respect to Alon–Tarsi orientation. Specifically, G is not $(k-1)$-AT, but every proper subgraph is; such graphs are *k-AT-critical.* The characterization of degree-AT graphs shows that, much like k-critical graphs, in a k-AT-critical graph G, every connected subgraph induced by vertices of degree $k-1$ must be a Gallai tree. Our main result characterizes the subgraphs that can be induced by vertices of degree $k-1$, together with a single vertex of degree k. Thus, it is natural to expect that this result will lead to improved lower bounds on the number of edges in n-vertex k-AT-critical graphs.

Similar to that for degree-AT, our characterization remains unchanged in the contexts of list-coloring and paintability, as we show in Section 4. We see a sharp contrast when we consider graphs G with two specified vertices x_1 and x_2 that are not f-AT, where $f(x_i) = d(x_i) - 1$ for each $i \in \{1,2\}$ and $f(v) = d(v)$ for all other $v \in V(G)$. For Alon–Tarsi orientations, we have more than 50 exceptional graphs on seven vertices or fewer. Furthermore, the characterizations for list-coloring, paintability, and Alon–Tarsi orientations all differ.

We consider graphs with vertices labeled by natural numbers; that is, pairs (G,h) where G is a graph and $h\colon V(G) \to \mathbb{N}$. We focus on the case when $h(x) = 1$ for some x and $h(v) = 0$ for all other v; we denote this labeling as h_x. We say that (G,h) *is AT* if G is $(d_G - h)$-AT. When H is an induced subgraph of G, we simplify notation by referring to the pair (H,h) when we really mean $\big(H, h{\restriction}_{V(H)}\big)$. h_x (G,h) is AT

Given a pair (G,h) and a specified edge $e \in E(G)$, when we *stretch* e, we form (G',h') from (G,h) by subdividing e twice and setting $h'(v_i) = 0$ for each of the two new vertices, v_1 and v_2 (and $h'(v) = h(v)$ for all other vertices v). In Section 2, we prove a Stretching Lemma, which shows that if (G,h) is not AT and $e \in E(G)$, then stretching e often yields another pair (G',h') that is also not AT. Thus, stretching plays a key role in our main result. stretch e

It is easy to check that the three pairs (G,h) shown in Figure 1 are not AT (and we do this below, in Proposition 1.1). Let $\mathcal{D}$ be the collection of all pairs formed from the graphs in Figure 1 by stretching each bold edge 0 or more times. The Stretching Lemma implies that each pair in $\mathcal{D}$ is not AT. Our Main Lemma is that these are the only pairs (G,h_x), where G is 2-connected and neither complete nor an odd cycle, such that (G,h_x) is not AT, for some vertex $x \in V(G)$. $\mathcal{D}$

Main Lemma. *Let G be 2-connected and $x \in V(G)$. Now (G,h_x) is AT if and only if*

(1) $d(x) = 2$ and $G - x$ is not a Gallai tree; or

(2) $d(x) \geqslant 3$, G is not complete, and $(G,h_x) \notin \mathcal{D}$.

The characterization of degree-choosable graphs has been applied to prove a variety of graph coloring results [2, 4, 8, 9, 12]. Likewise, we think our main results in this paper may be helpful in proving other results for Alon–Tarsi orientations, such as giving better lower bounds on the number of edges in k-AT-critical graphs.

To conclude this section, we show that each pair in $\mathcal{D}$ is not AT.

Proposition 1.1. *If $(G,h_x) \in \mathcal{D}$, then (G,h_x) is not AT.*

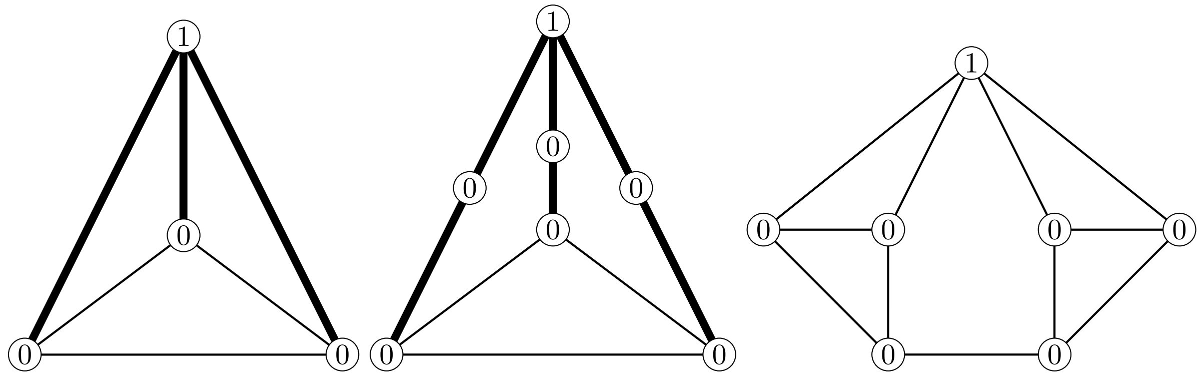

Figure 1: Three pairs (G, h_x) that are in $\mathcal{D}$. In each case x is labeled 1 and all other vertices are labeled 0. Each other pair in $\mathcal{D}$ can be formed from one of these pairs by repeatedly stretching one or more bold edges. Note that the rightmost graph is the Moser spindle.

Proof. For each pair $(G, h_x) \in \mathcal{D}$, we construct a list assignment L such that $|L(x)| = d(x) - 1$ and $|L(v)| = d(v)$ for all other $v \in V(G)$, but G has no proper coloring from L. Now (G, h_x) is not AT, by the contrapositive of Theorem B.

Let (G, h_x) be some stretching of the leftmost pair in Figure 1. Assign the list $\{1, 2, 3\}$ to each of the vertices on the unbolded triangle and assign the list $\{1, 2\}$ to each other vertex. If G has some coloring from these lists, then vertex x, labeled 1 in the figure, must get color 1 or 2; by symmetry, assume it is 1. Along each path from x to the triangle, colors must alternate $2, 1, \ldots$. Each of the paths from x to the triangle has odd length; thus, color 1 is forbidden from appearing on the triangle. So G has no coloring from L. Now let (G, h_x) be some stretching of the center pair in Figure 1. The proof is identical to the first case, except that each path has even length, so if x gets color 1, then color 2 is forbidden on the triangle.

Finally, consider the rightmost pair in Figure 1. Here $d(x) = 4$ and $d(v) = 3$ for all other $v \in V(G)$. Thus, it suffices to show that G is not 3-colorable. Assume that G has a 3-coloring and, by symmetry, assume that x is colored 1. Now colors 2 and 3 must each appear on two neighbors of x. Thus, the two remaining vertices must be colored 1. Since they are adjacent, this is a contradiction, which proves that G is not 3-colorable. $\square$

2 Subgraphs, subdivisions, and cuts

When Hladký, Kráľ, and Schauz characterized degree-AT graphs, their proof relied heavily on the observation that a connected graph G is degree-AT if and only if G has some induced subgraph H such that H is degree-AT. Below, we reprove this easy lemma, and also extend it to our setting of pairs (G, h_x).

Subgraph Lemma. *Let G be a connected graph and let H be an induced subgraph of G. If H is degree-AT, then also G is degree-AT. Similarly, if $x \in V(H)$ and (H, h_x) is AT,* *then also (G, h_x) is AT. Further, if $x \notin V(H)$, $d_G(x) \geqslant 2$, and (H, h_x) is AT, then (G, h_x) is AT.*

Proof. Suppose that H is degree-AT, and let D' be an orientation of H showing this. Let σ be an order of $V(G)$ such that if $v <_\sigma w$, then $\text{dist}(v, H) \leqslant \text{dist}(w, H)$. Extend D' to an orientation D of G by orienting each edge $vw \in E(G) \setminus E(H)$ as $\overrightarrow{vw}$ precisely when $v <_\sigma w$. For every subgraph J of D with $V(J) \nsubseteq V(H)$, the vertex $w \in V(J)$ that appears latest in σ is a sink in J. Thus, every directed cycle in D is also a directed cycle in D' (and vice versa), so G is degree-AT. The proof of the second statement is identical. The proof of the third statement is similar, but now for each edge wx with endpoints equidistant from H, we require that $w <_\sigma x$, so wx is oriented into x. □

Recall that, given a pair (G, h) and a specified edge $e \in E(G)$, when we *stretch* e, we form (G', h') from (G, h) by subdividing e twice and setting $h'(v_i) = 0$ for each of the two new vertices, v_1 and v_2 (and $h'(v) = h(v)$ for all other vertices v). By repeatedly stretching edges, starting from the three pairs in Figure 1, we form all pairs (G, h_x), where G is 2-connected and (G, h_x) is not AT. The following lemma will be useful for proving this.

Stretching Lemma. *Form (G', h') from (G, h) by stretching some edge $e \in E(G)$. Now*

(1) if (G, h) is AT, then (G', h') is AT; and

(2) if (G', h') is AT, then either (G, h) is AT or $(G - e, h)$ is AT.

Proof. Suppose $e = u_1u_2$ and call the new vertices v_1 and v_2 so that G' contains the induced path $u_1v_1v_2u_2$. For (1), let D be an orientation of G showing that (G, h) is AT. By symmetry we may assume $u_1u_2 \in E(D)$. Form an orientation D' of G' from D by replacing u_1u_2 with the directed path $u_1v_1v_2u_2$. We have a natural parity preserving bijection between the spanning Eulerian subgraphs of D and D', so we conclude that (G', h') is AT.

For (2), let D' be an orientation of G' showing that (G', h') is AT. Suppose G' contains the directed path $u_1v_1v_2u_2$ or the directed path $u_2v_2v_1u_1$. By symmetry, we can assume it is $u_1v_1v_2u_2$. Now form an orientation D of G by replacing $u_1v_1v_2u_2$ with the directed edge u_1u_2. As above, we have a parity preserving bijection between the spanning Eulerian subgraphs of D and D', so we conclude that (G, h) is AT. So suppose instead that G' contains neither of the directed paths $u_1v_1v_2u_2$ and $u_2v_2v_1u_1$. Now no spanning Eulerian subgraph of D' contains a cycle passing through v_1 and v_2. So, the spanning Eulerian subgraph counts of D' are the same as those of $D' - v_1 - v_2$. However, this gives an orientation of $G - e$ showing that $(G - e, h)$ is AT. □

Given a pair (G, h) that is not AT, the Stretching Lemma suggests a way to construct a larger graph G' such that (G', h') is not AT. In some cases, we can also use the Stretching Lemma to construct a smaller graph $\widehat{G}$ such that $(\widehat{G}, h)$ is not AT. Specifically, we have the following.

Corollary 2.1. *If e is an edge in G such that (G, h) is not AT and $(G - e, h)$ is not AT, then stretching e gives a pair (G', h') that is not AT. Further, let G be a graph with an induced path $u_1v_1v_2u_2$ such that $d_G(v_1) = d_G(v_2) = 2$. If (G, h) is AT, where $h(v_1) = h(v_2) = 0$, and $(G - v_1 - v_2, h)$ is not AT, then*

$$\big((G - v_1 - v_2) + u_1u_2, h\restriction_{V(G)\setminus\{v_1,v_2\}}\big) \text{ is AT.}$$

Proof. The first statement follows directly from part (2) of the Stretching Lemma. Now we prove the second. Suppose that (G, h) satisfies the hypotheses. Now part (2) of the Stretching Lemma also shows that either $\big(G - v_1 - v_2, h\restriction_{V(G)\setminus\{v_1,v_2\}}\big)$ is AT or else $\big((G - v_1 - v_2) + u_1u_2, h\restriction_{V(G)\setminus\{v_1,v_2\}}\big)$ is AT. By hypothesis, the former is false. Thus, the latter is true. □

With standard vertex coloring, we can easily reduce to the case where G is 2-connected. If G is a connected graph with two blocks, B_1 and B_2, meeting at a cutvertex x, then we can color each of B_1 and B_2 independently, and afterward we can permute colorings to match at x. For Alon–Tarsi orientations, the situation is not quite as simple. However, the following lemma plays a similar role for us.

Lemma 2.2. *Let $A_1, A_2 \subseteq V(G)$, and $x \in V(G)$ be such that $A_1 \cup A_2 = V(G)$ and $A_1 \cap A_2 = \{x\}$. If $G[A_i]$ is f_i-AT for each $i \in \{1, 2\}$, then G is f-AT, where $f(v) = f_i(v)$ for each $v \in V(A_i - x)$ and $f(x) = f_1(x) + f_2(x) - 1$. Going the other direction, if G is f-AT, then $G[A_i]$ is f_i-AT for each $i \in \{1, 2\}$, where $f_i(v) = f(v)$ for each $v \in V(A_i - x)$ and $f_1(x) + f_2(x) \leqslant f(x) + 1$.*

Proof. We begin with the first statement. For each $i \in \{1, 2\}$, choose an orientation D_i of A_i showing that A_i is f_i-AT. Together these D_i give an orientation D of G. Since no cycle has vertices in both $A_1 - x$ and $A_2 - x$, we have

$$\begin{aligned} EE(D) - EO(D) = {} & EE(D_1)EE(D_2) + EO(D_1)EO(D_2) \\ & - EE(D_1)EO(D_2) - EO(D_1)EE(D_2) \\ = {} & (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2)) \\ \neq {} & 0. \end{aligned}$$

Hence G is f-AT.

Now we prove the second statement. Suppose that G is f-AT and choose an orientation D of G showing this. Let $D_i = D[A_i]$ for each $i \in \{1, 2\}$. As above, we have $0 \neq EE(D) - EO(D) = (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2))$. Hence, $EE(D_1) - EO(D_1) \neq 0$ and $EE(D_2) - EO(D_2) \neq 0$. Since the indegree of x in D is the sum of the indegree of x in D_1 and the indegree of x in D_2, the lemma follows. □

3 When h is 1 for one vertex

In this section, we prove our Main Lemma. For a graph G and $x \in V(G)$ recall that $h_x \colon V(G) \to \mathbb{N}$ is defined as $h_x(x) = 1$ and $h_x(v) = 0$ for all $v \in V(G - x)$. We classify the connected graphs G such that (G, h_x) is AT for some $x \in V(G)$. We begin with the case when G is 2-connected, which takes most of the work. At the end of the section, we extend our characterization to all connected graphs.

We will show that for most 2-connected graphs G and vertices $x \in V(G)$, the pair (G, h_x) is AT. Specifically, this is true for all pairs except those in $\mathcal{D}$, defined in the introduction. In view of the Subgraph Lemma, for a 2-connected graph G and $x \in V(G)$, to show that (G, h_x) is AT it suffices to find some induced subgraph H such that (H, h_x) is AT. The subgraphs H that we consider all have $d_H(x) = 0$ or $d_H(x) \geqslant 3$. This motivates the next lemma, which allows us to reduce to the case $d_G(x) \geqslant 3$.

Lemma 3.1. *If G is a connected graph and $x \in V(G)$ with $d_G(x) = 2$, then (G, h_x) is AT if and only if $G - x$ is degree-AT.*

Proof. Let D be an orientation of G showing that (G, h_x) is AT. Now $d_D^-(x) = 2$, so no spanning Eulerian subgraph contains a cycle passing through x. Therefore, the Eulerian subgraph counts in $G - x$ are different and $G - x$ is degree-AT. The other direction is immediate from the Subgraph Lemma. $\square$

Recall that our Main Lemma relies on a characterization of degree-AT graphs. As we mentioned in the introduction, a description of degree-choosable graphs was first given by Borodin [3] and Erdős, Rubin, and Taylor [5]. Hladký, Kráľ, and Schauz [7] later extended the proof from [5] to Alon–Tarsi orientations. For reference, we record their result next.

Lemma 3.2. *A connected graph G is degree-AT if and only if it is not a Gallai tree.*

Lemmas 3.1 and 3.2 combine to prove Case (1) of our Main Lemma. Before we can prove Case (2), we need a few more definitions and lemmas. A *θ-graph* consists of two vertices joined by three internally disjoint paths, P_1, P_2, and P_3. When we write h_x for a θ-graph, we always assume that $d(x) = 3$. We will see shortly that if H is a θ-graph with $d_H(x) = 3$, then (H, h_x) is AT. Thus, the Subgraph Lemma implies that if (G, h_x) is not AT, then G has no induced θ-graph H with $d_H(x) = 3$. A *T-graph* is formed from vertices x, z_1, z_2, z_3, by making the z_i pairwise adjacent, and joining each vertex z_i to x by a path P_i (where the P_i are disjoint). Equivalently, a T-graph is formed from K_4 by subdividing each of the edges incident to x zero or more times. θ-graph T-graph

Similar to the proof characterizing degree-AT graphs in [7], our approach in proving our Main Lemma is to find an induced subgraph H such that (H, h_x) is AT, and apply the Subgraph Lemma. So we need the following lemma about pairs (H, h_x) that are AT.

Lemma 3.3. *The pair (G, h_x) is AT whenever (i) G is a θ-graph, (ii) G is a T-graph and two paths P_i have lengths of opposite parities, as in Figure 2, or (iii) G is formed from a T-graph by adding an extra vertex with neighborhood $\{z_1, z_2, z_3\}$, as in Figure 3.*

Proof. In each case, we give an AT orientation D of G such that $d_D^-(v) \geqslant h_x(v) + 1$ for each $v \in V(G)$.

Case (i). Orient the edges of each path P_i consistently, with P_1 and P_2 into x and P_3 out of x; this orientation satisfies the degree requirements. Further, it has exactly three

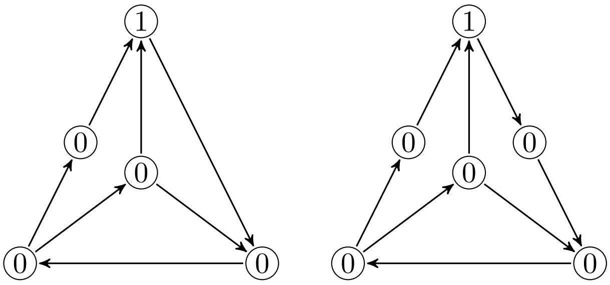

Figure 2: The pair (G, h_x) is AT, when G is formed from K_4 by subdividing one or two edges incident to x.

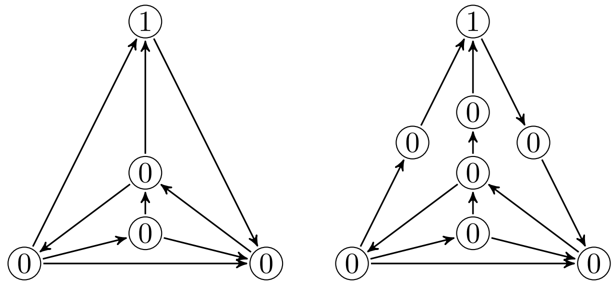

Figure 3: (a) The pair (G, h_x) is AT, where $G = K_5 - xy$. (b) The pair (G, h_x) is AT, where G is formed from $K_5 - e$ by subdividing each edge incident to x.

spanning Eulerian subgraphs, including the empty subgraph. Thus, $EE + EO$ is odd, so $EE \neq EO$.

Case (ii). Let P_1 and P_2 be two paths with opposite parities. As before, orient the edges of each path consistently, with P_1 and P_2 into x and P_3 out of x. Orient the three additional edges as $\overrightarrow{z_1z_2}$, $\overrightarrow{z_2z_3}$, and $\overrightarrow{z_3z_1}$. The resulting digraph D has four spanning Eulerian subgraphs, 3 of one parity and 1 of the other. Note that the empty subgraph and the subgraph $\{\overrightarrow{z_1z_2}, \overrightarrow{z_2z_3}, \overrightarrow{z_3z_1}\}$ have opposite parities. Further, the parities are the same for the two subgraphs consisting of the directed cycles $xP_3z_3z_1P_1$ and $xP_3z_3z_1z_2P_2$. So, $EE \neq EO$.

Case (iii). The simplest instance of this case is when $G = K_5 - e$, as on the left in Figure 3. Denote its vertices by $\{t, u, v, w, x\}$, where $w \not\leftrightarrow x$. Begin with the transitive orientation given by $t \to w \to u \to v \to x$, and reverse $\overrightarrow{ux}$ and $\overrightarrow{tv}$ to get $\overrightarrow{xu}$ and $\overrightarrow{vt}$. Call this orientation D. We use D to show that (G, h_x) is AT. Note that D satisfies the degree constraints. Consider the Eulerian subgraph counts. Since K_4 is not 3-colorable, $(G - x, h_x)$ is not AT. Thus, the even and odd Eulerian subgraph counts of $D - x$ are equal; in particular, the sum of these is even (in fact there are 2 even and 2 odd, but this is unimportant). These Eulerian subgraphs are precisely the Eulerian subgraphs of D with $d^+(x) = d^-(x) = 0$. Now consider an Eulerian subgraph H of D with $d^+(x) = d^-(x) = 1$. If $\overrightarrow{tx} \in E(H)$, then $E(H) = \{\overrightarrow{tx}, \overrightarrow{xu}, \overrightarrow{uv}, \overrightarrow{vt}\}$. If $\overrightarrow{tx} \notin E(H)$, then $\overrightarrow{vx}, \overrightarrow{xu}, \overrightarrow{uv} \in E(H)$. Further, $E(H)$ contains all or none of $\overrightarrow{vt}, \overrightarrow{tw}, \overrightarrow{wv}$ (and no other edges). Thus, D has 3 Eulerian subgraphs with $d^+(x) = d^-(x) = 1$. In total, $EE + EO$ is odd, so $EE \neq EO$.

To handle larger instances of this case, we repeatedly subdivide edges incident to x and orient each of the resulting paths consistently, and in the direction of the corresponding edge in D. The resulting orientation satisfies the degree requirements. Further, the sum $EE + EO$ remains unchanged, and thus odd. Hence, still $EE \neq EO$. □

Lemma 3.4. *Let G be a T-graph. Let P be a path of G where all internal vertices of P have degree 2 in G and one endvertex of P has degree 2 in G. Form G' from G by adding a path P' (of length at least 2) joining the endvertices of P. Now (G', h_x) is AT.*

Proof. We can assume that G is not AT; otherwise, we are done by the Subgraph Lemma. By symmetry, assume P is a subpath of P_3. First, we get an orientation of G with indegree at least 1 for all vertices and $d^-(x) = 2$. Orient P_1 from z_1 to x, P_2 from z_2 to x, P_3 from x to z_3, and the triangle as $\overrightarrow{z_1 z_2}$, $\overrightarrow{z_2 z_3}$, and $\overrightarrow{z_3 z_1}$. To get an orientation of G', orient the new path P' consistently, and opposite of P. Now the only directed cycle containing edges of P' is $P'P$. Since the Eulerian subgraph counts are equal for G, they differ by 1 for G'. □

Now we can prove Case (2) of our Main Lemma. For reference, we restate it.

Lemma 3.5. *Let G be 2-connected, and choose $x \in V(G)$ with $d(x) \geqslant 3$. Now (G, h_x) is AT if and only if G is not complete and $(G, h_x) \notin \mathcal{D}$.*

Proof. If G is complete, then G is not degree-AT, by Lemma 3.2, so clearly (G, h_x) is not AT. When $(G, h_x) \in \mathcal{D}$ the lemma holds by Proposition 1.1.

Now let G be 2-connected, choose $x \in V(G)$ with $d(x) \geqslant 3$, and suppose that $(G, h_x) \notin \mathcal{D}$. Since $G - x$ is connected, let H' be an induced subgraph of $G - x$ with as few vertices as possible such that H' is connected and contains three neighbors of x; call these neighbors w_1, w_2, and w_3. Consider a spanning tree T of H'. Since H' is minimum, each leaf of T is among $\{w_1, w_2, w_3\}$. So T is either (i) a path or (ii) a subdivision of $K_{1,3}$; if possible, choose T to be in case (ii). Suppose we are in case (i). If there exists $e \in E(H') - E(T)$, then either H' is not minimal, or we could be in case (ii), or $H' = K_3$. So either $H' = K_3$, H' is a path, or we are in case (ii). Assume we are in case (ii) and let s be the vertex with $d_T(s) = 3$. If $E(H') - E(T)$ has any edge with at least one endpoint outside of $N(s)$, then we can delete some vertex in $N(s)$ and remain connected, contradicting the minimality of H'. Similarly, if $N(s)$ contains at least two edges, then $H' - s$ still connects, w_1, w_2, and w_3. Now let H be the subgraph of G induced by $V(H') \cup \{x\}$. Note that H is either a θ-graph (if H' is a tree) or a T-graph (if H' has one extra edge in $N(s)$, or $H' = K_3$).

If H is a θ-graph, then (G, h_x) is AT, by Lemma 3.3.i and the Subgraph Lemma. So assume H is a T-graph. Let z_1, z_2, z_3 be the vertices of degree 3 (other than x), and let P_1, P_2, and P_3 denote the paths from x to z_1, z_2, and z_3; when we write $V(P_i)$, we exclude x and z_i, so possibly $V(P_i)$ is empty for one or more $i \in \{1, 2, 3\}$. If any two of P_1, P_2, and P_3 have lengths with opposite parities, then we are done by Lemma 3.3.ii; so assume not.

Now $(H, h_x) \in \mathcal{D}$, so we can assume that $V(G - H) \neq \emptyset$. Choose $u \in V(G - H)$, and let H_u be a minimal 2-connected induced subgraph of G that contains $V(H) \cup \{u\}$.

By the Subgraph Lemma and Lemma 3.2, $G - x$ is a Gallai tree. Thus, so is $H_u - x$; in particular, the block B_u of $H_u - x$ containing u is complete or an odd cycle. Therefore, we either have (i) $V(B_u) \cap V(H) = \{z_1, z_2, z_3\}$ or (ii) $V(B_u) \cap V(H) \subseteq P_i \cup \{z_i\}$ for some $i \in \{1, 2, 3\}$.

Suppose (i) happens. Now $N_G(u) \cap V(H_u - x) = \{z_1, z_2, z_3\}$. If $x \not\leftrightarrow u$, then (G, h_x) is AT by the Subgraph Lemma and Lemma 3.3.iii. If $x \leftrightarrow u$, then x must have odd length paths to each z_i, by Lemma 3.3.ii, with u in the role of some z_i. Further, $x \leftrightarrow z_i$ for all $i \in \{1, 2, 3\}$, since otherwise (G, h_x) is AT by the Subgraph Lemma, Lemma 3.3.iii, and the Stretching Lemma. So, $H = K_4$ and $H_u = K_5$. This implies that (ii) cannot happen for any vertex in $V(G - H)$, since if $V(B_u) \cap V(H) = \{z_i\}$ for some i, then (G, h_x) is AT by Lemma 3.3.i and the Subgraph Lemma. So (i) happens for every vertex in $V(G - H)$; in particular, $V(G - H)$ is joined to $\{x, z_1, z_2, z_3\}$. Since G is not complete, $G - x$ must contain an induced copy of Figure 3(a); hence, (G, h_x) is AT by Lemma 3.3.iii and the Subgraph Lemma.

So instead (ii) happens for every vertex in $V(G - H)$, including u. By symmetry, assume $V(B_u) \cap V(H) \subseteq P_1 \cup \{z_1\}$. Let $z_1 P_1 = v_1 v_2 \cdots v_\ell$, where $v_\ell \not\leftrightarrow x$. First, suppose B_u is an odd cycle of length at least 5. If there is $u' \in V(B_u) \setminus V(H)$ with $u' \leftrightarrow x$, then G contains a θ-graph and (G, h_x) is AT, by Lemma 3.3.i and the Subgraph Lemma. So, we assume $u' \not\leftrightarrow x$ for all $u' \in V(B_u) \setminus V(H)$. Now we are done by Lemma 3.4 and the Subgraph Lemma.

So instead we assume that B_u is complete. If $V(B_u) \cap V(H) = \{v_\ell\}$, then G has an induced θ-graph J, where $d_J(x) = d_J(v_\ell) = 3$, so we are done by Lemma 3.3.i and the Subgraph Lemma. Thus, we must have $V(B_u) \cap V(H) = \{v_j, v_{j+1}\}$ for some $j \in \{1, \ldots, \ell - 1\}$. In particular, B_u is a triangle. If $u \not\leftrightarrow x$, then (G, h_x) is AT by the Subgraph Lemma and Lemma 3.4. So we conclude that $u \leftrightarrow x$, which requires $j = \ell - 1$, by the minimality of H. Hence, H_u is formed from a T-graph by adding a vertex u that is adjacent to x and also to the vertices of a K_2 endblock D_u of $H - x$. Suppose there are distinct vertices $u_1, u_2 \in V(G - H)$ adjacent to vertices of the same K_2 endblock. Now G contains an induced copy of Figure 3(a), so (G, h_x) is AT by Lemma 3.3.iii and the Subgraph Lemma. Thus, each K_2 endblock has at most one such u.

Let $t = |V(G - H)|$, and note that $0 \leqslant t \leqslant 3$. Suppose $t \geqslant 2$. By symmetry, assume that for each $i \in \{1, 2\}$ there exists u_i such that $V(B_{u_i}) \cap V(H) \subseteq P_i \cup \{z_i\}$. Let G' denote the subgraph of G induced by $V(H - P_1) \cup \{u_2\}$. Now (G', h_x) is AT by Lemma 3.4, so (G, h_x) is AT by the Subgraph Lemma. If $t = 0$, then (since $(G, h_x) \notin \mathcal{D}$) graph G is a T-graph with two paths with lengths of opposite parities. Thus, we are done by Lemma 3.3.ii and the Subgraph Lemma.

Now assume $t = 1$, and let $\{u\} = V(G - H)$. By symmetry, assume $V(B_u) \cap V(H) \subseteq P_1 \cup \{z_1\}$. If each of P_2 and P_3 is a single edge and P_1 has length 3, then G is the Moser spindle (shown in Figure 1 on the right), contradicting that $(G, h_x) \notin \mathcal{D}$. Thus, some P_i is longer than this; suppose it is P_1. Let $G' = G - (V(P_1) - N(u))$; note that G' is an induced subgraph, and is formed from two θ-graphs by identifying a vertex of degree 2 in each. Further, this identified vertex is x. Since θ-graphs are not Gallai trees, they are degree-AT by Lemma 3.2. Thus, Lemma 2.2 shows that (G', h_x) is AT. So we are done by the Subgraph Lemma. So instead assume either P_2 or P_3 is longer than a single edge; by symmetry, assume it is P_2. Let $G' = G - V(P_2)$, and note that G' is an induced subgraph of G. Further, (G', h_x) is AT by Lemma 3.4. Thus, we are done by the Subgraph Lemma. □

Finally, Lemmas 3.1, 3.2, and 3.5 combine to prove our Main Lemma. However, this characterization requires that G be 2-connected. Now we extend our result to the more general case, when G need only be connected. We use the following two definitions. Let G be a graph, x a vertex of G, and B a block of G. An *x-lobe of G* is a maximal subgraph A such that $A - x$ is connected. A *B-lobe of G* is a maximal subgraph A such that $A - B$ is connected, and A includes a single vertex of B. x-lobe B-lobe

Theorem 3.6. *If G is connected and $x \in V(G)$, then (G, h_x) is not AT if and only if*

(1) G is a Gallai tree; or

(2) $d(x) = 1$; or

(3) $d(x) = 2$ and $G - x$ has a component that is a Gallai tree; or

(4) x is not a cutvertex, for the block B of G containing x, we have $(B, h_x) \in \mathcal{D}$, and every other block of G is complete or an odd cycle; or

(5) x is a cutvertex, all but at most one x-lobe of G, say A, is a Gallai tree, and either: (i) $d_A(x) = 1$; or (ii) $d_A(x) = 2$ and $A - x$ is a Gallai tree; or (iii) for the block B of A containing x, we have $(B, h_x) \in \mathcal{D}$ and all B-lobes of A are Gallai trees.

Proof. First, we check that if any of Cases (1)–(5) hold, then (G, h_x) is not AT. Cases (1) and (2) are immediate. Case (3) follows from Lemma 3.1. Consider Case (4). By Proposition 1.1, we know (B, h_x) is not AT. Now (G, h_x) is not AT by repeated application of Lemma 2.2. Finally, Case (5) follows from Cases (2), (3), and (4), by Lemma 2.2.

Now, for the other direction, suppose (G, h_x) is not AT and none of Cases (1)–(5) hold. By Lemma 3.1, and not (2) and not (3), we must have $d(x) \geqslant 3$. Suppose x is a cutvertex. Now, by not (5), either (a) at least two x-lobes of G are not Gallai trees or (b) (H, h_x) is AT for some x-lobe H of G. In each case, (G, h_x) is AT by Lemma 2.2, which is a contradiction.

So assume instead that x is not a cutvertex. Suppose the block B of G containing x is complete or $(B, h_x) \in \mathcal{D}$. By not (1) and not (4), some B-lobe H of G is not a Gallai tree. Since H is a subgraph of $G - x$, and $G - x$ is connected, Lemma 3.2 and the Subgraph Lemma imply that $G - x$ is degree-AT; hence, (G, h_x) is also AT. So, we conclude that B is not complete and $(B, h_x) \notin \mathcal{D}$. First suppose that $d(x) = 2$. By not (3), we know that $G - x$ is not a Gallai tree. Lemma 3.2 implies that $G - x$ is degree-AT. So, again, the Subgraph Lemma shows that (G, h_x) is AT. Now assume instead that $d(x) \geqslant 3$. Since B is not complete and $(B, h_x) \notin \mathcal{D}$, Lemma 3.5 now implies that (B, h_x) is AT; once more, the Subgraph Lemma implies that (G, h_x) is AT. □

4 Choosability and Paintability

As we mentioned in the introduction, Alon and Tarsi showed that if a graph G is f-AT, then G is also f-choosable. *Online list coloring*, also called *painting* is similar to list coloring, but now the list for each vertex is progressively revealed, as the graph is colored. Schauz [11] extended the Alon–Tarsi theorem, to show that if G is f-AT, then G is also f-paintable (which we define formally below). In this section, we use our characterization of pairs (G, h_x) that are not AT to prove characterizations of pairs (G, h_x) that are not paintable and that are not choosable. More precisely, a pair (G, h_x) is *choosable* if G has a proper coloring from its lists L whenever L is such that $|L(x)| = d(x) - 1$ and $|L(v)| = d(v)$ for all other v; otherwise (G, h_x) is *not choosable*. A pair being *paintable* is defined analogously. We characterize all pairs (G, h_x), where G is connected and (G, h_x) is not choosable (resp. not paintable). In fact, we will see that these characterizations, for both choosability and paintability, are identical to that for pairs that are not AT. choosable pair paintable pair

For completeness, we include the following definition of f-paintable. Schauz [10] gave a more intuitive (yet equivalent) definition, in terms of a two player game. We say that G is *f-paintable* if either (i) G is empty or (ii) $f(v) \geqslant 1$ for all $v \in V(G)$ and for every $S \subseteq V(G)$ there is an independent set $I \subseteq S$ such that $G - I$ is f'-paintable where $f'(v) := f(v)$ for all $v \in V(G) - S$ and $f'(v) := f(v) - 1$ for all $v \in S - I$. f-paintable

Since all pairs (G, h_x) that are AT are also both paintable and choosable, it suffices to show that every pair (G, h_x) that is not AT is also not choosable (here we use that if a pair is paintable, then it is also choosable).

Theorem 4.1. *For every connected graph G, the pair (G, h_x) is not choosable if and only if (G, h_x) is not AT. Thus, the same characterization holds for pairs that are not paintable.*

Proof. As noted above, every pair that is AT is also choosable and paintable. Thus, it suffices to show that each pair (G, h_x) in Theorem 3.6 is not choosable.

To show that Gallai trees are not degree-choosable, assign to each block B a list of colors L_B such that $|L_B| = d_B(x)$ for each $x \in V(B)$; further, for all distinct blocks B_1 and B_2, we require that L_{B_1} and L_{B_2} are disjoint. For each $v \in V(G)$, let $L(v) = \cup_{B_i \ni v} L_{B_i}$. To show that G is not colorable from these lists, we use induction on the number of blocks. Let B be an endblock and x a cutvertex in B. Let $G' = G \setminus (V(G) - x)$. Since B is complete or an odd cycle, B has no coloring from L_B. Thus any coloring φ of G from L does not use L_B on x. Hence, φ gives a coloring φ' of G' from its lists L', where $L'(x) = L(x) \setminus L_B$ and $L'(v) = L(v)$ for all $v \in V(G) \setminus V(B)$. This coloring φ' of G' contradicts the induction hypothesis. Thus, G has no coloring from L.

Here we use a similar approach. Consider a pair (G, h_x) that satisfies one of Cases (1)–(5) in Theorem 3.6. We show that (G, h_x) is not choosable. Case (1) is immediate by the previous paragraph. Case (2) is immediate, since $|L(x)| = 0$. For Case (3), give lists to the Gallai tree of $G - x$ as above; now let $L(x) = \{c\}$ for some new color c, and add c to the list of each neighbor of x. Again G cannot be colored from L. For Case (4), assign lists to $V(B)$ as in Proposition 1.1 and to the other blocks as above. Again, G has no coloring from these lists. Finally, consider Case (5). Assign lists for all blocks outside of A as above, and assign lists for A as above in Case (2), (3), or (4). $\square$

To conclude this section, we consider labelings $h_{x,y}$, where $h_{x,y}(x) = h_{x,y}(y) = 1$ and $h_{x,y}(v) = 0$ for all other $v \in V(G)$. We show that the set of pairs $(G, h_{x,y})$ that are not AT differs from the set of pairs that are not paintable. Further, both sets differ from the set of pairs that are not choosable. It suffices to give a pair $(G_1, h_{x,y})$ that is choosable but not paintable and a second pair $(G_2, h_{x,y})$ that is paintable but not AT.

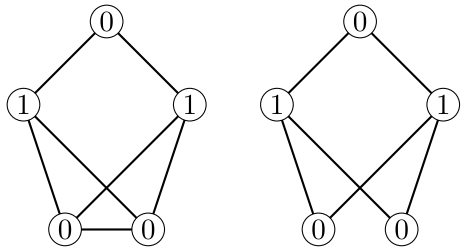

Figure 4: The pair on the left is choosable, but not paintable. The pair on the right is paintable, but not AT.

Proposition 4.2. *The pair $(G_1, h_{x,y})$ on the left in Figure 4 is choosable, but it is not paintable. The pair $(G_2, h_{x,y})$ on the right in Figure 4 is paintable, but it is not AT.*

Proof. Let $(G_1, h_{x,y})$ denote the pair on the left, where x and y are the vertices labeled 1. Let $(G_2, h_{x,y})$ denote the pair on the right, where x and y are the vertices labeled 1.

We first show that $(G_1, h_{x,y})$ is choosable. Let L denote the list assignment. If there exists $c \in L(x) \cap L(y)$, then use c to color x and y, and color the remaining vertices greedily. So suppose there does not exist such a color c. Let z be a vertex in both triangles and note that there exist $c \in (L(x) \cup L(y)) \setminus L(z)$. By symmetry, assume that $c \in L(x)$. Color x with c, and color $G_1 - x$ greedily, starting with the vertex of degree 2 and ending with z.

We now show that $(G_1, h_{x,y})$ is not paintable. Let S be the vertices of one triangle. By definition, there must be $I \subseteq S$ such that $G_1 - I$ is f'-paintable, where $f'(v) := f(v)$ for $v \in V(G_1) - S$ and $f'(v) := f(v) - 1$ for $v \in S - I$. I must have one vertex, w. There are two choices for w; either w is in two triangles or not. If w is not in two triangles, then $G_1 - w$ is a triangle with a pendant edge, where the vertices on the triangle all have list size 2, so $G_1 - w$ is not paintable. If w is one of the vertices in two triangles, then $G_1 - w$ is a 4-cycle with list sizes $1, 2, 2, 2$. Again $G_1 - I$ is not paintable (nor choosable).

To see that $(G_2, h_{x,y})$ is not AT, note that any good orientation would need indegrees summing to at least 7, but G_2 has only 6 edges. Now we show that $(G_2, h_{x,y})$ is paintable. Note that G_2 is isomorphic to $K_{2,3}$, the complete bipartite graph. Call the parts X and Y, with $|X| = 2$ and $|Y| = 3$. We color (take I to be) all vertices in whichever is larger of $X \cap S$ and $Y \cap S$, picking $X \cap S$ if their sizes are equal. If we colored at least two vertices, then it is easy to check that $G - I$ is paintable, since it induces either an independent set, or a path where each vertex has at least as many colors as neighbors and one endvertex has more colors than neighbors. So assume that we colored only a single vertex. If we colored a vertex of X, then the resulting graph is paintable, since it is a claw, $K_{1,3}$, with at most one leaf having a single color and all other vertices having two colors. Finally, suppose we colored a single vertex of Y. The remaining graph is C_4, which is degree-paintable (since it is degree-AT). $\square$

A graph is *unstretched* if it has no induced path $u_1v_1v_2u_2$ where $d(v_1) = d(v_2) = 2$ (as in Corollary 2.1). We finish with the following question. unstretched

Question. *Are there only finitely many unstretched, 2-connected graphs G such that $(G, h_{x,y})$ is not choosable (resp. paintable, AT)? More generally, let $h_{x_1,\dots,x_k}$ be a labeling that assigns 1 to vertices $x_1, \dots, x_k$ and 0 to all others. Are there only finitely many unstretched, 2-connected graphs G such that $(G, h_{x_1,\dots,x_k})$ is not choosable (resp. paintable, AT)?*

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