Ideals and quotients of diagonally quasi-symmetric functions

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Submitted: Dec 1, 2016; Accepted: Jun 29, 2017; Published: Jul 14, 2017
Mathematics Subject Classifications: 05E05

Abstract

In 2004, J.-C. Aval, F. Bergeron and N. Bergeron studied the algebra of diagonally quasi-symmetric functions DQSym in the ring $\mathbb{Q}[\mathbf{x},\mathbf{y}]$ with two sets of variables. They made conjectures on the structure of the quotient $\mathbb{Q}[\mathbf{x},\mathbf{y}]/\langle\mathsf{DQSym}^+\rangle$, which is a quasi-symmetric analogue of the diagonal harmonic polynomials. In this paper, we construct a Hilbert basis for this quotient when there are infinitely many variables i.e. $\mathbf{x} = x_1, x_2, \dots$ and $\mathbf{y} = y_1, y_2, \dots$. Then we apply this construction to the case where there are finitely many variables, and compute the second column of its Hilbert matrix.

1 Introduction

In the polynomial ring $\mathbb{Q}[\mathbf{x}_n] = \mathbb{Q}[x_1, \dots, x_n]$ with n variables, the ring of symmetric polynomials (cf. [13, 14]), Sym_n, is the subspace of invariants under the symmetric group S_n action

$$\sigma \cdot f(x_1, x_2, \dots, x_n) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}).$$

The quotient space $\mathbb{Q}[\mathbf{x}_n]/\langle\mathsf{Sym}_n^+\rangle$ over the ideal generated by symmetric polynomials with no constant term is thus called the coinvariant space of symmetric group. Classic results by Artin [5] and Steinberg [16] asserts that this quotient forms an S_n-module that is isomorphic to the left regular representation. Moreover, considering the natural scalar product

$$\langle f, g\rangle = \big(f(\partial x_1, \dots, \partial x_n)(g(x_1, \dots, x_n))\big)(0, 0, \dots, 0),$$

this quotient is equal to the orthogonal complement of Sym_n. In particular, the coinvariant space is killed by Laplacian operator $\Delta = \partial x_1^2 + \cdots + \partial x_n^2$. Hence, it is also known as the harmonic space.

One can show that $\{h_k(x_k, \ldots, x_n) : 1 \leqslant k \leqslant n\}$ forms a Gröbner basis of $\langle \mathsf{Sym}_n^+ \rangle$ with respect to the usual order $x_1 > \cdots > x_n$, where h_k is the complete homogeneous basis of degree k. As a result, the dimension of $\mathbb{Q}[\mathbf{x}_n]/\langle \mathsf{Sym}_n^+ \rangle$ is $n!$.

One generalization is the diagonal harmonic space. In the context of $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n] = \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, the diagonally symmetric functions, DSym_n, is the space of invariants under the diagonal action of S_n

$$\sigma \cdot f(x_1, \ldots, x_n, y_1, \ldots, y_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_{\sigma(1)}, \ldots, y_{\sigma(n)}).$$

The diagonal harmonics, $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]/\langle \mathsf{DSym}_n^+ \rangle$, was studied by Garsia and Haiman [9, 12] where it was used to prove the $n!$ conjecture and Macdonald positivity. In particular, its dimension turns out to be $(n+1)^{n-1}$. More interesting results and applications can be found in [6, 7, 11].

The ring of quasi-symmetric functions, QSym, was introduced by Gessel [8] as generating function for Stanley's P-partitions [15]. It soon shows great importance in algebraic combinatorics e.g. [4, 10]. In our context, QSym_n can be defined as the space of invariants in $\mathbb{Q}[\mathbf{x}_n]$ under the S_n-action of Hivert

$$\sigma \cdot \left(x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}\right) = x_{j_1}^{a_1} \cdots x_{j_k}^{a_k}$$

where $i_1 < \cdots < i_k$, $j_1 < \cdots < j_k$ and $\{j_1, \ldots, j_k\} = \{\sigma(i_1), \ldots, \sigma(i_k)\}$.

In a series of papers by Aval, F. Bergeron and N. Bergeron, the authors studied the quotient $\mathbb{Q}[\mathbf{x}_n]/\langle \mathsf{QSym}_n^+ \rangle$ over the ideal generated by quasi-symmetric polynomials with no constant term, which they called the super-covariant space of S_n. Their main result is that a basis of this quotient corresponds to Dyck paths, and the dimension of the quotient space is the n-th Catalan number C_n [1, 2].

After that, they extended QSym to diagonal setting, called diagonally quasi-symmetric functions, DQSym [3]. They described a Hopf algebra structure on DQSym, and made a conjecture about the linear structure of $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]/\langle \mathsf{DQSym}_n^+ \rangle$.

In this paper, we continue the study of the linear structure. We start with the case where there are infinitely many variables i.e. $R = \mathbb{Q}[[\mathbf{x}, \mathbf{y}]]$ is the ring of formal power series where $\mathbf{x} = x_1, x_2, \ldots$ and $\mathbf{y} = y_1, y_2, \ldots$. The main result is that we give a description of a Hilbert basis for the quotient space R/I where $I = \overline{\mathsf{DQSym}^+}$ is the closure of the ideal generated by DQSym without constant terms. This Hilbert basis gives an upper bound on the degree of $\mathbb{Q}[\mathbf{x}_n, \mathbf{y}_n]/\langle \mathsf{DQSym}_n^+ \rangle$. We then use it to compute the second column of the Hilbert matrix, which coincides with the conjecture in [3].

2 Definitions

2.1 Bicompositions

An element $\tilde{\alpha} = \begin{pmatrix} \tilde{\alpha}_{11} & \tilde{\alpha}_{12} & \cdots \\ \tilde{\alpha}_{21} & \tilde{\alpha}_{22} & \cdots \end{pmatrix} \in \mathbb{N}^{2\mathbb{N}}$ is called a generalized bicomposition if all but finitely many $(\tilde{\alpha}_{1k}, \tilde{\alpha}_{2k})$ are $(0, 0)$. Let k be the maximum number such that $(\tilde{\alpha}_{1k}, \tilde{\alpha}_{2k}) \neq$

$(0,0)$. The length of $\tilde{\alpha}$, denoted by $\ell(\tilde{\alpha})$, is k. The size of $\tilde{\alpha}$, denoted by $|\tilde{\alpha}|$, is the sum of all its entries. For simplicity, we usually write $\tilde{\alpha}$ as $\binom{\tilde{\alpha}_{11}\ \cdots\ \tilde{\alpha}_{1k}}{\tilde{\alpha}_{21}\ \cdots\ \tilde{\alpha}_{2k}}$. There also exists a generalized bicomposition with length 0 and size 0, called the zero bicomposition, denoted by $\binom{0}{0}$.

Every monomial in R can be expressed as $\mathbf{X}^{\tilde{\alpha}} = x_1^{\tilde{\alpha}_{11}} y_1^{\tilde{\alpha}_{21}} \cdots x_k^{\tilde{\alpha}_{1k}} y_k^{\tilde{\alpha}_{2k}}$ for some generalized bicomposition $\tilde{\alpha}$. A generalized bicomposition α is called a bicomposition if $\ell(\alpha) = 0$ or $(\alpha_{1j}, \alpha_{2j}) \neq (0,0)$ for all $1 \leqslant j \leqslant \ell(\alpha)$.

In this paper, we use Greek letters to denote bicompositions, and Greek letters with tilde to denote generalized bicompositions.

Let $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ be non-zero generalized bicompositions. We write $\tilde{\alpha} = \tilde{\beta}\tilde{\gamma}$ if $\tilde{\alpha}_{ij} = \tilde{\beta}_{ij}$ for all $1 \leqslant j \leqslant \ell(\tilde{\alpha}) - \ell(\tilde{\gamma})$, $\tilde{\beta}_{ij} = 0$ for all $j > \ell(\tilde{\alpha}) - \ell(\tilde{\gamma})$ and $\tilde{\alpha}_{i(j+\ell(\tilde{\alpha})-\ell(\tilde{\gamma}))} = \tilde{\gamma}_{ij}$ for all $j \geqslant 1$. We write $\tilde{\alpha} = \binom{0}{0}\tilde{\beta}$ if $\tilde{\alpha}_{11} = \tilde{\alpha}_{21} = 0$ and $\tilde{\alpha}_{i(j+1)} = \tilde{\beta}_{ij}$ for all $j \geqslant 2$.

Note that for each generalized bicomposition $\tilde{\alpha}$ that is not a bicomposition, there is a unique way to decompose it into $\tilde{\alpha} = \tilde{\beta}\binom{0}{0}\gamma$ for some generalized bicomposition $\tilde{\beta}$ and bicomposition γ.

2.2 Diagonally quasi-symmetric functions

The algebra of diagonally quasi-symmetric functions, DQSym, is a subalgebra of $\mathbb{Q}[[\mathbf{x}, \mathbf{y}]]$ spanned by monomials indexed by bicompositions

$$M_\alpha = \sum_{i_1 < \cdots < i_k} x_{i_1}^{\alpha_{11}} y_{i_1}^{\alpha_{21}} \cdots x_{i_k}^{\alpha_{1k}} y_{i_k}^{\alpha_{2k}}.$$

As a graded algebra, $\mathsf{DQSym} = \bigoplus_{n \geqslant 0} \mathsf{DQSym}_n$ where $\mathsf{DQSym}_n = \text{span-}\{M_\alpha : |\alpha| = n\}$ is the degree n component. The algebra structure is defined in [3].

2.3 The F basis

We define a partial ordering $\preceq$ on bicompositions: $\alpha \preceq \beta$ and β covers α if there exists a $1 \leqslant k \leqslant \ell(\alpha) - 1$ such that either $\alpha_{2k} = 0$ or $\alpha_{1(k+1)} = 0$, and

$$\beta = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1(k-1)} & \alpha_{1k} + \alpha_{1(k+1)} & \alpha_{1(k+2)} & \cdots & \alpha_{1\ell(\alpha)} \\ \alpha_{21} & \cdots & \alpha_{2(k-1)} & \alpha_{2k} + \alpha_{2(k+1)} & \alpha_{2(k+2)} & \cdots & \alpha_{2\ell(\alpha)} \end{pmatrix}.$$

By triangularity, $\left\{ F_\alpha = \sum_{\alpha \preceq \beta} M_\beta \right\}$ forms a basis for DQSym. For example,

$$F_{\binom{2}{2}} = M_{\binom{2}{2}} + M_{\binom{2\ 0}{0\ 2}} + M_{\binom{1\ 1}{0\ 2}} + M_{\binom{1\ 1\ 0}{0\ 0\ 2}} + M_{\binom{2\ 0}{1\ 1}} + M_{\binom{2\ 0\ 0}{0\ 1\ 1}} + M_{\binom{1\ 1\ 0}{0\ 1\ 1}} + M_{\binom{1\ 1\ 0\ 0}{0\ 0\ 1\ 1}}.$$

For convenience, we set $F_{\binom{0}{0}} = 1$. This basis has the following easy but important properties:

If $\alpha_{11} \geqslant 1$ and $\alpha_{11} + \alpha_{21} \geqslant 2$, then

$$F_\alpha = x_1 F_{\left(\begin{smallmatrix} \alpha_{11}-1 & \alpha_{12} & \cdots & \alpha_{1\ell(\alpha)} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2\ell(\alpha)} \end{smallmatrix}\right)} + F_\alpha(x_2, x_3, \ldots, y_2, y_3, \ldots); \tag{2.1}$$

If $\alpha_{11} = 1$ and $\alpha_{21} = 0$, then

$$F_\alpha = x_1 F_{\left(\begin{smallmatrix} \alpha_{12} & \cdots & \alpha_{1\ell(\alpha)} \\ \alpha_{22} & \cdots & \alpha_{2\ell(\alpha)} \end{smallmatrix}\right)}(x_2, x_3, \ldots, y_2, y_3, \ldots) + F_\alpha(x_2, x_3, \ldots, y_2, y_3, \ldots); \tag{2.2}$$

If $\alpha_{11} = 0$ and $\alpha_{21} \geqslant 2$, then

$$F_\alpha = y_1 F_{\left(\begin{smallmatrix} 0 & \alpha_{12} & \cdots & \alpha_{1\ell(\alpha)} \\ \alpha_{21}-1 & \alpha_{22} & \cdots & \alpha_{2\ell(\alpha)} \end{smallmatrix}\right)} + F_\alpha(x_2, x_3, \ldots, y_2, y_3, \ldots); \tag{2.3}$$

If $\alpha_{11} = 0$ and $\alpha_{21} = 1$, then

$$F_\alpha = y_1 F_{\left(\begin{smallmatrix} \alpha_{12} & \cdots & \alpha_{1\ell(\alpha)} \\ \alpha_{22} & \cdots & \alpha_{2\ell(\alpha)} \end{smallmatrix}\right)}(x_2, x_3, \ldots, y_2, y_3, \ldots) + F_\alpha(x_2, x_3, \ldots, y_2, y_3, \ldots). \tag{2.4}$$

3 The G basis

In this section, we define a basis $\{G_{\tilde{\epsilon}}\}$ indexed by generalized bicompositions for $\mathbb{Q}[[\mathbf{x}, \mathbf{y}]]$. Base cases: $G_{\binom{0}{0}} = 1$ and $G_{\tilde{\epsilon}} = F_{\tilde{\epsilon}}$ if $\tilde{\epsilon}$ is a bicomposition. Otherwise, let $\tilde{\epsilon} = \tilde{\alpha}\binom{0}{0}\beta$ where β is a non-zero bicomposition. Let $k = \ell(\tilde{\epsilon}) - \ell(\beta) - 1$.
If $\beta_{11} > 0$,

$$G_{\tilde{\epsilon}} = G_{\tilde{\alpha}\beta} - x_{k+1} G_{\tilde{\alpha}\left(\begin{smallmatrix} \beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)}. \tag{3.1}$$

If $\beta_{11} = 0$,

$$G_{\tilde{\epsilon}} = G_{\tilde{\alpha}\beta} - y_{k+1} G_{\tilde{\alpha}\left(\begin{smallmatrix} 0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)}. \tag{3.2}$$

Inductively, $\{G_{\tilde{\epsilon}}\}$ is defined for all generalized bicomposition $\tilde{\epsilon}$. Clearly $G_{\tilde{\epsilon}}$ is homogeneous in degree $|\tilde{\epsilon}|$. Hence, we have a notion of leading monomial of $G_{\tilde{\epsilon}}$, denoted by $LM(G_{\tilde{\epsilon}})$ with respect to the lexicographic order with $x_1 > y_1 > x_2 > y_2 > \cdots$. To show that $\{G_{\tilde{\epsilon}}\}$ form a basis, it suffices to prove the leading monomial of $G_{\tilde{\epsilon}}$ is $\mathbf{X}^{\tilde{\epsilon}}$.

Lemma 3.1. *Let $\tilde{\alpha} = \binom{a}{b}\tilde{\beta}$ be a generalized bicomposition,*

1. *if $a = b = 0$, then $G_{\tilde{\alpha}} = G_{\tilde{\beta}}(x_2, x_3, \ldots, y_2, y_3, \ldots)$,*
2. *if $a > 0$, then $G_{\tilde{\alpha}} = x_1 G_{\binom{a-1}{b}\tilde{\beta}} + P(x_2, x_3, \ldots, y_2, y_3, \ldots)$,*
3. *if $a = 0$ and $b > 0$, then $G_{\tilde{\alpha}} = y_1 G_{\binom{0}{b-1}\tilde{\beta}} + P(x_2, x_3, \ldots, y_2, y_3, \ldots)$*

for some $P \in \mathbb{Q}[[\mathbf{x}, \mathbf{y}]]$.

Proof. We prove by induction on the length of $\tilde{\alpha}$.

1. If $\tilde{\alpha} = \binom{0}{0}$, then $G_{\tilde{\alpha}} = 1$ and we are done.

2. If $\tilde{\beta} = \beta$ is a bicomposition,

   (a) if $a = b = 0$ and β non-zero,

      i. if $\beta_{11} \geqslant 1$ and $\beta_{11} + \beta_{21} \geqslant 2$, using (2.1) and (3.1), we get

$$\begin{aligned} G_{\tilde{\alpha}} &= G_\beta - x_1 G_{\left(\begin{smallmatrix} \beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)} \\ &= F_\beta - x_1 F_{\left(\begin{smallmatrix} \beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)} \\ &= F_\beta(x_2, x_3, \ldots, y_2, y_3, \ldots) = G_\beta(x_2, x_3, \ldots, y_2, y_3, \ldots) \end{aligned}$$

      and the lemma follows.

      ii. if $\beta_{11} = 1$ and $\beta_{21} = 0$, using (2.2), (3.1) and induction on $\ell(\tilde{\beta})$, we get

$$\begin{aligned} G_{\tilde{\alpha}} &= G_\beta - x_1 G_{\left(\begin{smallmatrix} 0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ 0 & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)} \\ &= G_\beta - x_1 G_{\left(\begin{smallmatrix} \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)}(x_2, x_3, \ldots, y_2, y_3, \ldots) \\ &= F_\beta - x_1 F_{\left(\begin{smallmatrix} \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)}(x_2, x_3, \ldots, y_2, y_3, \ldots) \\ &= F_\beta(x_2, x_3, \ldots, y_2, y_3, \ldots) = G_\beta(x_2, x_3, \ldots, y_2, y_3, \ldots) \end{aligned}$$

      and the lemma follows.

      iii. if $\beta_{11} = 0$ and $\beta_{21} \geqslant 2$, using (2.3) and (3.2), we get

$$\begin{aligned} G_{\tilde{\alpha}} &= G_\beta - y_1 G_{\left(\begin{smallmatrix} 0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)} \\ &= F_\beta - y_1 F_{\left(\begin{smallmatrix} 0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)} \\ &= F_\beta(x_2, x_3, \ldots, y_2, y_3, \ldots) = G_\beta(x_2, x_3, \ldots, y_2, y_3, \ldots) \end{aligned}$$

      and the lemma follows.

      iv. if $\beta_{11} = 0$ and $\beta_{21} = 1$, using (2.4), (3.2) and induction on $\ell(\tilde{\beta})$, we get

$$\begin{aligned} G_{\tilde{\alpha}} &= G_\beta - y_1 G_{\left(\begin{smallmatrix} 0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ 0 & \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)} \\ &= G_\beta - y_1 G_{\left(\begin{smallmatrix} \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)}(x_2, x_3, \ldots, y_2, y_3, \ldots) \\ &= F_\beta - y_1 F_{\left(\begin{smallmatrix} \beta_{12} & \cdots & \beta_{1\ell(\beta)} \\ \beta_{22} & \cdots & \beta_{2\ell(\beta)} \end{smallmatrix}\right)}(x_2, x_3, \ldots, y_2, y_3, \ldots) \\ &= F_\beta(x_2, x_3, \ldots, y_2, y_3, \ldots) = G_\beta(x_2, x_3, \ldots, y_2, y_3, \ldots) \end{aligned}$$

      and the lemma follows.

(b) if $a \geqslant 1$ and $a+b \geqslant 2$, by definition $G_{\tilde{\alpha}} = F_{\tilde{\alpha}}$. Using (2.1), we get

$$G_{\tilde{\alpha}} = F_{\tilde{\alpha}} = x_1 F_{\binom{a-1}{b}\beta} + F_{\tilde{\alpha}}(x_2, x_3, \ldots, y_2, y_3, \ldots)$$

and the lemma follows, with $P = F_{\tilde{\alpha}}(x_2, x_3, \ldots, y_2, y_3, \ldots)$.

(c) if $a = 1$ and $b = 0$, by definition $G_{\tilde{\alpha}} = F_{\tilde{\alpha}}$. Using (2.2) and (2a). we get

$$\begin{aligned} G_{\tilde{\alpha}} = F_{\tilde{\alpha}} &= x_1 F_{\beta}(x_2, x_3, \ldots, y_2, y_3, \ldots) + F_{\tilde{\alpha}}(x_2, x_3, \ldots, y_2, y_3, \ldots) \\ &= x_1 G_{\binom{0}{0}\beta} + F_{\tilde{\alpha}}(x_2, x_3, \ldots, y_2, y_3, \ldots) \end{aligned}$$

and the lemma follows with $P = F_{\tilde{\alpha}}(x_2, x_3, \ldots, y_2, y_3, \ldots)$.

(d) if $a = 0$ and $b \geqslant 2$, by definition $G_{\tilde{\alpha}} = F_{\tilde{\alpha}}$. Using (2.3), we get

$$G_{\tilde{\alpha}} = F_{\tilde{\alpha}} = y_1 F_{\binom{a}{b-1}\beta} + F_{\tilde{\alpha}}(x_2, x_3, \ldots, y_2, y_3, \ldots)$$

and the lemma follows, with $P = F_{\tilde{\alpha}}(x_2, x_3, \ldots, y_2, y_3, \ldots)$.

(e) if $a = 0$ and $b = 1$, by definition $G_{\tilde{\alpha}} = F_{\tilde{\alpha}}$. Using (2.4) and (2a). we get

$$\begin{aligned} G_{\tilde{\alpha}} = F_{\tilde{\alpha}} &= y_1 F_{\beta}(x_2, x_3, \ldots, y_2, y_3, \ldots) + F_{\tilde{\alpha}}(x_2, x_3, \ldots, y_2, y_3, \ldots) \\ &= y_1 G_{\binom{0}{0}\beta} + F_{\tilde{\alpha}}(x_2, x_3, \ldots, y_2, y_3, \ldots) \end{aligned}$$

and the lemma follows with $P = F_{\tilde{\alpha}}(x_2, x_3, \ldots, y_2, y_3, \ldots)$.

3. In the general case, let $\tilde{\alpha} = \tilde{\gamma}\binom{0}{0}\beta$ where β is a non-empty bicomposition and $k = \ell(\tilde{\alpha}) - \ell(\beta) - 1$. We prove by induction on k. If $k = 1$, then we are back in case (2a) above. Hence, we assume $k > 1$ and $\tilde{\gamma} = \binom{a}{b}\tilde{\mu}$.

   (a) If $a = b = 0$,

      i. if $\beta_{11} \geqslant 1$, by induction and (3.1), we have

$$\begin{aligned} G_{\tilde{\alpha}} =& G_{\binom{0}{0}\tilde{\mu}\binom{0}{0}\beta} = G_{\binom{0}{0}\tilde{\mu}\beta} - x_k G_{\binom{0}{0}\tilde{\mu}\binom{\beta_{11}-1 \ \beta_{12} \ \cdots \ \beta_{1\ell(\beta)}}{\beta_{21} \ \beta_{22} \ \cdots \ \beta_{2\ell(\beta)}}} \\ =& G_{\tilde{\mu}\beta}(x_2, x_3, \ldots, y_2, y_3, \ldots) \\ & - x_{(k-1)+1} G_{\tilde{\mu}\binom{\beta_{11}-1 \ \beta_{12} \ \cdots \ \beta_{1\ell(\beta)}}{\beta_{21} \ \beta_{22} \ \cdots \ \beta_{2\ell(\beta)}}}(x_2, x_3, \ldots, y_2, y_3, \ldots) \\ =& G_{\tilde{\mu}\binom{0}{0}\beta}(x_2, x_3, \ldots, y_2, y_3, \ldots) \end{aligned}$$

      and the lemma follows.

      ii. if $\beta_{11} = 0$, by induction and (3.2), we have

$$G_{\tilde{\alpha}} = G_{\binom{0}{0}\tilde{\mu}\binom{0}{0}\beta} = G_{\binom{0}{0}\tilde{\mu}\beta} - y_k G_{\binom{0}{0}\tilde{\mu}\binom{0 \ \beta_{12} \ \cdots \ \beta_{1\ell(\beta)}}{\beta_{21}-1 \ \beta_{22} \ \cdots \ \beta_{2\ell(\beta)}}}$$

$$
\begin{aligned}
=&G_{\tilde{\mu}\beta}(x_2,x_3,\dots,y_2,y_3,\dots)\\
&-y_{(k-1)+1}G_{\tilde{\mu}\left(\begin{smallmatrix}0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)}\\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)}(x_2,x_3,\dots,y_2,y_3,\dots)\\
=&G_{\tilde{\mu}\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\beta}(x_2,x_3,\dots,y_2,y_3,\dots)
\end{aligned}
$$

and the lemma follows.

(b) If $a \geqslant 1$,

i. if $\beta_{11} \geqslant 1$, by induction and (3.1), we have

$$
\begin{aligned}
G_{\tilde{\alpha}} =&G_{\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\beta} = G_{\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\tilde{\mu}\beta} - x_k G_{\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix}\beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)}\\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)}\\
=&x_1 G_{\left(\begin{smallmatrix}a-1\\b\end{smallmatrix}\right)\tilde{\mu}\beta} + P_1(x_2,x_3,\dots,y_2,y_3,\dots)\\
&-x_k\Big(x_1 G_{\left(\begin{smallmatrix}a-1\\b\end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix}\beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)}\\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)}\\
&+P_2(x_2,x_3,\dots,y_2,y_3,\dots)\Big)\\
=&x_1\left(G_{\left(\begin{smallmatrix}a-1\\b\end{smallmatrix}\right)\tilde{\mu}\beta} - x_k G_{\left(\begin{smallmatrix}a-1\\b\end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix}\beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)}\\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)}\right)\\
&+P(x_2,x_3,\dots,y_2,y_3,\dots)\\
=&x_1 G_{\left(\begin{smallmatrix}a-1\\b\end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\beta} + P(x_2,x_3,\dots,y_2,y_3,\dots)
\end{aligned}
$$

and the lemma follows with $P = P_1 - x_k P_2$.

ii. if $\beta_{11} = 0$, by induction and (3.2), we have

$$
\begin{aligned}
G_{\tilde{\alpha}} =&G_{\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\beta} = G_{\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\tilde{\mu}\beta} - y_k G_{\left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix}0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)}\\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)}\\
=&x_1 G_{\left(\begin{smallmatrix}a-1\\b\end{smallmatrix}\right)\tilde{\mu}\beta} + P_1(x_2,x_3,\dots,y_2,y_3,\dots)\\
&-y_k\Big(x_1 G_{\left(\begin{smallmatrix}a-1\\b\end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix}0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)}\\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)}\\
&+P_2(x_2,x_3,\dots,y_2,y_3,\dots)\Big)\\
=&x_1\left(G_{\left(\begin{smallmatrix}a-1\\b\end{smallmatrix}\right)\tilde{\mu}\beta} - y_k G_{\left(\begin{smallmatrix}a-1\\b\end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix}0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)}\\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)}\right)\\
&+P(x_2,x_3,\dots,y_2,y_3,\dots)\\
=&x_1 G_{\left(\begin{smallmatrix}a-1\\b\end{smallmatrix}\right)\tilde{\mu}\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)\beta} + P(x_2,x_3,\dots,y_2,y_3,\dots)
\end{aligned}
$$

and the lemma follows with $P = P_1 - y_k P_2$.

(c) If $a = 0$ and $b \geqslant 1$,

i. if $\beta_{11} \geqslant 1$, by induction and (3.1), we have

$$\begin{aligned}
G_{\tilde{\alpha}} =& G_{\binom{0}{b}\tilde{\mu}\binom{0}{0}\beta} = G_{\binom{0}{b}\tilde{\mu}\beta} - x_k G_{\binom{0}{b}\tilde{\mu}\left(\begin{smallmatrix}\beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)}\\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)}\\
=& y_1 G_{\binom{0}{b-1}\tilde{\mu}\beta} + P_1(x_2, x_3, \dots, y_2, y_3, \dots)\\
& - x_k\Big(y_1 G_{\binom{0}{b-1}\tilde{\mu}\left(\begin{smallmatrix}\beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)}\\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)}\\
& + P_2(x_2, x_3, \dots, y_2, y_3, \dots)\Big)\\
=& y_1\left(G_{\binom{0}{b-1}\tilde{\mu}\beta} - x_k G_{\binom{0}{b-1}\tilde{\mu}\left(\begin{smallmatrix}\beta_{11}-1 & \beta_{12} & \cdots & \beta_{1\ell(\beta)}\\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)}\right)\\
& + P(x_2, x_3, \dots, y_2, y_3, \dots)\\
=& y_1 G_{\binom{0}{b-1}\tilde{\mu}\binom{0}{0}\beta} + P(x_2, x_3, \dots, y_2, y_3, \dots)
\end{aligned}$$

and the lemma follows with $P = P_1 - x_k P_2$.

ii. if $\beta_{11} = 0$, by induction and (3.2), we have

$$\begin{aligned}
G_{\tilde{\alpha}} =& G_{\binom{0}{b}\tilde{\mu}\binom{0}{0}\beta} = G_{\binom{0}{b}\tilde{\mu}\beta} - y_k G_{\binom{0}{b}\tilde{\mu}\left(\begin{smallmatrix}0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)}\\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)}\\
=& y_1 G_{\binom{0}{b-1}\tilde{\mu}\beta} + P_1(x_2, x_3, \dots, y_2, y_3, \dots)\\
& - y_k\Big(y_1 G_{\binom{0}{b-1}\tilde{\mu}\left(\begin{smallmatrix}0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)}\\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)}\\
& + P_2(x_2, x_3, \dots, y_2, y_3, \dots)\Big)\\
=& y_1\left(G_{\binom{0}{b-1}\tilde{\mu}\beta} - y_k G_{\binom{0}{b-1}\tilde{\mu}\left(\begin{smallmatrix}0 & \beta_{12} & \cdots & \beta_{1\ell(\beta)}\\ \beta_{21}-1 & \beta_{22} & \cdots & \beta_{2\ell(\beta)}\end{smallmatrix}\right)}\right)\\
& + P(x_2, x_3, \dots, y_2, y_3, \dots)\\
=& y_1 G_{\binom{0}{b-1}\tilde{\mu}\binom{0}{0}\beta} + P(x_2, x_3, \dots, y_2, y_3, \dots)
\end{aligned}$$

and the lemma follows with $P = P_1 - y_k P_2$. □

Corollary 3.2. *Let $\tilde{\epsilon}$ be a generalized bicomposition, then the leading monomial of $G_{\tilde{\epsilon}}$ is $\mathbf{X}^{\tilde{\epsilon}}$. Hence, $\{G_{\tilde{\alpha}}\}$ forms a Hilbert basis for R.*

Proof. We prove by induction on $\ell(\tilde{\epsilon})$ and $|\tilde{\epsilon}|$. If $\tilde{\epsilon} = \binom{0}{0}$, by definition $G_{\tilde{\epsilon}} = 1 = X^{\tilde{\epsilon}}$. Otherwise, let $\tilde{\epsilon} = \binom{a}{b}\tilde{\beta}$.

1. If $a = b = 0$ and $\tilde{\beta}$ non-zero, by induction on $\ell(\tilde{\epsilon})$ and Lemma 3.1, we have

$$LM(G_{\tilde{\epsilon}}) = LM(G_{\tilde{\beta}}(x_2, x_3, \dots, y_2, y_3, \dots)) = (x_2, x_3, \dots, y_2, y_3, \dots)^{\tilde{\beta}} = \mathbf{X}^{\tilde{\epsilon}}.$$

2. If $a \geqslant 1$, by induction on $|\tilde{\epsilon}|$ and Lemma 3.1, we have

$$LM(G_{\tilde{\epsilon}}) = LM\left(x_1 G_{\left(\begin{smallmatrix} a-1 \\ b \end{smallmatrix}\right)\tilde{\beta}}\right) = \mathbf{X}^{\tilde{\epsilon}}.$$

3. If $a = 0$ and $b \geqslant 1$, by induction on $|\tilde{\epsilon}|$ and Lemma 3.1, we have

$$LM(G_{\tilde{\epsilon}}) = LM\left(y_1 G_{\left(\begin{smallmatrix} 0 \\ b-1 \end{smallmatrix}\right)\tilde{\beta}}\right) = \mathbf{X}^{\tilde{\epsilon}}.$$

□

4 The Hilbert Basis

The set $\{x^{\tilde{\alpha}} F_\beta\}$ is a spanning set of the ideal I. For each $\tilde{\alpha}$ and β, we write $x^{\tilde{\alpha}} F_\beta$ in terms of the G basis by the following rules.

(1) We reorder the product $x^{\tilde{\alpha}} F_\beta$ as $\cdots \left(x_2^{\tilde{\alpha}_{21}} \left(y_2^{\tilde{\alpha}_{22}} \left(x_1^{\tilde{\alpha}_{11}} \left(y_1^{\tilde{\alpha}_{21}} F_\beta\right)\right)\right)\right)$.

(2) We reduce the above product recursively using (3.1)

$$x_i G_{\tilde{\gamma}} = x_i G_{\left(\begin{smallmatrix} \cdots & \tilde{\gamma}_{1i} & \cdots \\ \cdots & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)} = G_{\left(\begin{smallmatrix} \cdots & \tilde{\gamma}_{1i}+1 & \cdots \\ \cdots & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)} - G_{\left(\begin{smallmatrix} \cdots & 0 & \tilde{\gamma}_{1i}+1 & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)}; \tag{4.1}$$

or using (3.2) when $\tilde{\gamma}_{1i} = 0$ for some i,

$$y_i G_{\tilde{\gamma}} = y_i G_{\left(\begin{smallmatrix} \cdots & \tilde{\gamma}_{1i} & \cdots \\ \cdots & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)} = G_{\left(\begin{smallmatrix} \cdots & \tilde{\gamma}_{1i} & \cdots \\ \cdots & \tilde{\gamma}_{2i}+1 & \cdots \end{smallmatrix}\right)} - G_{\left(\begin{smallmatrix} \cdots & 0 & \tilde{\gamma}_{1i} & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2i}+1 & \cdots \end{smallmatrix}\right)}. \tag{4.2}$$

(3) When $\tilde{\gamma}_{1i} = a > 0$, we reduce $y_i G_{\tilde{\gamma}}$ as

$$\begin{aligned} y_1 G_{\tilde{\gamma}} = y_i G_{\left(\begin{smallmatrix} \cdots & a & \cdots \\ \cdots & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)} &= y_i \left(G_{\left(\begin{smallmatrix} \cdots & 0 & a & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)} + x_i G_{\left(\begin{smallmatrix} \cdots & a-1 & \cdots \\ \cdots & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)}\right) \\ &= y_i G_{\left(\begin{smallmatrix} \cdots & 0 & a & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)} + x_i \left(y_i G_{\left(\begin{smallmatrix} \cdots & a-1 & \cdots \\ \cdots & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)}\right) = \cdots \\ &= \sum_{k=0}^{a-1} x_i^k \left(y_i G_{\left(\begin{smallmatrix} \cdots & 0 & a-k & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)}\right) + x_i^a \left(y_i G_{\left(\begin{smallmatrix} \cdots & 0 & \cdots \\ \cdots & \tilde{\gamma}_{2i} & \cdots \end{smallmatrix}\right)}\right). \end{aligned} \tag{4.3}$$

The "$\cdots$" above means $\tilde{\gamma}_{11} \ \cdots \ \tilde{\gamma}_{1(i-1)}$, $\tilde{\gamma}_{1(i+1)} \ \cdots \ \tilde{\gamma}_{1\ell(\tilde{\gamma})}$, $\tilde{\gamma}_{21} \ \cdots \ \tilde{\gamma}_{2(i-1)}$ or $\tilde{\gamma}_{2(i+1)} \ \cdots \ \tilde{\gamma}_{1\ell(\tilde{\gamma})}$ with respect to their positions in the generalized bicomposition. For example,

$$\begin{aligned} y_1 F_{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} &= y_1 \left(G_{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} + x_1 G_{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)}\right) = y_1 G_{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)} + x_1 y_1 G_{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)} \\ =& G_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)} - G_{\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)} + x_1 \left(G_{\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)} - G_{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)}\right) \\ =& G_{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)} - G_{\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)} + G_{\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)} - G_{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)} - G_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} + G_{\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)}. \end{aligned}$$

For each of the above rule, we choose one $G_{\tilde{\eta}}$ as leading basis element. We define a function ϕ from $(\{x_i\} \times \{G_{\tilde{\gamma}}\}) \cup (\{y_i\} \times \{G_{\tilde{\gamma}}\})$ to $\{G_{\tilde{\gamma}}\}$ as follows. In the case of rules (4.1), (4.2), we choose $\phi(x_i, G_{\tilde{\gamma}}) = G_{\begin{pmatrix} \cdots & 0 & \tilde{\gamma}_{1i}+1 & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2i} & \cdots \end{pmatrix}}$ and $\phi(y_i, G_{\tilde{\gamma}}) = G_{\begin{pmatrix} \cdots & 0 & \tilde{\gamma}_{1i} & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2i}+1 & \cdots \end{pmatrix}}$. In the case of rule (4.3), we choose $\phi(y_i, G_{\tilde{\gamma}}) = \phi\left(y_i, G_{\begin{pmatrix} \cdots & 0 & a & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2i} & \cdots \end{pmatrix}}\right) = G_{\begin{pmatrix} \cdots & 0 & 0 & a & \cdots \\ \cdots & 0 & 1 & \tilde{\gamma}_{2i} & \cdots \end{pmatrix}}$. In the other words, at each step of the expansion, we choose the lexicographically smallest $\tilde{\eta}$ such that $G_{\tilde{\eta}}$ appears as a term in the expansion.

Lemma 4.1. *The above process of choosing is invertible, i.e. ϕ is injective.*

Proof. Since each time we multiply x_i or y_i, the chosen term contains a $\binom{0}{0}$ at position i. Combining this fact with the rule that we have to multiply y_i before x_i, we have the following inverse function.

Let i be the largest number that $(\tilde{\gamma}_{1i}, \tilde{\gamma}_{2i}) = (0,0)$ and $0 < i < \ell(\tilde{\gamma})$.

(1) If $\tilde{\gamma}_{1(i+1)} > 0$, then $\phi^{-1}\left(G_{\begin{pmatrix} \cdots & 0 & \tilde{\gamma}_{1(i+1)} & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2(i+1)} & \cdots \end{pmatrix}}\right) = x_i G_{\begin{pmatrix} \cdots & \tilde{\gamma}_{1(i+1)}-1 & \cdots \\ \cdots & \tilde{\gamma}_{2(i+1)} & \cdots \end{pmatrix}}$.

(2) If $\tilde{\gamma}_{1(i+1)} = 0$ and, $\tilde{\gamma}_{1(i+2)} = 0$ or $\tilde{\gamma}_{2(i+1)} > 1$, then
$\phi^{-1}\left(G_{\begin{pmatrix} \cdots & 0 & \tilde{\gamma}_{1(i+1)} & \cdots \\ \cdots & 0 & \tilde{\gamma}_{2(i+1)} & \cdots \end{pmatrix}}\right) = y_i G_{\begin{pmatrix} \cdots & \tilde{\gamma}_{1(i+1)} & \cdots \\ \cdots & \tilde{\gamma}_{2(i+1)}-1 & \cdots \end{pmatrix}}$.

(3) If $\tilde{\gamma}_{1(i+1)} = 0$, $\tilde{\gamma}_{2(i+1)} = 1$ and $\tilde{\gamma}_{1(i+2)} > 0$, then
$\phi^{-1}\left(G_{\begin{pmatrix} \cdots & 0 & 0 & \tilde{\gamma}_{1(i+2)} & \cdots \\ \cdots & 0 & 1 & \tilde{\gamma}_{2(i+2)} & \cdots \end{pmatrix}}\right) = y_i G_{\begin{pmatrix} \cdots & \tilde{\gamma}_{1(i+2)} & \cdots \\ \cdots & \tilde{\gamma}_{2(i+2)} & \cdots \end{pmatrix}}$. ☐

Then, we can construct a map $\Phi : \{X^{\tilde{\alpha}} F_\beta : |\beta| \geqslant 1\} \to \{G_{\tilde{\gamma}}\}$ that is defined by "composing" ϕ with itself $(|\tilde{\alpha}|-1)$ times. By the above Lemma, we also have Φ is injective. For simplicity, we define $\phi^{-1}(G_{\tilde{\gamma}})$ (or $\Phi^{-1}(G_{\tilde{\gamma}})$) to be $X^{\tilde{\alpha}} G_{\tilde{\beta}}$ (or $X^{\tilde{\alpha}} F_\beta$) if $\phi(X^{\tilde{\alpha}} G_{\tilde{\beta}}) = G_{\tilde{\gamma}}$ (or $\Phi(X^{\tilde{\alpha}} F_\beta) = G_{\tilde{\gamma}}$ respectively).

Lemma 4.2. *In the expansion of $X^{\tilde{\alpha}} F_\beta$ in the G basis using the rules above, the term $\Phi(X^{\tilde{\alpha}} F_\beta)$ appears only once. In particular, it has coefficients 1 or -1.*

Proof. We begin with the claim that if $\tilde{\mu} \neq \tilde{\nu}$, then $\phi(x_i G_{\tilde{\mu}})$ and $\phi(y_i G_{\tilde{\mu}})$ do not appear in the expansion of $x_i G_{\tilde{\nu}}$ and $y_i G_{\tilde{\nu}}$ respectively.

Let k be the smallest integer such that $(\tilde{\mu}_{k1}, \tilde{\mu}_{k2}) \neq (\tilde{\nu}_{k1}, \tilde{\nu}_{k2})$. In rules (4.1), (4.2) and (4.3), for all $G_{\tilde{\gamma}}$ in the expansion of $x_i G_{\tilde{\mu}}$ or $y_i G_{\tilde{\mu}}$, the first $i-1$ columns of $\tilde{\gamma}$ is the same as that of $\tilde{\mu}$. Hence, the claim follows if $k < i$.

If $k = i$, and if we are multiplying x_i using rules (4.1) or (4.2), then the claim holds because either the i-th or the $i+1$-th columns of $x_i G_{\tilde{\mu}}$ will be different from terms in expansions of $x_i G_{\tilde{\nu}}$. If we are multiplying by y_i, then note that if the $i - th$ column of μ is $(0,0)$, then $\mu_{(i+1)1}$ must be 0 because otherwise, that means we multiplied an x_i or x_j or y_j with $j > i$ before y_i, which violates our rule. And the same condition applies to ν. With this restriction, it is easy to check that the claim holds.

If $k > i$, in both cases, if we choose any term in the expansion that is not $\phi(x_iG_{\tilde{\nu}})$ or $\phi(y_iG_{\tilde{\nu}})$, then the i or $i+1$ column of its index must be different from that of $\phi(x_iG_{\tilde{\mu}})$ or $\phi(y_iG_{\tilde{\mu}})$. If we choose $\phi(x_iG_{\tilde{\nu}})$ or $\phi(y_iG_{\tilde{\nu}})$, we also have $\phi(x_iG_{\tilde{\mu}}) \neq \phi(x_iG_{\tilde{\nu}})$ and $\phi(y_iG_{\tilde{\mu}})\phi(y_iG_{\tilde{\nu}})$ because $\mu \neq \nu$.

Since each term in the expansion of $X^{\tilde{\alpha}}F_\beta$ corresponds to a sequence of choice using rules (4.1), (4.2) or (43), if at some point, we choose a term that is different from the choice in Φ, then a recursive use of the claim asserts that $\Phi(X^{\tilde{\alpha}}F_\beta)$ will not appear again. $\square$

We now define an order ($<_G$) on the set of generalized bicompositions as follows

1. If $\tilde{\alpha}$ and $\tilde{\beta}$ are bicompositions, then $\tilde{\alpha} <_G \tilde{\beta}$ if $\tilde{\alpha} <_{lex} \tilde{\beta}$.
2. If $\tilde{\alpha}$ is a bicomposition and $\tilde{\beta}$ is not, then $\tilde{\alpha} <_G \tilde{\beta}$.
3. If $\tilde{\alpha} = \tilde{\mu}\binom{0}{0}\alpha'$, $\tilde{\beta} = \tilde{\nu}\binom{0}{0}\beta'$ where α' and β' are bicompositions, let $u = \ell(\tilde{\alpha}) - \ell(\alpha') - 1$, $v = \ell(\tilde{\alpha}) - \ell(\beta') - 1$, then $\tilde{\alpha} <_G \tilde{\beta}$ if
   (a) $u < v$, or
   (b) $u = v$, $\alpha'_{11} > 0$ and $\beta'_{11} = 0$, or
   (c) $u = v$, $\alpha'_{11} > 0$, $\beta'_{11} > 0$ (or $\alpha'_{11} = 0$, $\beta'_{11} = 0$) and $\overleftarrow{\phi}(G_{\tilde{\alpha}}) <_G \overleftarrow{\phi}(G_{\tilde{\beta}})$ where we define $\overleftarrow{\phi}(G_{\tilde{\delta}})$ to be $\tilde{\gamma}$ if $\phi(x_iG_{\tilde{\gamma}}) = G_{\tilde{\delta}}$ or $\phi(y_iG_{\tilde{\gamma}}) = G_{\tilde{\delta}}$ for some i.

Lemma 4.3. *The order defined above is a total order on the set of generalized bicompositions such that if $G_{\tilde{\gamma}} = \Phi(X^{\tilde{\alpha}}F_\beta)$, then for all $G_{\tilde{\delta}}$ that appears in the expansion of $X^{\tilde{\alpha}}F_\beta$, we have $\tilde{\gamma} \geqslant_G \tilde{\delta}$.*

Proof. Clearly this is a total order. If $\tilde{\alpha} < \tilde{\beta}$ by (1) or (2), then $\tilde{\beta}$ cannot appear in the expansion of $\Phi^{-1}(\tilde{\alpha}) = \tilde{\alpha}$.

If $\tilde{\alpha} < \tilde{\beta}$ by (3a), that means $\phi^{-1}(\tilde{\alpha}) = x_{u+1}G_{\tilde{\gamma}}$ or $y_{u+1}G_{\tilde{\gamma}}$ for some $\tilde{\gamma}$. Hence, $\tilde{\beta}$ cannot appear in the expansion of $\Phi^{-1}(\tilde{\alpha})$ because $\tilde{\beta}_{(v+1)1} = \tilde{\beta}_{(v+1)2} = 0$ cannot be created.

If $\tilde{\alpha} < \tilde{\beta}$ by (3b), that means $\phi^{-1}(\tilde{\alpha}) = x_{u+1}G_{\tilde{\gamma}}$ for some $\tilde{\gamma}$. Hence, $\tilde{\beta}$ cannot appear in the expansion of $\Phi^{-1}(\tilde{\alpha})$ because it is not in that of $x_{u+1}G_{\tilde{\delta}}$ for any $\tilde{\delta}$. $\square$

With this ordering, there is a unique leading $G_{\tilde{\delta}}$ for each expansion of $X^{\tilde{\alpha}}F_\beta$.

Theorem 4.4. *The set $A = \{G_{\tilde{\alpha}} \mid G_{\tilde{\alpha}} \notin Img(\Phi)\}$ forms a Hilbert basis for the quotient space R/I.*

Proof. For any polynomial p in R, we write p in terms of the G basis with $<_G$ order. For each term $G_{\tilde{\alpha}} \in Img(\Phi)$, we subtract p by $\Phi^{-1}(G_{\tilde{\alpha}}) \in I$ and $G_{\tilde{\alpha}}$ is cancelled. If we repeat this process (possibly countably many times), we can express p as a series of A. $\square$

5 Finitely many variables case

In the case that there are finitely many variables, $R_n = \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, the above constructions of $\mathsf{DQSym}(x_1, \ldots, x_n, y_1, \ldots, y_n)$, the F, G bases and the ideal $I_n =< \mathsf{DQSym}^+(x_1, \ldots, x_n, y_1, \ldots, y_n) >$ remain the same by taking $x_i = y_i = 0$ for $i > n$. In this case, $LM(G_{\tilde{\alpha}}) = X^{\tilde{\alpha}}$ whenever $\ell(\tilde{\alpha}) \leqslant n$ and hence $\{G_{\tilde{\alpha}} : \ell(\tilde{\alpha}) \leqslant n\}$ spans R_n.

Let $R_n^{i,j}$ be the span of $\{X^{\tilde{\alpha}} : \ell(\tilde{\alpha}) \leqslant n, \sum_k \tilde{\alpha}_{1k} = i, \sum_k \tilde{\alpha}_{2k} = j\}$. Since I_n is bihomogeneous in $\mathbf{x}$ and $\mathbf{y}$, $I_n = \bigoplus_{i,j} I_n^{i,j}$ where $I_n^{i,j} = I_n \cap R_n^{i,j}$, and $R_n/I_n = \bigoplus_{i,j} V_n^{i,j}$ where $V_n^{i,j} = R_n/I_n \cap R_n^{i,j}$.

The Hilbert matrix corresponding to R_n/I_n is the matrix $M_n(i,j) = \dim(V_n^{i-1,j-1})$.

The goal of this section is to compute the second column of the Hilbert matrix. The proof is slight generalization of the one in [2].

Lemma 5.1. *The set $\{G_{\tilde{\alpha}} \mid G_{\tilde{\alpha}} \notin Img(\Phi), \ell(\tilde{\alpha}) \leqslant n\}$ spans the quotient R_n/I_n.*

Proof. Among all $\tilde{\alpha}$ such that $G_{\tilde{\alpha}} \in Img(\Phi)$, $\ell(\tilde{\alpha}) \leqslant n$ and $G_{\tilde{\alpha}}$ cannot be reduced to 0, let $\tilde{\beta}$ be the smallest one with respect to the $<_G$ order. Then,

$$\begin{aligned} G_{\tilde{\beta}} &= G_{\tilde{\beta}} - \Phi^{-1}(G_{\tilde{\beta}}) + \Phi^{-1}(G_{\tilde{\beta}}) \\ &\equiv G_{\tilde{\beta}} - \Phi^{-1}(G_{\tilde{\beta}}) \mod I_n \end{aligned}$$

But since $G_{\tilde{\beta}}$ is the leading term in $\Phi^{-1}(G_{\tilde{\beta}})$, terms in $G_{\tilde{\beta}} - \Phi^{-1}(G_{\tilde{\beta}})$ are strictly smaller than $G_{\tilde{\beta}}$, and thus they reduce to 0. This contradicts to our assumption on $\tilde{\beta}$. □

Let B_n be the set of generalized bicompositions $\{\tilde{\alpha}\}$ such that $\sum_{i=1}^{k}(\tilde{\alpha}_{1i} + \tilde{\alpha}_{2i}) < k$ for all $1 \leqslant k \leqslant n$ and $\ell(\tilde{\alpha}) \leqslant n$. Clearly from the definition of G basis, if $\tilde{\alpha} \notin B_n$, then $G_{\tilde{\alpha}} \in I_n$. Therefore, the set $\{X^{\tilde{\alpha}} : \tilde{\alpha} \in B_n\}$ spans R_n/I_n, the proof is essentially the same as Lemma 5.1. In particular, $X^{\tilde{\alpha}} \in I_n$ for all $|\tilde{\alpha}| \geqslant n$.

Lemma 5.2. *The set $\{X^{\tilde{\alpha}} F_\beta : \tilde{\alpha} \in B_n, |\beta| \geqslant 0\}$ spans R_n.*

Proof. We already have $X^{\tilde{\epsilon}} \equiv \sum_{\tilde{\alpha} \in B_n} X^{\tilde{\alpha}} \mod I_n$, which means $X^{\tilde{\epsilon}} = \sum_{\tilde{\alpha} \in B_n} X^{\tilde{\alpha}} + \sum_{|\beta| \geqslant 1} P_\beta F_\beta$ for some polynomial P_β. If we reduce each monomial P_β using the above rule, and write the product of F basis in terms of F basis, the claim will be satisfied in a finite number of steps. □

For a generalized bicomposition $\tilde{\alpha}$ with $\ell(\tilde{\alpha}) \leqslant n$, we define its reverse $\overline{\alpha}$ to be the generalized bicomposition such that $\overline{\alpha}_{1i} = \tilde{\alpha}_{1(n-i+1)}$ and $\overline{\alpha}_{2i} = \tilde{\alpha}_{2(n-i+1)}$ for all $1 \leqslant i \leqslant n$.

We denote the set $\{X^{\tilde{\alpha}} : \overline{\alpha} \in B_n\}$ by A_n. The endomorphism of R_n that sends x_i to x_{n-i+1} and y_i to y_{n-i+1} is clearly an algebra isomorphism that fixes $\mathsf{DQSym}(\mathbf{x}, \mathbf{y})$, in fact, it sends M_α to $M_{\alpha'}$ where α' is the reversed bicomposition of α. Therefore, by Lemma 5.2, the set $\{X^{\tilde{\alpha}} F_\beta : \tilde{\alpha} \in A_n, |\beta| \geqslant 0\}$ spans R_n.

Hence, $I_n = \langle F_\gamma : |\gamma| \geqslant 1 \rangle$ is spanned by $\{X^{\tilde{\alpha}} F_\beta F_\gamma : \tilde{\alpha} \in A_n, |\beta| \geqslant 0, |\gamma| \geqslant 1\}$, which means it is spanned by $\{X^{\tilde{\alpha}} F_\beta : \tilde{\alpha} \in A_n, |\beta| \geqslant 1\}$.

Lemma 5.3. *For $X^{\tilde{\alpha}}F_\beta \in R_n^{i,1}$ with $\tilde{\alpha} \in A_n$, $|\beta| \geqslant 1$ and $|\tilde{\alpha}|+|\beta| < n$, let $G_{\tilde{\gamma}} = \Phi(X^{\tilde{\alpha}}F_\beta)$, then $\ell(\tilde{\gamma}) \leqslant n$.*

Proof. First, rules (4.1) and (4.2) increase the length by 1 while (4.3) increase the length by 2. Now, we need to track $\tilde{\gamma}_{\ell(\tilde{\gamma})}$. If $\tilde{\gamma}_{\ell(\tilde{\gamma})}$ comes from $\beta_{\ell(\beta)}$ and gets shifted, since we can use (4.3) at most once, we can make at most $|\tilde{\alpha}| + 1$ steps to the right. Therefore, $\ell(\tilde{\gamma}) \leqslant |\tilde{\alpha}| + 1 + \ell(\beta) \leqslant |\tilde{\alpha}| + 1 + |\beta| \leqslant n$.

If $\tilde{\gamma}_{\ell(\tilde{\gamma})}$ is 1 which comes from multiplying x_k or y_k to $G_{\tilde{\epsilon}}$ with $k > \ell(\tilde{\epsilon})$, since $\tilde{\alpha} \in A_n$, we have $\sum_{i\geqslant k}(\tilde{\alpha}_{1i} + \tilde{\alpha}_{2i}) < n - k + 1$. In this process, we use rules (4.1) and (4.2) only and each increases the length by 1. Therefore, $\tilde{\gamma}_{\ell(\tilde{\gamma})}$ can be shifted to at most position $k + n - k = n$. □

Corollary 5.4. *Let M_n be the Hilbert matrix of R_n/I_n, then $M_n(n-1, 2) = \frac{1}{n}\binom{2n-2}{n-1}$, $M_n(i, 2) = \sum_{1\leqslant j\leqslant i, 1\leqslant k\leqslant 2} M_{n-1}(j, k)$ for $1 \leqslant i \leqslant n-2$, and $M_n(2, 1) = 0$ for $i \geqslant n$.*

Proof. Lemma 5.1 shows that $C_i = \{G_{\tilde{\alpha}} \in V_n^{i,1} : G_{\tilde{\alpha}} \notin Img(\Phi)\}$ spans $V_n^{i,1}$. Suppose there is a linear dependence $P = \sum_{G_{\tilde{\alpha}}\in C_i} a_{\tilde{\alpha}} G_{\tilde{\alpha}} \in I_n^{i,1}$. Since $I_n^{i,1}$ is spanned by $D = \{X^{\tilde{\alpha}}F_\beta \in R_n^{i,1} : \tilde{\alpha} \in A_n, |\beta| \geqslant 1\}$, we have $P = \sum_{X^{\tilde{\alpha}}F_\beta\in D} b_{\tilde{\alpha}\beta} X^{\tilde{\alpha}}F_\beta$. This means the leading term of P when we expand in G basis is some $G_{\tilde{\gamma}}$ such that $\tilde{\gamma} \in Img(\Phi)$ and by Lemma 5.3 $\ell(\tilde{\gamma}) \leqslant n$, which is absurd. Therefore, C_i is a linear basis for $V_n^{i,1}$.

Now, $M_n(i, 1) = \dim V_n^{i-1,1} = |C_{i-1}|$. Let $G_{\tilde{\gamma}} \in V_n^{i,1}$ and k be the unique number that $\tilde{\gamma}_{k2} = 1$. First, from definition of G, $\tilde{\gamma} \notin B_n$ implies $G_{\tilde{\gamma}} \in I_n$ and $G_{\tilde{\gamma}} \in Img(\Phi)$.

If $i = n - 1$, then $|\tilde{\gamma}| = n - 1$. If $k < \ell(\tilde{\gamma})$, since $\sum_{j=k+1}^{n} \tilde{\gamma}_{1j} \geqslant n - k$, we will be using rules (4.3) when applying ϕ^{-1}. This reduces the length by 2 while the size by 1, which means $G_{\tilde{\gamma}} \in Img(\Phi)$. If $k = \ell(\tilde{\gamma})$, we only use rules (4.1) and (4.2) when applying ϕ^{-1}. In this case, $G_{\tilde{\gamma}} \notin Img(\Phi)$ whenever $\tilde{\gamma} \in B_n$. Therefore, $|C_{n-2}|$ is the Catalan number $\frac{1}{n}\binom{2n-2}{n-1}$.

If $1 \leqslant i \leqslant n - 2$, $|\tilde{\gamma}| \leqslant n - 2$. From the definition of ϕ, $G_{\tilde{\gamma}} \notin Img(\Phi)$ if and only if $G_{\left(\begin{smallmatrix}\tilde{\gamma}_{11} & \cdots & \tilde{\gamma}_{1(n-1)}\\ \tilde{\gamma}_{21} & \cdots & \tilde{\gamma}_{2(n-1)}\end{smallmatrix}\right)} \in V_{n-1}^{j,k} \setminus Img(\Phi)$ for some $1 \leqslant j \leqslant i, 1 \leqslant k \leqslant 2$. Therefore, $M_n(i, 2) = \sum_{1\leqslant j\leqslant i, 1\leqslant k\leqslant 2} M_{n-1}(j, k)$ for $1 \leqslant i \leqslant n - 2$. □

By the symmetry $M_n(a, b) = M_n(b, a)$, we obtain the first to rows of the Hilbert matrix, namely $M_n(2, n-1) = \frac{1}{n}\binom{2n-2}{n-1}$, $M_n(2, i) = \sum_{1\leqslant j\leqslant i, 1\leqslant k\leqslant 2} M_{n-1}(k, j)$ for $1 \leqslant i \leqslant n-2$, and $M_n(2, i) = 0$ for $i \geqslant n$.

This method can be applied directly to some other terms. To be more specific, for $2i + j \leqslant n$, the set $\{G_{\tilde{\alpha}} \mid G_{\tilde{\alpha}} \notin Img(\Phi), \ell(\tilde{\alpha}) \leqslant n\}$ is a linear basis in $V_n^{i,j}$. Therefore, the formula for each column stabilizes when the number of variables is large enough. However, it fails in some other terms and this set is not a linear basis in general.

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