Quasi-Eulerian Hypergraphs

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Abstract

We generalize the notion of an Euler tour in a graph in the following way. An *Euler family* in a hypergraph is a family of closed walks that jointly traverse each edge of the hypergraph exactly once. An *Euler tour* thus corresponds to an Euler family with a single component. We provide necessary and sufficient conditions for the existence of an Euler family in an arbitrary hypergraph, and in particular, we show that every 3-uniform hypergraph without cut edges admits an Euler family. Finally, we show that the problem of existence of an Euler family is polynomial on the class of all hypergraphs.

This work complements existing results on rank-1 universal cycles and 1-overlap cycles in triple systems, as well as recent results by Lonc and Naroski, who showed that the problem of existence of an Euler tour in a hypergraph is NP-complete.

Keywords: Hypergraph, Euler tour, eulerian hypergraph, Euler family, quasi-eulerian hypergraph, (g, f)-factor

1 Introduction

As first claimed by Euler in 1741 [6], and proved by Hierholzer and Wiener in 1873 [9], it is now well known that a connected graph admits an Euler tour — that is, a closed walk traversing each edge exactly once — if and only if it has no vertices of odd degree. In this paper, we are concerned with the analogous problem for hypergraphs. As we shall see, this extended problem is much more complex; in fact, there is more than one natural way to generalize the notion of an Euler tour to hypergraphs. We shall consider three such natural extensions, one being the main focus.

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An *Euler tour* of a hypergraph H is a closed walk traversing every edge of H exactly once, and a hypergraph is called *eulerian* if it admits and Euler tour. Not much has been previously known about eulerian properties of hypergraphs. The most in-depth treatment to date can be found in [12], where Euler tours in k-uniform hypergraphs are considered. In particular, the authors of [12] determine some necessary conditions for the existence of an Euler tour in a hypergraph, and show that these are also sufficient for certain classes of k-uniform hypergraphs. They also show that the problem of determining whether or not a given hypergraph is eulerian is NP-complete on the class of k-uniform hypergraphs (for each $k \geqslant 3$), and even just on the class of 3-uniform hypergraphs with a connected "skeleton" [12, Theorem 7]. Thus, there is little hope to characterize eulerian (uniform) hypergraphs in general.

The results of this paper will also be of interest to those who study universal cycles for set systems. This concept was introduced by Chung, Diaconis, and Graham [4], and generalizes de Bruijn cycles to various combinatorial structures. In this context, an Euler tour in a 3-uniform hypergraph has also been called a *rank-1 universal cycle* [5] and *1-overlap cycle* [11]. Thus, the results of [5, 11] show that certain classes of triple systems are eulerian. In a recent work, Wagner and the second author applied our results (Theorem 5.2) to show that in fact all triple systems with at least two blocks are eulerian.

We generalize the notion of an Euler tour as follows. An *Euler family* of a hypergraph H is a family of closed walks of H such that each edge of H lies in exactly one component (member) of the family. A hypergraph is called *quasi-eulerian* if it admits an Euler family. Clearly, every eulerian hypergraph is quasi-eulerian, but the converse does not hold.

The main contributions of this paper are as follows. In Sections 3 and 4, respectively, we present some necessary and some sufficient conditions for a hypergraph to be eulerian or quasi-eulerian. In Section 5, we show that every 3-uniform hypergraph without cut edges is quasi-eulerian. In Section 6, we exhibit a close relationship between the rich theory of factors in graphs and quasi-eulerian hypergraphs, and give necessary and sufficient conditions for the existence of an Euler family in an arbitrary hypergraph. Finally, in Section 7 we show that the problem of existence of an Euler family is polynomial on the class of all hypergraphs. We conclude the paper with a brief discussion of closely related concepts such as flag-traversing tours and cycle decompositions of hypergraphs.

2 Preliminaries

For basic hypergraph terminology, we refer the reader to [2] or [3]; the latter being a longer version of the present article.

A hypergraph $H = (V, E)$ is called *trivial* if it has only one vertex. The *degree* of a vertex $v \in V$ (denoted by $\deg_H(v)$ or simply $\deg(v)$ if no ambiguity can arise) is the number of edges $e \in E$ such that $v \in e$. A vertex of degree 0 is called *isolated*, and a vertex of degree 1 is called *pendant*. A hypergraph $H = (V, E)$ is called *even* if all of its vertices are of even degree, and *r-regular on a set* $V' \subseteq V$ if $\deg_H(v) = r$ for all $v \in V'$.

A *walk* $W = v_0e_1v_1e_2v_2 \ldots v_{k-1}e_kv_k$ in a hypergraph $H = (V, E)$ is an alternating sequence of vertices and edges with $\{v_{i-1}, v_i\} \subseteq e_i$ and $v_{i-1} \neq v_i$ for all $i = 1, \ldots, k$. Such a walk W is called a *strict trail* if $e_1, \ldots, e_k$ are pairwise distinct, and *closed* if $k \geqslant 2$ and $v_0 = v_k$. Thus, each component in an Euler family is a closed strict trail. A closed strict trail $W = v_0 e_1 v_1 e_2 v_2 \ldots v_{k-1} e_k v_0$ is called a *cycle* if the vertices $v_0, v_1, \ldots, v_{k-1}$ are pairwise distinct.

Let $\mathrm{c}(H)$ denote the number of connected components of a hypergraph $H = (V, E)$. A *cut edge* of H is an edge $e \in E$ such that $\mathrm{c}(H - e) > \mathrm{c}(H)$. A graph with a cut edge obviously does not admit an Euler tour. The issue is more complex for hypergraphs. First, we need to distinguish between two types of cut edges. If e is a cut edge in a hypergraph $H = (V, E)$, then $\mathrm{c}(H - e) \leqslant \mathrm{c}(H) + |e| - 1$ [2, Lemma 3.15]. A cut edge that achieves the upper bound in this inequality is called *strong*; all other cut edges are called *weak*. Observe that a cut edge has cardinality at least two, and that any cut edge of cardinality two (and hence any cut edge in a graph) is necessarily strong.

The *incidence graph* $\mathcal{I}(H)$ of a hypergraph $H = (V, E)$ is a bipartite simple graph with bipartition $\{V, E\}$ and edge set $\{ve : v \in V, e \in E, v \in e\}$. The elements of V and E are called the *v-vertices* and *e-vertices*, respectively.

The following lemma will be one of our main tools in the study of quasi-eulerian hypergraphs. We leave the proof to the reader.

Lemma 1. *Let $H = (V, E)$ be a hypergraph and G its incidence graph. Then H is quasi-eulerian if and only if G has an even subgraph that is 2-regular on E. Moreover, H is eulerian if and only if G has an even subgraph with a single non-trivial connected component that is 2-regular on E.*

3 Necessary Conditions

It is easy to see that a quasi-eulerian hypergraph has at least two edges. Here is a more interesting result.

Lemma 2. *Let $H = (V, E)$ be a quasi-eulerian hypergraph, and let V_{odd} be the set of odd-degree vertices in H. Then the following hold.*

(a) Each edge of H contains at least two non-pendant vertices.

(b) $|E| \leqslant \sum_{v \in V} \lfloor \deg_H(v)/2 \rfloor$.

(c) $|V_{odd}| \leqslant \sum_{e \in E} (|e| - 2)$.

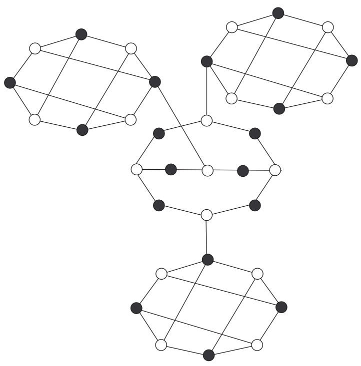
(d) H has no strong cut edges.

Moreover, statements (b) and (c) are equivalent.

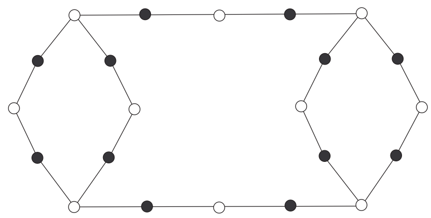
Proof. Statement (a) follows immediately from the definition of an Euler family. Statements (b) and (c), and their equivalence, are straightforward extensions of a result by Lonc and Naroski [12, Proposition 1].

To prove (d), suppose to the contrary that e is a strong cut edge of H. Since H admits an Euler family, the edge e lies in a closed strict trail, and consequently in a cycle of H. However, by [2, Theorem 3.18], no strong cut edge lies in a cycle — a contradiction. $\square$

It is easy to see that for graphs, the necessary conditions in Lemma 2 are also sufficient (Note that some of these conditions alone are not sufficient. In fact, conditions (b) and (c) alone are sufficient, but conditions (a) and (d) alone are not). This is not the case for general hypergraphs. Figures (A) and (B) show the incidence graphs of hypergraphs that satisfy conditions (a)–(d) of Lemma 2, but are not quasi-eulerian. (Note that in all figures, black dots represent the v-vertices.) Observe that the hypergraph on the left (A) has only weak cut edges, while the hypergraph on the right (B) has no cut edges.

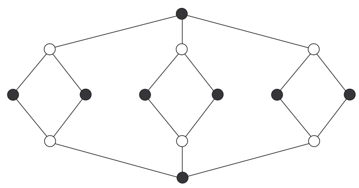


(A)



(B)

Very recently, Stamplecoskie [15] showed that for every $c \geqslant 2$ and $m \geqslant 5$, there exists a connected hypergraph with m edges, each of size at least c, that has no cut edges, satisfies the necessary conditions from Lemma 2, but is not quasi-eulerian. In Theorem 5.2 we shall see that every 3-uniform hypergraph without cut edges is quasi-eulerian. However, Figure (C) shows that not all 3-uniform hypergraphs without cut edges are eulerian.



(C)

Of course, for a hypergraph to be eulerian, conditions (a)–(d) of Lemma 2 are also necessary. The following result elaborates on (d).

Proposition 3.1. *Let $H = (V, E)$ be an eulerian hypergraph with a cut edge e. Then $H - e$ has at most one non-trivial connected component.*

Proof. Suppose, to the contrary, that $H - e$ has at least two non-trivial connected components. Since e is a cut edge of H, it is a cut e-vertex in its incidence graph $G = \mathcal{I}(H)$ [2, Theorem 3.23], and the connected components of the vertex-deleted graph $G \backslash e$ are the incidence graphs of the connected components of $H - e$ [2, Lemma 2.8 and Corollary

3.12]. Hence, by assumption, $G\backslash e$ has at least two connected components with e-vertices. Since H has an Euler tour, G has a closed trail T traversing each e-vertex (including e) exactly once. Hence $T\backslash e$ is a trail that traverses every e-vertex in $G\backslash e$, contradicting the above. $\square$

We conclude this section with the following open question.

Question 3.2. Does there exist a connected k-uniform hypergraph with $k \geqslant 4$ that has no cut edges and that satisfies the necessary conditions from Lemma 2 but is not quasi-eulerian?

4 Sufficient Conditions

For any positive integer ℓ, we define the *ℓ-intersection graph* $\mathcal{L}_\ell(H)$ of the hypergraph $H = (V, E)$ as the graph with vertex set E and edge set $\{ee' : e, e' \in E, e \neq e', |e \cap e'| = \ell\}$. Lonc and Naroski [12, Theorem 2] showed the following.

Theorem 4.1. *[12, Theorem 2] Let $H = (V, E)$ be a hypergraph with a connected $(k-1)$-intersection graph. Then H is eulerian if and only if $|V_{odd}| \leqslant \sum_{e \in E}(|e| - 2)$.*

Using the techniques of [12, Theorem 3] in the proof of Case $k > 3$ of Theorem 4.1, we extend this result to quasi-eulerian hypergraphs, as well as a to a larger family of eulerian (not necessarily uniform) hypergraphs; see Theorems 4.2 and 4.3 below, respectively.

For any hypergraph $H = (V, E)$, define a digraph $\mathcal{D}_3(H)$ as follows: its vertex set is E and its arc set is $\{(e, f) : e, f \in E, |f - e| = 1, |e \cap f| \geqslant 3\}$. Recall that an *arborescence* is a directed graph whose underlying undirected graph is a tree, and whose arcs are all directed towards a root.

Theorem 4.2. *Let $H = (V, E)$ be a hypergraph such that its digraph $\mathcal{D}_3(H)$ has a spanning subdigraph that is a vertex-disjoint union of non-trivial arborescences. Then H is quasi-eulerian.*

Proof. For convenience, we say that a digraph satisfies Property P if has a spanning subdigraph that is a vertex-disjoint union of non-trivial arborescences. We shall prove by induction on the number of edges that every hypergraph H whose digraph $\mathcal{D}_3(H)$ satisfies Property P possesses an Euler family. Observe that such a hypergraph necessarily has at least two edges.

First, let $H = (V, E)$ be a hypergraph with $E = \{e, f\}$ such that its digraph $\mathcal{D}_3(H)$ satisfies Property P. Then $\mathcal{D}_3(H)$ must have a spanning arborescence. Moreover, we have that $|e \cap f| \geqslant 3$. Take any $u, v \in e \cap f$ such that $u \neq v$. Then $T = uevfu$ is an Euler tour of H. Thus H possesses an Euler family as claimed.

Assume that for some $m \geqslant 2$, every hypergraph H with at least m edges whose digraph $\mathcal{D}_3(H)$ satisfies Property P possesses an Euler family. Let $H = (V, E)$ be a hypergraph with $|E| = m + 1$ such that its digraph $\mathcal{D}_3(H)$ has a spanning subdigraph D' that is a vertex-disjoint union of non-trivial arborescences. If each arborescence in D' is of order

2, then (just as in the base case above) each gives rise to a closed strict trail of length 2 in H, and the union of all these strict trails is an Euler family in H.

Hence assume that D' has a weakly connected component A that is an arborescence of order at least 3. Let $e \in E$ be a leaf (that is, vertex of indegree 0 and outdegree 1) of A and f its outneighbour in A. Then $|f - e| = 1$ and $|e \cap f| \geqslant 3$. Now $\mathcal{D}_3(H - e)$ has a spanning digraph $D' \setminus e$ that is a vertex-disjoint union of non-trivial arborescences. Hence by the induction hypothesis, the hypergraph $H - e$ possesses an Euler family $\mathcal{F}$. Let $T = ufvW$ — where u and v are distinct vertices in f, and W is an appropriate (v, u)-walk — be a closed strict trail in $\mathcal{F}$. We now reroute T to include the edge e, resulting in a closed strict trail T' of H, as follows.

Since $|f - e| = 1$, at least one of u and v — say v without loss of generality — is also in e, and since $|e \cap f| \geqslant 3$, there exists $w \in e \cap f$ such that $w \neq u, v$. Then $T' = ufwevW$ is a closed strict trail of H. Finally, replace T in $\mathcal{F}$ by T' to obtain an Euler family of H.

The result follows by induction. □

With very minor changes to the above proof we obtain the following.

Theorem 4.3. *Let $H = (V, E)$ be a hypergraph such that its digraph $\mathcal{D}_3(H)$ has a non-trivial spanning arborescence. Then H is eulerian.*

Using Lemma 1, certain families of hypergraphs can be easily seen to be eulerian or quasi-eulerian. The first of the following corollaries is immediate.

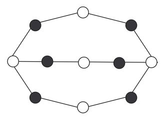
Corollary 4.4. *Let H be a hypergraph with the incidence graph G. If G has a 2-factor, then H is quasi-eulerian. If G is hamiltonian, then H is eulerian.*

Corollary 4.5. *Let H be an r-regular r-uniform hypergraph for $r \geqslant 2$. Then H is quasi-eulerian.*

Proof. The incidence graph G of H is an r-regular bipartite graph with $r \geqslant 2$. Therefore, as a corollary of Hall's Theorem [8], G admits two edge-disjoint perfect matchings, and hence a 2-factor. Thus H is quasi-eulerian by Corollary 4.4. □

Corollary 4.6. *Let H be a $2k$-uniform even hypergraph. Then H is quasi-eulerian.*

Proof. Let G be the incidence graph of H. In G, every e-vertex has degree $2k$, and every v-vertex has even degree. A result by Hilton [10, Theorem 8] then shows that G has an *evenly equitable* k-edge colouring; that is, a k-edge colouring such that (i) every vertex is incident with an even number of edges of each colour, and (ii) for each vertex, the numbers of edges of any two colours that are incident with this vertex differ by at most two. Hence the i-th colour class, for $i = 1, 2, \ldots, k$, induces an even subgraph G_i of G that is 2-regular on E. By Lemma 1, each G_i corresponds to an Euler family $\mathcal{F}_i$ of H, and H is quasi-eulerian. □



(D)

5 3-uniform Hypergraphs

Recall from Lemma 2 that a quasi-eulerian hypergraph cannot have strong cut edges, while a hypergraph with weak cut edges may or may not be quasi-eulerian. The main result of this section (Theorem 5.2 below) completes the picture for 3-uniform hypergraphs. The key ingredient in the proof is the following result by Fleischner.

Theorem 5.1. *[7] Every graph without cut edges and of minimum degree at least 3 has a spanning even subgraph without isolated vertices.*

Theorem 5.2. *Let $H = (V, E)$ be a 3-uniform hypergraph without cut edges. Then H is quasi-eulerian.*

Proof. First, we show that since H has no cut edges, its incidence graph $G = \mathcal{I}(H)$ has no cut edges either. Suppose, to the contrary, that ve is a cut edge of G (where $v \in V$ and $e \in E$), and let G_v and G_e be the connected components of $G - ve$ containing vertex v and e, respectively. Since $|e| > 1$, the component G_e must contain a v-vertex w, and v and w are disconnected in $G - ve$. Hence they are disconnected in $H - e$, showing that e is a cut edge of H, a contradiction. Therefore G has no cut edges as claimed.

Note that we may assume that H, and hence G, has no isolated vertices. Clearly, G has no vertices of degree 1, since the edge incident with such a vertex would necessarily be a cut edge. Suppose G has a vertex of degree 2. Then it must be a v-vertex, since $|e| = 3$ for all $e \in E$. Obtain a graph G^* from G by replacing, for every vertex v of degree 2, the 2-path e_1ve_2 in G with an edge e_1e_2. Observe that in G^*, all e-vertices have degree 3, and all v-vertices have degree at least 3. Moreover, G^* has no cut edges since G does not. Therefore, by Theorem 5.1, G^* has a spanning even subgraph G_1^* without isolated vertices. We construct a subgraph G_1 of G as follows: for any vertex v of degree 2 in G, and its incident edges e_1 and e_2, if e_1e_2 is an edge of G_1^*, then replace it with the 2-path e_1ve_2. The resulting graph G_1 is an even subgraph of G without isolated e-vertices. Since every e-vertex of G has degree 3 in G, it has degree 2 in G_1. Thus, by Lemma 1, G_1 gives rise to an Euler family of H. □

The reader may have noticed that if H has no isolated vertices, then the Euler family of H constructed in the proof of Theorem 5.2 traverses every vertex of H except possibly some of the vertices of degree 2. Observe also that Theorem 5.2 does not generally hold for graphs — for example, a cycle with a chord — nor for hypergraphs in which every edge has size 2 or 3 — an example is given in Figure (D).

Corollary 5.3. *Let $H = (V, E)$ be a 3-uniform hypergraph with at least two edges such that each pair of vertices lie together in at least one edge. Then H is quasi-eulerian.*

Proof. By Theorem 5.2, it suffices to show that H has no cut edges. If $|V| = 3$, then clearly none of the edges are cut edges. Hence assume $|V| \geqslant 4$, and suppose that $e \in E$ is a cut edge of H. Let $u_1, u_2 \in V$ be vertices of e that lie in distinct connected components of $H - e$, and consider any vertex $w \notin e$. Then there exist edges e_1, e_2 such that $w, u_i \in e_i$ for $i = 1, 2$. Since obviously $e_1, e_2 \neq e$, vertex w must lie in the same connected component of $H - e$ as both u_1 and u_2, a contradiction. □

Recall that a *triple system TS(n,λ)* is a 3-uniform hypegraph of order n such that every pair of vertices lie together in exactly λ edges.

Corollary 5.4. *Every triple system TS(n,λ) with $(n, \lambda) \neq (3, 1)$ is quasi-eulerian.*

Very recently, applying the results of this section, Wagner and the second author [14] proved that all triple systems TS(n,λ), except for TS(3,1), are in fact eulerian, thus extending various results on rank-1 universal cycles [5] and 1-overlap cycles [11].

The proof of Theorem 4.1 [12, Theorem 2] for the case $k = 3$ is very long and technical; as another corollary of our Theorem 5.2, we show that every 3-uniform hypergraph with a connected 2-intersection graph is eulerian provided that it has no pendant vertices.

Corollary 5.5. *Every 3-uniform hypergraph with a connected 2-intersection graph and without pendant vertices is eulerian.*

Proof. Let $H = (V, E)$ be a 3-uniform hypergraph with a connected 2-intersection graph L and without pendant vertices. Hence H has at least 2 edges. Suppose it has a cut edge e. Since L is connected, e shares exactly two of its vertices with another edge; consequently, these two vertices lie in the same connected component of $H-e$. Thus $H-e$ has exactly two connected components; let H_1 be the connected component containing a single vertex, w, of e. By assumption, w is not a pendant vertex in H, so $E(H_1) \neq \emptyset$. Take any $e_1 \in E(H_1)$ and any $e_2 \in E - E(H_1)$. Then $e_1 \cap e_2 \subseteq \{w\}$, whence $e_1 e_2 \notin E(L)$. It follows that L is disconnected, a contradiction.

We conclude that H has no cut edges, and hence is quasi-eulerian by Theorem 5.2. Let $\mathcal{F} = \{T_1, \ldots, T_k\}$ be an Euler family of H with a minimum number of components, and suppose $k \geqslant 2$. Let G be the incidence graph of H, let G' be the subgraph of G corresponding to $\mathcal{F}$, and $G_1, \ldots, G_k$ the connected components of G' corresponding to the closed strict trails $T_1, \ldots, T_k$ of H. Since L is connected, without loss of generality, there exist e-vertices e_1 of G_1 and e_2 of G_2 that are adjacent in L, and hence in G have two common neighbours, say v_1 and v_2. Since e_1 and e_2 are of degree 3 in G, and of degree 2 in G', each is adjacent to at least one of v_1 and v_2 in G', and since they lie in distinct connected components of G', we may assume without loss of generality that $v_1 e_1, v_2 e_2 \in E(G')$. Obtain G'' by replacing these two edges of G' with edges $v_1 e_2$ and $v_2 e_2$. Then G'' is an even subgraph of G that is 2-regular on E, so it corresponds to an Euler family of H. But since G'' has one fewer connected component than G', it contradicts the minimality of $\mathcal{F}$.

We conclude that $k = 1$, that is, H admits an Euler tour. □

6 Necessary and Sufficient Conditions via (g, f)-factors

Recall that by Lemma 1, a hypergraph $H = (V, E)$ admits an Euler family if and only if its incidence graph $\mathcal{I}(H)$ has an even subgraph that is 2-regular on E. We shall now combine this observation with Lovasz's (g, f)-factor Theorem [13] to give more easily verifiable necessary and sufficient conditions.

For a graph G and functions $f, g : V(G) \to \mathbb{N}$, a (g, f)*-factor* of G is a spanning subgraph F of G such that $g(x) \leqslant \deg_F(x) \leqslant f(x)$ for all $x \in V(G)$. An f*-factor* is simply an (f, f)-factor. For any subgraph G_1 of G and any sets $V_1, V_2 \subseteq V(G)$, let $\varepsilon_{G_1}(V_1, V_2)$ denote the number of edges of G_1 with one end in V_1 and the other in V_2.

Theorem 6.1. *[13] Let G be a graph and let $f, g : V(G) \to \mathbb{N}$ be functions such that $g(x) \leqslant f(x)$ and $g(x) \equiv f(x) \pmod 2$ for all $x \in V(G)$. Then G has a (g, f)-factor F such that $\deg_F(x) \equiv f(x) \pmod 2$ for all $x \in V(G)$ if and only if all disjoint subsets S and T of $V(G)$ satisfy*

$$\sum_{x \in S} f(x) + \sum_{x \in T} (\deg_G(x) - g(x)) - \varepsilon_G(S, T) - q(S, T) \geqslant 0, \tag{1}$$

where $q(S, T)$ is the number of connected components C of $G \setminus (S \cup T)$ such that

$$\sum_{x \in V(C)} f(x) + \varepsilon_G(V(C), T) \quad \textit{is odd.}$$

Corollary 6.2. *Let $H = (V, E)$ be a hypergraph and $G = \mathcal{I}(H)$ its incidence graph. Then H is quasi-eulerian if and only if all disjoint sets $S \subseteq E$ and $T \subseteq V \cup E$ satisfy*

$$2|S| + \sum_{x \in T} \deg_G(x) - 2|T \cap E| - \varepsilon_G(S, T \cap V) - q(S, T) \geqslant 0, \tag{2}$$

where $q(S, T)$ is the number of connected components C of $G \setminus (S \cup T)$ such that $\varepsilon_G(V(C), T)$ is odd.

Proof. By Lemma 1, H has an Euler family if and only if G has an even subgraph G' that is 2-regular on E. Define functions $f, g : V \cup E \to \mathbb{N}$ as follows:

$$g(x) = \begin{cases} 0 & \text{if } x \in V \\ 2 & \text{if } x \in E \end{cases} \qquad \text{and} \qquad f(x) = \begin{cases} K & \text{if } x \in V \\ 2 & \text{if } x \in E \end{cases},$$

where K is a sufficiently large even integer. Observe that f and g satisfy the assumptions of Theorem 6.1. Moreover, a subgraph G' of G with the required properties is a (g, f)-factor F of G with $\deg_F(x) \equiv f(x) \pmod 2$ for all $x \in V(G)$, and conversely.

For any subsets S and T of $V \cup E$, if $S \cap V \neq \varnothing$, then $\sum_{x \in S} f(x)$ is very large, and Condition (1) clearly holds for S and T. Thus Theorem 6.1 asserts that G has an (f, g)-factor if and only if Condition (1) holds for all disjoint sets $S \subseteq E$ and $T \subseteq V \cup E$ of $V(G)$.

Observing that $\sum_{x \in V(C)} f(x) + \varepsilon_G(V(C), T) \equiv \varepsilon_G(V(C), T) \pmod 2$, it is then straightforward to show that Condition (1) in Theorem 6.1 is equivalent to Condition (2) in the statement of this corollary. The result follows as claimed. □

To express the necessary conditions in Corollary 6.2 in the language of the hypergraph itself, we refer the reader to [3, Corollary 2.36].

7 Complexity

By the *bigness* of a hypergraph we mean the maximum of the order (number of vertices) and the size (number of edges) of the hypergraph. A hypergraph is called *linear* if every pair of distinct edges intersect in at most one vertex. In this section, we study the following two decision problems.

Problem 3. Euler Tour
Given: A hypergraph H.
Decide: Does H have an Euler tour?

Problem 4. Euler Family
Given: A hypergraph H.
Decide: Does H have an Euler family?

Lonc and Naroski [12, Theorem 1] showed that Euler Tour is NP-complete on the set of k-uniform hypergraphs for any $k \geqslant 3$ (as well as on the set of 3-uniform hypergraphs with a connected "skeleton"). Their proof for $k = 3$ actually demonstrates the following stronger statement.

Theorem 7.1. *[12] Let $\mathcal{LH}_3^2$ denote the family of linear 2-regular 3-uniform hypergraphs. Then* Euler Tour *is NP-complete on $\mathcal{LH}_3^2$.*

Next we show that Euler Family is a polynomial problem on the set of all hypergraphs.

Theorem 7.2. *Let $\mathcal{H}$ be the family of all hypergraphs. Then* Euler Family *is polynomial on $\mathcal{H}$.*

Proof. Let $H = (V, E)$ be a hypergraph, and G its incidence graph. By Lemma 1, H admits an Euler family if and only if G has an even subgraph G' that is 2-regular on E; in this proof, we shall call such a subgraph G' an *EF-factor* of G.

Starting from G, construct a graph G^* by appending $\lfloor \frac{\deg_G(v)}{2} \rfloor$ loops to each $v \in V$. Then define a function $f : V \cup E \to \mathbb{N}$ by $f(e) = 2$ for all $e \in E$, and $f(v) = 2\lfloor \frac{\deg_G(v)}{2} \rfloor$ for all $v \in V$.

We claim that G has an EF-factor G' if and only if G^* has an f-factor. Indeed, take an EF-factor G' of G. Appending $\frac{1}{2}(f(v) - \deg_{G'}(v))$ loops to each vertex $v \in V$ results in an f-factor of G^*. Conversely, removing the loops from any f-factor of G^* will result in an EF-factor G' of G. This conversion is clearly polynomial in the order of G, and hence in the bigness of H.

By [1, Theorem 6.2], the problem of finding an f-factor in the graph G^* is polynomial in the order of G^*. Hence the problem of finding an Euler family in H is polynomial in the bigness of H. We conclude that Euler Family is polynomial on $\mathcal{H}$. □

8 Final Remarks

The reader will have noticed that the conditions on the existence of an Euler tour and Euler family are equivalent for a connected 2-uniform hypergraph (that is, a connected graph without loops). In addition, observe that an Euler tour in a graph $H = (V, E)$ also traverses each ordered pair (v, e) such that $v \in V$, $e \in E$, and $v \in e$ (called a *flag*) exactly once. Hence we have another natural way to generalize the concept of an Euler tour to hypergraphs: we define a *flag-traversing tour* of a hypergraph H as a closed walk of H traversing each flag of H exactly once. As it turns out (see Theorem 8.1 below), the problem of existence of a flag-traversing tour in a hypergraph is easily solved. The proof, which we leave to the reader, is straightforward, relying on the fact that the flags of a hypergraph correspond to the edges of its incidence graph.

Theorem 8.1. *A connected hypergraph $H = (V, E)$ has a flag-traversing tour if and only if its incidence graph has an Euler tour, that is, if and only if $\deg_H(v)$ and $|e|$ are even for all $v \in V$ and $e \in E$.*

If the incidence graph is eulerian, then, by the well-known Veblen's Theorem [16], it admits a decomposition into cycles; hence the following easy corollary.

Corollary 8.2. *Let $H = (V, E)$ be a hypergraph such that $\deg_H(v)$ and $|e|$ are even for all $v \in V$ and $e \in E$. Then H is admits a collection of cycles such that each flag of H is traversed by exactly one of these cycles.*

As an analogue of Veblen's Theorem for hypergraphs, however, we have the following.

Theorem 8.3. *A hypergraph is quasi-eulerian if and only if it admits a decomposition into cycles.*

Proof. Let $H = (V, E)$ be a quasi-eulerian hypergraph and G its incidence graph. By Lemma 1, G has an even subgraph G' that is 2-regular on E. Hence G' admits a decomposition into cycles, $\mathcal{C}_G$, and every e-vertex lies in exactly one of the cycles in $\mathcal{C}_G$. Let $\mathcal{C}_H$ be the corresponding family of cycles in H. Then every $e \in E$ lies in exactly one of the cycles in $\mathcal{C}_H$, so $\mathcal{C}_H$ is a cycle decomposition of H.

Conversely, assume that $H = (V, E)$ is a hypergraph with a cycle decomposition $\mathcal{C}$. Then $\mathcal{C}$ is also an Euler family for H. □

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