On t-common list-colorings

Hojin Choi
Department of Mathematics, KAIST, Korea
hojinchoi@kaist.ac.kr

Young Soo Kwon*
Department of Mathematics, Yeungnam University, Korea
ysookwon@ynu.ac.kr

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Abstract

In this paper, we introduce a new variation of list-colorings. For a graph G and for a given nonnegative integer t, a *t-common list assignment* of G is a mapping L which assigns each vertex v a set $L(v)$ of colors such that given set of t colors belong to $L(v)$ for every $v \in V(G)$. The *t-common list chromatic number* of G denoted by $ch_t(G)$ is defined as the minimum positive integer k such that there exists an L-coloring of G for every t-common list assignment L of G, satisfying $|L(v)| \geqslant k$ for every vertex $v \in V(G)$. We show that for all positive integers k, ℓ with $2 \leqslant k \leqslant \ell$ and for any positive integers $i_1, i_2, \ldots, i_{k-2}$ with $k \leqslant i_{k-2} \leqslant \cdots \leqslant i_1 \leqslant \ell$, there exists a graph G such that $\chi(G) = k$, $ch(G) = \ell$ and $ch_t(G) = i_t$ for every $t = 1, \ldots, k-2$. Moreover, we consider the t-common list chromatic number of planar graphs. From the four color theorem [1, 2] and the result of Thomassen [9], for any $t = 1$ or 2, the sharp upper bound of t-common list chromatic number of planar graphs is 4 or 5. Our first step on t-common list chromatic number of planar graphs is to find such a sharp upper bound. By constructing a planar graph G such that $ch_1(G) = 5$, we show that the sharp upper bound for 1-common list chromatic number of planar graphs is 5. The sharp upper bound of 2-common list chromatic number of planar graphs is still open. We also suggest several questions related to t-common list chromatic number of planar graphs.

Keywords: graph coloring, list coloring, planar graph

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1 Introduction

Throughout this paper, all graphs are finite, undirected, and simple. For a graph G, let $V(G)$ and $E(G)$ be the vertex set and the edge set of G, respectively. The neighborhood of a vertex $v \in V(G)$, denoted by $N(v)$, is the set of vertices adjacent to v.

For a given graph G, a proper k-coloring $\phi : V(G) \to \{1, 2, \ldots, k\}$ of a graph G is an assignment of colors to the vertices of G so that any two adjacent vertices receive distinct colors. The *chromatic number* $\chi(G)$ of a graph G is the least positive integer k such that there exists a proper k-coloring of G. If G has a proper k-coloring, namely $\chi(G) \leqslant k$, then we say that G is k-colorable. A *list assignment* of a graph G is a mapping L which assigns each vertex v a set $L(v)$ of colors. An *L-coloring* of G is a proper vertex coloring ϕ of G such that $\phi(v) \in L(v)$ for each v. We say G is *L-colorable* if there exists an L-coloring of G. For a positive integer k, we say G is *k-choosable* if G has an L-coloring for every list assignment L satisfying $|L(v)| \geqslant k$ for every $v \in V(G)$. The *list chromatic number* or *choice number* $ch(G)$ of G is the minimum positive integer k such that G is k-choosable. Clearly $ch(G) \geqslant \chi(G)$ for every graph G.

For a graph G and for a given nonnegative integer t, a *t-common list assignment* of a graph G is a mapping L which assigns each vertex v a set $L(v)$ of colors such that given set of t colors belong to every $L(v)$, namely $|\bigcap_{v \in V(G)} L(v)| \geqslant t$. Note that 0-common list assignment is just a list assignment. The *t-common list chromatic number* of G denoted by $ch_t(G)$ is defined as the minimum positive integer k such that G is L-colorable for every t-common list assignment L of G satisfying $|L(v)| \geqslant k$ for every vertex v. Clearly, $ch_t(G) = t$ for every integer $t \geqslant \chi(G)$.

Before exploring this topic, we describe an application of t-common list-coloring. A company has n chemicals they have manufactured that need to be stored. Some pairs of chemicals are incompatible. For this reason, such pairs should be kept in distinct storage vessels. Say t storage vessels can keep all chemicals while other storage vessels can only keep certain chemicals because of storage vessel's conditions. Determine minimum positive integer k such that all chemicals can be stored if the number of possible storage vessels for each chemical is at least k. In other words, each chemical can potentially be stored in at least k vessels, given the restriction that some storage vessels can only store certain chemicals. We can convert this storage problem into a t-common list-coloring problem on a graph. Consider a graph $G = (V, E)$ with all chemicals as a vertex set, and an edge between chemicals x, y if and only if x and y are incompatible. For every vertex $v \in V$, let $L(v)$ be the set of all storage vessels which can keep the chemical corresponding to v. Now the list assignment L is a t-common list assignment and the above question corresponds to find t-common list chromatic number $ch_t(G)$ of G.

In the next section of this paper, we investigate several properties of the t-common list chromatic numbers. Furthermore we show that for all positive integers k, ℓ with $2 \leqslant k \leqslant \ell$

and for any positive integers $i_1, i_2, \ldots, i_{k-2}$ with $k \leqslant i_{k-2} \leqslant \cdots \leqslant i_1 \leqslant \ell$, there exists a graph G such that $\chi(G) = k$, $ch(G) = \ell$ and $ch_t(G) = i_t$ for all $t = 1, \ldots, k-2$. In Section 3, we consider the t-common list chromatic number of planar graphs. By constructing a planar graph G such that $ch_1(G) = 5$, we show that the sharp upper bound for 1-common list chromatic number of planar graphs is 5. Furthermore, we suggest several questions related to t-common list chromatic number of planar graphs.

2 Some properties of t-common list colorings

In this section, we consider several properties of the t-common list chromatic number. For a graph G with connected components $G_1, G_2, \ldots, G_i$ and for every nonnegative integer t, one can easily see that $ch_t(G) = \max\{ch_t(G_j) \mid j = 1, \ldots, i\}$. So from now on, we will consider the t-common list chromatic number of a connected graph. As with other graph coloring parameters, it holds that for every subgraph H of G and for every nonnegative integer t, $ch_t(H) \leqslant ch_t(G)$. The next lemma gives some relationships among $\chi(G), ch_t(G)$ and $ch(G)$.

Lemma 1. *Let G be a connected graph with $V(G) = \{v_1, \ldots, v_n\}$ and let $\chi(G) = k$. The following properties hold.*

(1) $\chi(G) = ch_{k-1}(G) \leqslant ch_{k-2}(G) \leqslant \cdots \leqslant ch_1(G) \leqslant ch(G)$.

(2) *For every nonnegative integer t with $t \geqslant k$, $ch_t(G) = t$.*

Proof. (1) Let t be a positive integer such that $t \leqslant k-1$. Note that the chromatic number $\chi(G)$ is the minimum i such that G has an L-coloring for $L(v_1) = \cdots = L(v_n) = \{c_1, c_2, \ldots, c_i\}$. Since the above list assignment L is a special t-common list assignment of G, we have $\chi(G) \leqslant ch_t(G)$. Note that every t-common list assignment is a $(t-1)$-common list assignment. So $\chi(G) \leqslant ch_{k-1}(G) \leqslant ch_{k-2}(G) \leqslant \cdots \leqslant ch_1(G) \leqslant ch(G)$.

Let L be a $(k-1)$-common list assignment such that $c_1, \ldots, c_{k-1} \in L(v_i)$ and $|L(v_i)| = k$ for all $i = 1, \ldots, n$. Since $\chi(G) = k$, the vertex set $V(G)$ can be partitioned into k independent sets $I_1, \ldots, I_k$. For all $j = 1, \ldots, k-1$, assign the color c_j to every vertex in I_j and for every $v \in I_k$, assign the color $c_v \in L(v) \setminus \{c_1, \ldots, c_{k-1}\}$ to v. This assignment is an L-coloring, and so $ch_{k-1}(G) \leqslant \chi(G)$. This implies that $ch_{k-1}(G) = \chi(G)$.
(2) By definition of $ch_t(G)$, one can easily show that $ch_t(G) = t$ for every t with $t \geqslant k$. □

By Lemma 1, we have the following corollary.

Corollary 2. *For a connected graph G and a nonnegative integer t, $ch_t(G) = t+1$ if and only if $t = \chi(G) - 1$.*

For a graph G, a list assignment L of G is called a *maximal unavailable list assignment* of G if G has no L-coloring and $|L(v)| = ch(G) - 1$ for every $v \in V(G)$. For example, a cycle C_3 of length 3 with vertex set $\{v_1, v_2, v_3\}$ has a maximal unavailable list assignment L with $L(v_1) = L(v_2) = L(v_3) = \{a, b\}$. Note that $ch(C_3) = 3$ and L is the unique maximal unavailable list assignment up to permutation of colors.

It is known that for all positive integers k and ℓ with $2 \leqslant k \leqslant \ell$, there exists a graph G such that $\chi(G) = k$ and $ch(G) = \ell$ [5]. In the remaining part of this section, we generalize this result as follows: for all positive integers k, ℓ with $2 \leqslant k \leqslant \ell$ and for any positive integers $i_1, i_2, \ldots, i_{k-2}$ with $k \leqslant i_{k-2} \leqslant \cdots \leqslant i_1 \leqslant \ell$, there exists a graph G such that $\chi(G) = k$, $ch(G) = \ell$ and $ch_t(G) = i_t$ for all $t = 1, \ldots, k-2$. For this purpose, we introduce two graph operations. The first graph operation is defined here, and the second is defined later. For every graph G with $V(G) = \{v_1, \ldots, v_n\}$, the *duplication* $D(G)$ of G is defined as follows:

$$
\begin{aligned}
V(D(G)) &= V(G) \cup \{v_{i,j} \mid i, j = 1, \ldots, n\} \quad \text{and} \\
E(D(G)) &= E(G) \cup \{\{v_{i,r}, v_{i,s}\} \mid i = 1, \ldots, n, \quad \{v_r, v_s\} \in E(G)\} \\
&\quad \cup \{\{v_i, v_{i,j}\} \mid i, j = 1, \ldots, n\}.
\end{aligned}
$$

Namely $D(G)$ is obtained by the following ways: With G, construct n more copies $G_1, G_2, \ldots, G_n$ of G, which correspond to vertices of G, and add edges between v_i and every vertex in the corresponding copy G_i for all $i = 1, \ldots, n$. For convenience, let G_i be the induced subgraph of $D(G)$ with vertex set $\{v_{i,j} \mid j = 1, \ldots, n\}$ for each $i \in \{1, \ldots, n\}$. Note that G_i is isomorphic to G.

Lemma 3. *Let G be a connected graph with $V(G) = \{v_1, \ldots, v_n\}$ and let $\chi(G) = k$. Now the following properties hold.*

(1) $\chi(D(G)) = \chi(G) + 1$.

(2) $ch(D(G)) = ch(G) + 1$.

(3) *For every nonnegative integer t with $1 \leqslant t \leqslant k$, $ch_t(D(G)) = ch_{t-1}(G) + 1$.*

Proof. (1) Let H be the induced subgraph of $D(G)$ with $V(H) = \{v_1\} \cup \{v_{1,j} \mid j = 1, \ldots, n\}$. Now H is isomorphic to a graph join of a trivial graph and G. So $\chi(H) = k+1$, which implies that $\chi(D(G)) \geqslant \chi(G) + 1$.

Let $\phi : V(G) \to \{c_1, \ldots, c_k\}$ be a proper k-coloring of G. For all $i = 1, \ldots, n$, let $C_i = \{c_1, \ldots, c_k, c_{k+1}\} - \{\phi(v_i)\}$. Now G_i has a proper k-coloring ϕ_i with the color set C_i. These proper k-colorings define a proper $(k+1)$-coloring of $D(G)$. Hence $\chi(D(G)) \leqslant k+1$ and so $\chi(D(G)) = \chi(G) + 1$.

(2) Let L_1 be a maximal unavailable list assignment of G. Choose a color c which does not belong to $L_1(v)$ for any $v \in V(G)$. Let L_2 be a list assignment of $D(G)$ defined by $L_2(v_j) = L_2(v_{i,j}) = L_1(v_j) \cup \{c\}$ for all $i, j = 1, \ldots, n$. Suppose that $D(G)$ has an L_2-coloring. Now, c should be assigned to at least one of $v_1, \ldots, v_n$, say v_i, because L_1 is a maximal unavailable list assignment of G, and hence G_i has a proper coloring ϕ such

that $\phi(v_{i,j}) \in L_1(v_j)$. Since G_i is isomorphic to G, this implies that G has an L_1-coloring. This is a contradiction. So $ch(D(G)) \geqslant ch(G) + 1$.

Let L be a list assignment of $D(G)$ such that $|L(u)| = ch(G)+1$ for every $u \in V(D(G))$. Consider the induced subgraph G of $D(G)$. Now, G has an L-coloring ϕ. For all $i, j = 1, \dots, n$, let $L_3(v_{i,j}) = L(v_{i,j}) - \{\phi(v_i)\}$. Since for all $i, j = 1, \dots, n$, $|L_3(v_{i,j})| \geqslant ch(G)$, G_i has an L_3-coloring, and hence $D(G)$ has an L-coloring. So $ch(D(G)) \leqslant ch(G) + 1$. Therefore $ch(D(G)) = ch(G) + 1$.

(3) Let L_4 be a $(t-1)$-common list assignment of G such that G has no L_4-coloring and $|L_4(v)| = ch_{t-1}(G) - 1$ for every $v \in V(G)$. Choose a color c which does not belong to $L_4(v)$ for any $v \in V(G)$. Let L_5 be a t-common list assignment of $D(G)$ defined by $L_5(v_j) = L_5(v_{i,j}) = L_4(v_j) \cup \{c\}$ for all $i, j = 1, \dots, n$. Suppose that $D(G)$ has an L_5-coloring. Then c should be assigned to some $v_i \in \{v_1, \dots, v_n\}$, and hence G_i has a proper coloring ψ such that $\psi(v_{i,j}) \in L_4(v_j)$. This implies that G has an L_4-coloring, a contradiction. So $ch_t(D(G)) \geqslant ch_{t-1}(G) + 1$.

Let L' be a t-common list assignment of $D(G)$ such that $|L'(u)| = ch_{t-1}(G) + 1$ for every $u \in V(D(G))$. Since a t-common list assignment of G is also a $(t-1)$-common list assignment, G has an L'-coloring ϕ'. For all $i, j = 1, \dots, n$, let $L_6(v_{i,j}) = L'(v_{i,j}) - \{\phi'(v_i)\}$. Now the restriction of L_6 onto G_i is a $(t-1)$-common list assignment such that $|L_6(v_{i,j})| \geqslant ch_{t-1}(G)$. This implies that for all $i = 1, \dots, n$, G_i has an L_6-coloring, and hence $D(G)$ has an L'-coloring. So $ch_t(D(G)) \leqslant ch_{t-1}(G) + 1$. Therefore $ch_t(D(G)) = ch_{t-1}(G) + 1$. □

For complete bipartite graph $K_{n,n}$, $\chi(K_{n,n}) = 2$ and $ch(K_{n,n})$ approaches infinity as n goes to the infinity. In particular, $ch(K_{n,n}) \geqslant k + 1$ for $n = \binom{2k-1}{k}$. One can easily show that $ch(K_{n+1,n+1}) = ch(K_{n,n})$ or $ch(K_{n,n}) + 1$. So for any integer k with $k \geqslant 2$, there exists a smallest positive integer n such that $ch(K_{n,n}) = k$. We denote such an integer by $\gamma(k)$.

Now, we introduce the second graph operation. Let G be a connected graph and let k be a positive integer. Let H be a complete bipartite graph with $\gamma(k)$ vertices on each part. For a vertex $v \in V(G)$, an *attachment* $A(G, v, k)$ is a graph defined by

$$\begin{aligned} V(A(G,v,k)) &= V(G) \cup V(H) \cup \{x\} \quad \text{and} \\ E(A(G,v,k)) &= E(G) \cup E(H) \cup \{\{v,x\},\{x,u\}\}, \end{aligned}$$

where u is a vertex in H. Namely $A(G, v, k)$ is obtained by connecting G and H with a path of length 2 whose ends are v and a vertex u in H. For convenience, we use U and V, where $u \in U$, to refer to the vertex sets in the bipartition of $V(H)$.

The following lemma gives the chromatic number, the list chromatic number, and the t-common list chromatic number of $A(G, v, k)$ for a connected graph G with $\chi(G) \geqslant 2$.

Lemma 4. *Let G be a connected graph with $\chi(G) \geqslant 2$. For every $v \in V(G)$ and for every positive integer k, the following properties hold.*

(1) $\chi(A(G,v,k)) = \chi(G)$.

(2) $ch(A(G,v,k)) = \max\{ch(G), k\}$.

(3) *For every nonnegative integer* t*,* $ch_t(A(G,v,k)) = ch_t(G)$.

Proof. (1) By the definition of $A(G,v,k)$, the chromatic number of $A(G,v,k)$ is the maximum of $\chi(G)$ and $\chi(K_{\gamma(k),\gamma(k)})$. Since the chromatic number of a complete bipartite graph is 2, $\chi(A(G,v,k)) = \chi(G)$.

(2) If k is 2, then $\gamma(2) = 1$, namely $A(G,v,k)$ is a graph obtained by attaching a path of length 3 to v. So $ch(A(G,v,k)) = ch(G)$, which is the maximum of $ch(G)$ and 2. Assume that $k \geqslant 3$. Since both G and $K_{\gamma(k),\gamma(k)}$ are subgraphs of $A(G,v,k)$, $ch(A(G,v,k)) \geqslant \max\{ch(G), k\}$. Let L be a list assignment of $A(G,v,k)$ such that $|L(w)| = \max\{ch(G), k\}$ for every $w \in V(A(G,v,k))$. Now both G and $K_{\gamma(k),\gamma(k)}$ as subgraphs of $A(G,v,k)$ have L-colorings ϕ_1 and ϕ_2, respectively. By assigning a color $c \in L(x) - \{\phi_1(v), \phi_2(u)\}$ to x, we have L-coloring of $A(G,v,k)$. So $ch(A(G,v,k)) \leqslant \max\{ch(G), k\}$. Therefore $ch(A(G,v,k)) = \max\{ch(G), k\}$.

(3) Since G is a subgraph of $A(G,v,k)$, $ch_t(A(G,v,k)) \geqslant ch_t(G)$. Let L_1 be a t-common list assignment of $A(G,v,k)$ such that $|L_1(w)| = ch_t(G)$ for every $w \in V(A(G,v,k))$. Let c be a color belonging to $L_1(w)$ for every vertex $w \in V(A(G,v,k))$. Now G has an L_1-coloring ϕ_2. If $\phi_2(v) = c$, then for every $u' \in U$, let $\phi_3(u') = c$ and for every $y \in V \cup \{x\}$, choose a color c' in $L_1(y) - \{c\}$ and let $\phi_3(y) = c'$. Now ϕ_2 and ϕ_3 give an L_1-coloring of $A(G,v,k)$. When $\phi_2(v) \neq c$, assign c to every vertex in $V \cup \{x\}$ and for every $u' \in U$, assign an arbitrary color c' in $L_1(u') - \{c\}$. Now ϕ_2 and this assignment give an L_1-coloring of $A(G,v,k)$. So $ch_t(A(G,v,k)) \leqslant ch_t(G)$, and hence $ch_t(A(G,v,k)) = ch_t(G)$. □

Finally, we have the following theorem.

Theorem 5. *For all positive integers* k, ℓ *with* $2 \leqslant k \leqslant \ell$ *and for any positive integers* $i_1, i_2, \ldots, i_{k-2}$ *with* $k \leqslant i_{k-2} \leqslant \cdots \leqslant i_1 \leqslant \ell$*, there exists a graph* G *such that* $\chi(G) = k$*,* $ch(G) = \ell$ *and* $ch_t(G) = i_t$ *for every* $t = 1, \ldots, k-2$*.*

Proof. Let k, ℓ be positive integers satisfying $2 \leqslant k \leqslant \ell$ and let $i_1, i_2, \ldots, i_{k-2}$ be positive integers such that $k \leqslant i_{k-2} \leqslant \cdots \leqslant i_1 \leqslant \ell$. Let $H_0 = K_{\gamma(i_{k-2}-k+2),\gamma(i_{k-2}-k+2)}$ and choose a vertex v in H_0. For every $j = 1, \ldots, k-3$, let $H_j = A(D(H_{j-1}), v, i_{k-j-2} - k + j + 2)$ and let $G = A(D(H_{k-3}), v, \ell)$. The rest is to prove that $\chi(G) = k$, $ch(G) = \ell$ and $ch_t(G) = i_t$ for every $t = 1, \ldots, k-2$.

By Lemmas 3 and 4,

$$\chi(G) = \chi(D(H_{k-3})) = \chi(H_{k-3}) + 1 = \chi(D(H_{k-4})) + 1 = \cdots = \chi(H_0) + k - 2 = k.$$

Note that $ch(H_0) = i_{k-2} - k + 2$ and

$$ch(H_1) = \max\{ch(D(H_0)), i_{k-3} - k + 3\} = \max\{ch(H_0) + 1, i_{k-3} - k + 3\} = i_{k-3} - k + 3.$$

It can also be shown that for every $j \leqslant t$ $(1 \leqslant t \leqslant k-4)$, $ch(H_j) = i_{k-j-2} - k + j + 2$. It follows that

$$\begin{aligned} ch(H_{t+1}) &= \max\{ch(D(H_t)), i_{k-t-3} - k + t + 3\} \\ &= \max\{ch(H_t) + 1, i_{k-t-3} - k + t + 3\} = i_{k-t-3} - k + t + 3. \end{aligned}$$

Therefore for every $j = 1, \ldots, k-3$, $ch(H_j) = i_{k-j-2} - k + j + 2$. Furthermore we have

$$ch(G) = \max\{ch(D(H_{k-3})), \ell\} = \max\{ch(H_{k-3}) + 1, \ell\} = \max\{i_1, \ell\} = \ell.$$

Now $ch_1(G) = ch_1(D(H_{k-3})) = ch(H_{k-3}) + 1 = i_1$ and for every $t = 2, \ldots, k-2$,

$$ch_t(G) = ch_{t-1}(H_{k-3}) + 1 = ch_{t-2}(H_{k-4}) + 2 = \cdots = ch(H_{k-t-2}) + t = i_t. \qquad \square$$

3 On t-common list colorings of planar graphs

By the famous four color theorem, every planar graph is known to be 4-colorable [1, 2]. Voigt [10] gave an example of a non-4-choosable planar graph and Thomassen [9] showed that every planar graph is 5-choosable. So for every planar graph G, $ch_2(G) \leqslant ch_1(G) \leqslant 5$. From this inequality, one can ask whether there is a planar graph with $ch_1(G) = 5$. We prove that 5 is the sharp upper bound for 1-common list chromatic number of planar graphs. To this end, we first introduce the following lemma.

Lemma 6. *Let G_1 be the graph drawn in Figure 1. Suppose that L is a list assignment with $L(x) = L(y) = L(u_1) = L(u_2) = \{1, 2, 3, 4\}$, $L(v_1) = L(w_1) = \{1, 3, 4, 5\}$, $L(v_2) = L(w_2) = \{2, 3, 4, 5\}$, and $L(v_3) = L(w_3) = \{1, 2, 4, 5\}$. Then G_1 has no L-coloring ϕ with $\phi(x) = 1$, $\phi(y) = 2$.*

Proof. Suppose that G_1 has such an L-coloring ϕ. By simple observation, we can check that $\{\phi(u_1), \phi(u_2)\} = \{3, 4\}$, and we may assume that $\phi(u_1) = 4$ by symmetry. This implies that the cycle with vertices v_1, v_2, and v_3 is 2-colorable, which is a contradiction. Therefore, G_1 is not L-colorable with x, y being colored $1, 2$, respectively. $\square$

Now, the following theorem provides a planar graph G such that $ch_1(G) = 5$ and $ch_2(G) = 4$.

Theorem 7. *Let G be the graph drawn in Figure 2, while dashed arrows are copies of G_1 as mentioned in Figure 1. The graph G is a planar graph satisfying $ch_1(G) = 5$ and $ch_2(G) = 4$.*

Proof. One can easily check that G is a planar graph. Note that $ch_1(G) \leqslant 5$ by Lemma 1. Let L be a list assignment of G with $L(x_i) = \{1, 2, 3, 4\}$ for all i, and as defined in Lemma 6 for the vertices on the copies of G_1. Note that the color 4 belongs to all lists.

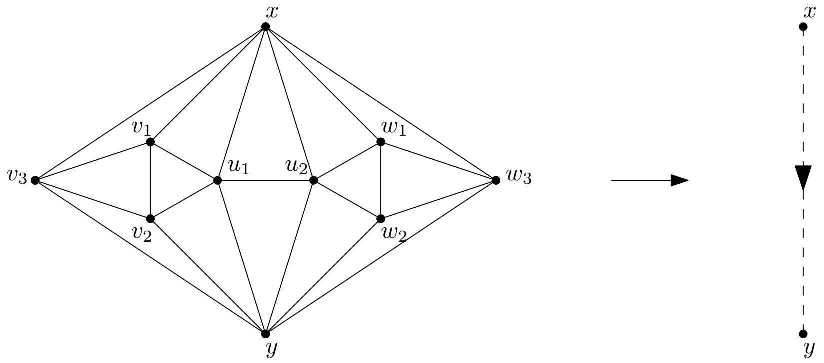

Figure 1: The graph G_1 is on the left. For two fixed vertices x, y of G_1, we simply draw a dashed arrow from x to y to represent the graph G_1, as on the right.

Suppose that G has an L-coloring ϕ. There exist x_i and x_j such that $\phi(x_i) = 1$ and $\phi(x_j) = 2$. By Lemma 6, the copy of G_1 corresponding to the dashed arrow from x_i to x_j has no L-coloring, which is a contradiction. Therefore G is not L-colorable and hence $ch_1(G) = 5$.

Now, we prove that for every 2-common list assignment L of G with $L(v) \geqslant 4$ for every vertex v, there exists an L-coloring of G. To this end, it is enough to find a bipartition of the vertex set of G into two sets U, V such that U induces a bipartite subgraph of G and V induces a 2-choosable subgraph.

For clarity, we use $G_{i,j}$ to refer to a copy of G_1 corresponding to the dashed arrow from x_i to x_j for all $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$. Moreover, for each i, j, and k, we call vertices of $G_{i,j}$ corresponding to u_k, v_k, and w_k by $u_k^{i,j}, v_k^{i,j}$, and $w_k^{i,j}$, respectively.

Let

$$\begin{aligned} S_0 &= \{x_1, x_2\}, \quad S_1 = \{v_1^{1,2}, v_2^{1,2}, w_1^{1,2}, w_2^{1,2}\} \cup \{v_1^{2,1}, v_2^{2,1}, w_1^{2,1}, w_2^{2,1}\} \\ S_2 &= \bigcup_{s \in \{1,2\}, t \in \{3,4\}} \left(\{u_1^{s,t}, v_2^{s,t}, v_3^{s,t}, w_1^{s,t}, w_2^{s,t}\} \cup \{u_1^{t,s}, v_1^{t,s}, v_3^{t,s}, w_1^{t,s}, w_2^{t,s}\}\right) \\ S_3 &= \{u_1^{3,4}, u_2^{3,4}, v_1^{3,4}, v_3^{3,4}, w_1^{3,4}, w_3^{3,4}\} \cup \{u_1^{4,3}, u_2^{4,3}, v_1^{4,3}, v_3^{4,3}, w_1^{4,3}, w_3^{4,3}\}. \end{aligned}$$

Let $S = S_0 \cup S_1 \cup S_2 \cup S_3$. Now the subgraph $G[S]$ induced by S is a bipartite graph, and $G \setminus S$ is a forest, which is always 2-choosable. Therefore $ch_2(G) = 4$. $\square$

It is unknown whether there exists a planar graph G satisfing $ch_2(G) = 5$. So we propose the following question.

Question 1. *Is there a planar graph G such that $ch_2(G) = 5$ or does it hold that $ch_2(G) \leqslant 4$ for every planar graph G?*

The well-known theorem of Grötzsch [6] states that every planar triangle-free graph is 3-colorable. This theorem was later slightly sharpened by Grünbaum [7] and Aksionov

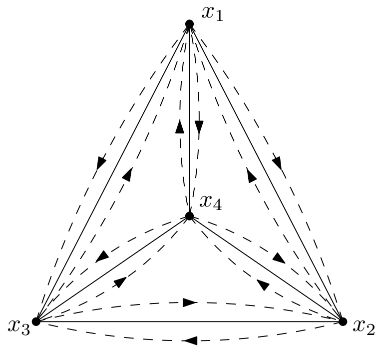

Figure 2: The graph G. Each dashed arrow represents a copy of G_1.

[3], who showed that every planar graph with at most 3 triangles is 3-colorable. The case of list coloring is different. Voigt [11], Gutner [8], Glebova et. al [4] gave examples of triangle-free planar graphs that are not 3-choosable. One can check that for each such example G, there exists an independent set S such that $G - S$ is a forest. This implies the 1-common list chromatic number $ch_1(G)$ is 3. Hence we propose the following question.

Question 2. *Is there a triangle-free planar graph* G *such that* $ch_1(G) = 4$ *or does it hold that* $ch_1(G) \leqslant 3$ *for every triangle-free planar graph* G*?*

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