Gröbner bases techniques for an S-packing k-coloring of a graph

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Abstract

In this paper, polynomial ideal theory is used to deal with the problem of the S-packing coloring of a finite undirected and unweighted graph by introducing a family of polynomials encoding the problem. A method to find the S-packing colorings of the graph is presented and illustrated by examples.

Keywords: Gröbner basis; Zero dimensional ideal; S-packing colorings; Shape Lemma

1 Introduction

Graph coloring has many applications like scheduling problems [17], register allocation [6], pattern matching [22] among others. In its simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color; this is called a vertex coloring. Similarly, an edge coloring assigns a color to each edge so that no two adjacent edges share the same color. This kind of coloring is called a proper coloring.

Problems like the frequency assignment problem, which is to assign a frequency to a given group of televisions or radio transmitters so that interfering transmitters are assigned a frequency with at least a minimum allowed separation, have inspired a variety of graphical coloring problems that generalize the notion of a proper coloring. One can see [13] or [21] for a survey of frequency assignment problems.

Goddard et al. [14] generalizes the ordinary colorings and some other coloring parameters such as the broadcast chromatic number to an *S-packing k-coloring*.

An *S-packing k-coloring* of an undirected and unweighted graph G is a mapping c from $V(G)$, the set of vertices, to $\{0, \ldots, k-1\}$ such that any two vertices with color

i are at mutual distance greater than a_i for $0 \leqslant i \leqslant k-1$, where $S = (a_0, a_1, \ldots)$ is a non-decreasing sequence of positive integers. That is,

$$\forall\, u \neq v \in V(G)^2,\ \forall i \in \{0, \ldots, k-1\},\ c(u) = c(v) = i \Longrightarrow \mathrm{dist}(u, v) > a_i$$

where $\mathrm{dist}(u, v)$ is the distance between the vertices u and v. So, a $(1, 1, \ldots)$-packing k-coloring is just the usual proper coloring. The S-packing chromatic number $\chi_S(G)$ of G is the smallest integer k such that G has an S-packing k-coloring. When the graph is finite, the S-packing chromatic number $\chi_S(G)$ never exceeds $n = |V(G)|$ since, with n different colors, one can always assign a different color to each vertex.

In this paper, we use the polynomial ideal theory to study the *S-packing k-coloring* problem of a finite graph G. We profit from the algorithmic aspect of this theory and especially the Gröbner basis tool to decide whether the graph G has an *S-packing k-coloring* for a given k and to determine the S-packing chromatic number $\chi_S(G)$ of G. This can be done by modeling the problem of the *S-packing k-coloring* by a radical and zero-dimensional polynomial ideal of a specific polynomial ring. The important geometric and algebraic properties of the modeling polynomial ideal of being zero-dimensional and radical allow us to find effectively an *S-packing k-coloring* of the graph G

In the particular case of a proper coloring, this problem has been studied, (see e.g. [1], [18] and [19]), through the following polynomials:

$$\begin{cases} x_j^k - 1 & \text{for} \quad 0 \leqslant j \leqslant n-1 \\ \frac{x_j^k - x_\ell^k}{x_j - x_\ell} & \text{for} \quad (v_j, v_\ell) \in E, \end{cases} \tag{1}$$

where $(v_j, v_\ell) \in E$ means that v_j and v_ℓ are adjacent vertices, and the ideal $\mathcal{J}_k$ generated by these polynomials is in the polynomial ring $\mathcal{R} = \mathbb{F}[x_0, \ldots, x_{n-1}]$ for some field $\mathbb{F}$. The graph G is of course finite and $V(G) = \{v_0, \ldots, v_{n-1}\}$ is its set of vertices. A proper k-coloring of the graph G is determined by a zero of the ideal $\mathcal{J}_k$ in some algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$. So, in order to determine the k-colorings of the graph G, one needs to work in the extension field $\overline{\mathbb{F}}$. The Gröbner basis tool can be used to decide whether the ideal $\mathcal{J}_k$ has zeros in $\overline{\mathbb{F}}$. Many other properties of the graph can be studied through these polynomials. One can survey [18], [19] and references therein.

The same idea is used here to study the problem of an *S-packing k-coloring* of the graph G through a different combinatorial encoding:

$$\begin{cases} -1 + \displaystyle\sum_{i=0}^{k-1} x_{i+kj} & \text{for} \quad 0 \leqslant j \leqslant n-1, \\ x_{i+kj} x_{q+kj} & \text{for} \quad 0 \leqslant i < q \leqslant k-1, 0 \leqslant j \leqslant n-1, \end{cases} \tag{2}$$

together with:

$$\left\{ \; x_{i+kj} x_{i+k\ell} \quad \text{for} \quad 0 \leqslant i \leqslant k-1, 0 \leqslant j < \ell \leqslant n-1 \text{ and } \mathrm{dist}(v_j, v_\ell) \leqslant a_i. \right. \tag{3}$$

The polynomials (2) and (3) in the polynomial ring $\mathcal{R} = \mathbb{F}[x_0, \ldots, x_{kn-1}]$ generate the modeling ideal $\mathcal{J}_k(S)$. It will shown, Theorem 13 page 8, that the existence of an *S-packing k-coloring* of the graph G is equivalent to the fact that the ideal $\mathcal{J}_k(S)$ has zeros in $\mathbb{F}$ and there will be no need to use an extension field; the zeros of the ideal $\mathcal{J}_k(S)$, when they exist, have components 0 or 1 and the correspondence between these zeros and the *S-packing k-colorings* of the graph G is one-to-one. The Gröbner basis method, with respect to a lexicographical ordering, and the Shape Lemma will be used to provide a method to find the *S-packing k-colorings* of the graph G. It is true that lexicographical orderings are not a good choice for computing Gröbner bases, but for zero-dimensional ideals like the ideal $\mathcal{J}_k(S)$ there are efficient algorithms [8] which transform a Gröbner basis of a zero-dimensional ideal with respect to any given ordering into a Gröbner basis with respect to any other ordering.

2 Preliminaries

The main purpose of this section is to provide a necessary description of the basic notions in polynomial ideal theory. Namely, the powerful tool of Gröbner basis which was invented by Bruno Buchberger in 1965. For more details, one can see [2], [3], [4], [5], [7] or [16].

Let $\mathbb{F}$ be a field and $\mathcal{R} = \mathbb{F}[x_1, \dots, x_n]$ be the ring of polynomials in $x_1, \dots, x_n$ over $\mathbb{F}$. A term order or monomial ordering on $\mathcal{R}$ is a total order $\prec$ on the set of all monomials $x^a = x_1^{a_1} \cdots x_n^{a_n}$ which has the following two properties:

- $x^a \prec x^b$ implies $x^{a+c} \prec x^{b+c}$ for all $a, b, c \in \mathbf{N}^n$, that is $\prec$ is multiplicative.
- $1 \prec x^a$ for all $a \in \mathbf{N}^n \setminus \{0\}$, that is the constant monomial is the smallest.

As an example of monomial orderings, we recall the one that will be used in our examples; the lexicographical ordering $>$. If $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$ are in $\mathbf{N}^n$ then, $x^a > x^b$ if and only if there is $1 \leqslant i \leqslant n$ such that $a_i > b_i$ and $b_j = a_j$ for $1 \leqslant j \leqslant i-1$. For a given monomial ordering, every non-zero polynomial f has a unique leading monomial $\mathrm{LM}(f) = x^a$ that is the largest monomial x^a which occurs with a nonzero coefficient in the expansion of f. If $\mathcal{I}$ is an ideal of R, its monomial ideal $\mathrm{LM}(\mathcal{I})$ is the ideal generated by the leading monomials of all the polynomials in $\mathcal{I}$; $\mathrm{LM}(\mathcal{I}) = \langle \mathrm{LM}(f); f \in \mathcal{I} \rangle$. A finite set F of generators of an ideal $\mathcal{I}$ is said to be a Gröbner basis of $\mathcal{I}$ with respect to some monomial ordering if the leading monomials of the elements in F generate the monomial ideal $\mathrm{LM}(\mathcal{I})$; $\mathrm{LM}(\mathcal{I}) = \langle \mathrm{LM}(F) \rangle$. If, in addition, the coefficient of $\mathrm{LM}(f)$ is equal to 1 and no monomial in f lies in $\langle \mathrm{LM}(F \setminus \{f\}) \rangle$ for every $f \in F$, then F is called a reduced Gröbner basis of the ideal $\mathcal{I}$.

Proposition 1 ([7])**.** *Let $\mathcal{I}$ be a non-zero polynomial ideal. Then, for a given monomial ordering, $\mathcal{I}$ has a unique reduced Gröbner basis.*

Among the numerous applications of Gröbner basis, one can cite the ideal membership problem; that is whether a given polynomial g belongs to some ideal $\mathcal{I}$. When $g = 1$, this problem is related to the celebrated weak Nullstellensatz theorem, which identifies exactly which ideals correspond to the empty variety. Recall that the variety of an ideal $\mathcal{I}$ of $\mathcal{R}$ in some extension field $\mathbb{L}$ of $\mathbb{F}$ is the subset $V_{\mathbb{L}}(\mathcal{I})$ of $\mathbb{L}^n$ defined by: $V_{\mathbb{L}}(\mathcal{I}) = \{\xi = (\xi_1, \dots, \xi_n) \in \mathbb{L}^n\,;\ f(\xi) = 0 \text{ for all } f \in \mathcal{I}\}$.

Theorem 2 ([7])**.** *If $\mathbb{F}$ is algebraically closed and $\mathcal{I}$ is an ideal of $\mathcal{R}$, then the variety $V_{\mathbb{F}}(\mathcal{I})$ is empty if and only if the ideal $\mathcal{I}$ is unit, that is $\mathcal{I} = \mathcal{R}$.*

One way to test whether a given ideal is unit is the use of the Gröbner basis tool. In fact, in practice an ideal $\mathcal{I}$ is always given by some finite basis F. One can compute the Gröbner basis G of $\mathcal{I}$ from F with respect to some monomial ordering, using Buchberger's algorithm or new algorithms like Faugère's algorithm [9] or [10]. Then, the ideal $\mathcal{I}$ is unit if and only if $G \cap \mathbb{F} \neq \varnothing$.

When dealing with radical ideals (multiplicity-free) and zero-dimensional ideals (finite variety), many arguments in algebraic geometry and commutative algebra are simplified. The ideals in this paper are not an exception and we are interested in finding their zeros. In this regard, we recall the following definition:

Definition 3 ([2],[16])**.** An ideal $\mathcal{I}$ is said to be in normal position with respect to some variable x_i or in normal x_i-position if the x_i-components of the zeros of $\mathcal{I}$ in $\overline{\mathbb{F}}^n$ are pairwise different.

The last definition can equivalently be stated as: The ideal $\mathcal{I}$ is in normal x_i-position if the map $\pi_i : (\xi_1, \dots, \xi_n) \mapsto \xi_i$ from $V_{\overline{\mathbb{F}}}(\mathcal{I})$ into $\overline{F}$ is injective. It may happen that an ideal is not in normal position with respect to any variable like the ideal $\mathcal{J}$ generated by the polynomials $x_1^2 - x_1$ and $x_2^2 - x_2$ in $\mathbb{F}[x_1, x_2]$ since $V_{\overline{\mathbb{F}}}(\mathcal{J}) = \{(0,0), (0,1), (1,0), (1,1)\}$ and the maps π_1 and π_2 are not injective. In this situation, a suitable linear change of coordinates transforms the ideal into an ideal in normal position with respect to some new variable.

Proposition 4 ([2],[16])**.** *Suppose that $\mathcal{I}$ is a zero-dimensional ideal of $\mathcal{R}$, let $d = \dim(\mathcal{R}/\mathcal{I})$ and assume that $\mathbb{F}$ contains more than $\binom{d}{2}$ elements. Then there exists a tuple $(\alpha_1, \dots, \alpha_n) \in \mathbb{F}^n$ such that the linear change of coordinates $x_i \mapsto x_i$ for $i \leqslant n-1$ and $x_n \mapsto \alpha_1 x_1 + \cdots + \alpha_n x_n$ such that for any two different zeros $\xi = (\xi_1, \dots, \xi_n)$ and $\nu = (\nu_1, \dots, \nu_n)$ of $\mathcal{I}$, one has*

$$\alpha_1 \xi_1 + \cdots + \alpha_n \xi_n \neq \alpha_1 \nu_1 + \cdots + \alpha_n \nu_n. \tag{4}$$

Proof. See [16] Proposition 3.7.22 page 255. □

Together with Gröbner basis tool, the following theorem, which is known as the Shape Lemma, can help to find the zeros of a zero-dimensional radical ideal. The field $\mathbb{F}$ is supposed to be perfect; every irreducible polynomial over $\mathbb{F}$ has distinct roots. Note that fields of characteristic zero or finite fields are perfect.

Theorem 5 ([2],[12],[16])**.** *Suppose that $\mathbb{F}$ is a perfect field, let $\mathcal{I}$ be a zero-dimensional radical ideal of $\mathcal{R}$ in normal x_n-position, let g_n be the monic generator of $\mathcal{I} \cap \mathbb{F}[x_n]$ and let d be its degree. Then, the reduced Gröbner basis according to the lexicographical ordering $x_1 > \cdots > x_n$ has always the following form:*

$$\{x_1 - g_1(x_n), \dots, x_{n-1} - g_{n-1}(x_n), g_n(x_n)\} \tag{5}$$

where $g_i \in \mathbb{F}[x_n]$, $\deg(g_i) < d$ *for* $i < n$. *The polynomial* g_n *has distinct zeros* $z_1, \ldots, z_d$ *in the algebraic closure* $\overline{\mathbb{F}}$ *of* $\mathbb{F}$ *and*

$$V_{\overline{\mathbb{F}}}(\mathcal{I}) = \{(g_1(z_i), \ldots, g_{n-1}(z_i), z_i) \mid i = 1, \ldots, d\}.$$

Proof. See [16] Theorem 3.7.25 page 257. □

We have already seen above that the ideal $\mathcal{J} = \langle x_1^2 - x_1, x_2^2 - x_2 \rangle$ in $\mathbb{F}[x_1, x_2]$ is not normal position with respect to any variable. For scalars α_1 and α_2 in $\mathbb{F}$, the augmented ideal $\mathcal{I} = \langle x_1^2 - x_1, x_2^2 - x_2, x_3 - \alpha_1 x_1 - \alpha_2 x_2 \rangle$ in $\mathbb{F}[x_1, x_2, x_3]$ is in normal x_3-position if and only if $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\alpha_1 \neq \alpha_2$ and $\alpha_1 \neq -\alpha_2$ since

$$V_{\overline{\mathbb{F}}}(\mathcal{I}) = \{(0,0,0), (0,1,\alpha_2), (1,0,\alpha_1), (1,1,\alpha_1+\alpha_2)\}.$$

According to the lexicographical ordering $x_1 > x_2 > x_3$, the reduced Gröbner basis of $\mathcal{I}$ is formed by the following polynomials

$$\begin{aligned}
B_1 &= x_3^4 - 2(\alpha_1+\alpha_2)x_3^3 + (3\alpha_1\alpha_2 + \alpha_1^2 + \alpha_2^2)\, x_3^2 - \alpha_1\alpha_2(\alpha_1+\alpha_2)x_3 \\
&= x_3\,(x_3 - \alpha_1)\,(x_3 - \alpha_2)\,(x_3 - \alpha_1 - \alpha_2)\,, \\
B_2 &= x_1 - g_1\,(x_3)\,, \\
B_3 &= x_2 - g_2\,(x_3)\,,
\end{aligned}$$

where

$$\begin{aligned}
g_1(x) &= \frac{2}{\alpha_1(\alpha_2^2 - \alpha_1^2)}x^3 - \frac{3}{\alpha_1(\alpha_2 - \alpha_1)}x^2 + \frac{\alpha_2(\alpha_2 + 3\alpha_1)}{\alpha_1(\alpha_2^2 - \alpha_1^2)}x, \\
g_2(x) &= \frac{2}{\alpha_2(\alpha_1^2 - \alpha_2^2)}x^3 - \frac{3}{\alpha_2(\alpha_1 - \alpha_2)}x^2 + \frac{\alpha_1(\alpha_1 + 3\alpha_2)}{\alpha_2(\alpha_1^2 - \alpha_2^2)}x.
\end{aligned}$$

Recall that an ideal of $\mathcal{R}$ is radical if it contains every element of $\mathcal{R}$ whenever it contains some positive power of this element. A sufficient condition for a zero-dimensional ideal of $\mathcal{R}$ to be radical is given by Seidenberg's Lemma:

Lemma 6 ([2])**.** *Let* $\mathcal{I}$ *be a zero-dimensional ideal of* $\mathcal{R}$ *and assume that for* $1 \leqslant i \leqslant n$, $\mathcal{I}$ *contains a polynomial* $f_i \in \mathbb{F}[x_i]$ *with* $\gcd(f_i, f_i') = 1$. *Then* $\mathcal{I}$ *is an intersection of finitely many maximal ideals. In particular,* $\mathcal{I}$ *is radical.*

Proof. See [2] Lemma 8.13 page 341. □

3 Main results

Let $S = (a_0, a_1, \ldots)$ be a non-decreasing sequence of positive integers. Let G be a finite undirected and unweighted graph with $V(G) = \{v_0, \ldots, v_{n-1}\}$ the set of its vertices and let k be a positive integer. Let also $\mathcal{J}_k(S)$ be the ideal of the polynomial ring $\mathbb{F}[x_0, \ldots, x_{kn-1}]$ generated by the polynomials:

$$\begin{cases}
-1 + \displaystyle\sum_{i=0}^{k-1} x_{i+kj} & \text{for} \quad 0 \leqslant j \leqslant n-1, \\
x_{i+kj}x_{q+kj} & \text{for} \quad 0 \leqslant i < q \leqslant k-1, 0 \leqslant j \leqslant n-1, \\
x_{i+kj}x_{i+k\ell} & \text{for} \quad 0 \leqslant i \leqslant k-1, 0 \leqslant j < \ell \leqslant n-1 \text{ and } \operatorname{dist}(v_j, v_\ell) \leqslant a_i,
\end{cases} \tag{6}$$

where $\mathbb{F}$ is an arbitrary field and $\mathrm{dist}(v_j, v_\ell)$ is the distance between the vertices v_j and v_ℓ which is the length of the smallest path joining the two vertices.

Before establishing the link between the ideal $\mathcal{J}_k(S)$ and the S-packing k-coloring of the graph, we give some useful algebraic properties of this ideal.

Proposition 7. *The ideal $\mathcal{J}_k(S)$ is either the polynomial ring $\mathcal{R}$ or a zero-dimensional ideal of $\mathcal{R}$.*

Proof. Let t be any index from 0 to $nk-1$. There are unique i and j such that $t = i + kj$ and $0 \leqslant i \leqslant k-1$ and $0 \leqslant j \leqslant n-1$ that are respectively the remainder and the quotient of the integer division of t by k. Since the ideal $\mathcal{J}_k(S)$ contains $x_{i+kj}x_{q+kj}$ for $q \neq i$ and $-1 + \sum_{q=0}^{k-1} x_{q+kj}$, it also contains the nonzero polynomial

$$x_t^2 - x_t \;=\; x_t\left(-1 + \sum_{q=0}^{k-1} x_{q+kj}\right) - \sum_{q\neq i} x_t x_{q+kj}.$$

This shows that $\mathcal{J}_k(S) \cap \mathbb{F}[x_t] \neq \{0\}$ and then the ideal $\mathcal{J}_k(S)$ is either the polynomial ring $\mathcal{R}$ or a zero-dimensional ideal of $\mathcal{R}$. Additionally this shows that if $(\xi_0, \xi_1, \ldots, \xi_{nk-1})$ is a zero of the ideal $\mathcal{J}_k(S)$, then $\xi_i \in \{0, 1\}$. □

Proposition 8. *The ideal $\mathcal{J}_k(S)$ is radical.*

Proof. If the ideal $\mathcal{J}_k(S)$ is not unit, it is zero-dimensional. From the proof of Proposition 7, we know that the ideal $\mathcal{J}_k(S)$ contains polynomial $x_t^2 - x_t$ for each t. From another hand, the greatest common divisor of this polynomial and its derivative $2x_t - 1$ is one. According to Seidenberg's Lemma 6, we conclude that the ideal $\mathcal{J}_k(S)$ is radical. □

Remark 9. In the examples, we will be interested in finding S-packing k-colorings of some graphs. In this regard, Theorem 5 and Proposition 4 will be used to achieve this goal. In fact, when choosing a suitable change of coordinates as in Proposition 4, one can introduce a new variable x_{kn} and add the new polynomial $x_{kn} - \alpha_0 x_0 - \cdots - \alpha_{kn-1}x_{kn-1}$ to the generators of $\mathcal{J}_k(S)$ to get an extended ideal, say $\mathcal{J}'_k(S)$, of the polynomial ring $\mathbb{F}[x_0, \ldots, x_{kn}]$. This new ideal is zero-dimensional, radical and in normal x_{kn}-position. By Theorem 5 and Proposition 4, the reduced Gröbner basis of $\mathcal{J}'_k(S)$, according to the lexicographical ordering

$$x_0 > \cdots > x_{kn-1} > x_{kn}, \tag{7}$$

has always the following form:

$$\{x_0 - g_0(x_{kn}), \ldots, x_{kn-1} - g_{kn-1}(x_{kn}), g_{kn}(x_{kn})\}\,. \tag{8}$$

The zeros of the ideal $\mathcal{J}_k(S)$ will then be recovered from the distinct roots of the univariate polynomial g_{kn}. The challenging problem is to find a suitable change of coordinates. In [12], methods to find such change of coordinates are proposed. These methods are based on factorization of polynomials or on random choices of $\alpha_0, \ldots, \alpha_{kn-1}$.

In the following result, we propose a suitable change of coordinates:

Theorem 10. *Suppose that the ideal $\mathcal{J}_k(S)$ is not unit and that the field $\mathbb{F}$ is of characteristic zero or $p \geqslant k^n$. Then the ideal $\mathcal{J}'_k(S)$ defined in Remark 9 is in normal position with respect to the variable x_{kn} if we take $\alpha_t = r(t)k^{q(t)}$, where $q(t)$ and $r(t)$ are respectively the quotient and the remainder of the division of t by k for all $t = 0, \dots, kn-1$.*

Proof. Let $\xi = (\xi_0, \dots, \xi_{kn})$ be a zero of $\mathcal{J}'_k(S)$. Then

$$\xi_{kn} = \sum_{t=0}^{kn-1} r(t)k^{q(t)}\xi_t = \sum_{v=0}^{n-1}\sum_{u=0}^{k-1} uk^v \xi_{u+kv}$$

since ξ is a zero of $x_{kn} - \alpha_0 x_0 - \cdots - \alpha_{kn-1}x_{kn-1}$. Moreover, for any v in $\{0, \dots, n-1\}$, there is a unique u in $\{0, \dots, k-1\}$ such that $\xi_{u+kv} = 1$ and $\xi_{u'+kv} = 0$ for $u' \neq u$ since $(\xi_0, \dots, \xi_{kn-1})$ is a zero of $\mathcal{J}_k(S)$. Then we can look at ξ_{kn} as an element of $\mathbb{F}$ or as a natural integer. When we look at it as a natural integer, a bound of it is

$$\begin{aligned} \xi_{kn} &= \sum_{v=0}^{n-1}\sum_{u'=0}^{k-1} u'k^v \xi_{u'+kv} \\ &\leqslant \sum_{v=0}^{n-1} (k-1)k^v = k^n - 1. \end{aligned}$$

So, with this bound and under the assumption that the field $\mathbb{F}$ is of characteristic zero or $p \geqslant k^n$, we can treat ξ_{kn} just as natural integer. Let now $\nu = (\nu_0, \dots, \nu_{kn})$ be a zero of $\mathcal{J}'_k(S)$ different from ξ. In order to show that $\xi_{kn} \neq \nu_{kn}$ and thus the ideal $\mathcal{J}'_k(S)$ is in normal x_{kn}-position, we introduce the following integer:

$$j = \max\{q(t) \text{ such that } \xi_t \neq \nu_t \text{ and } 0 \leqslant t \leqslant kn-1\}.$$

There are unique integers i and s such that $\xi_{i+kj} = 1$ and $\nu_{s+kj} = 1$. From the definition of j, one has $i \neq s$. Now

$$\begin{aligned} \nu_{kn} - \xi_{kn} &= \sum_{v=0}^{n-1}\sum_{u=0}^{k-1} uk^v(\nu_{u+kv} - \xi_{u+kv}) \\ &= \sum_{v=0}^{j}\sum_{u=0}^{k-1} uk^v(\nu_{u+kv} - \xi_{u+kv}) \\ &= \sum_{u=0}^{k-1} uk^j(\nu_{u+kj} - \xi_{u+kj}) + \sum_{v=0}^{j-1}\sum_{u=0}^{k-1} uk^v(\nu_{u+kv} - \xi_{u+kv}) \\ &= k^j(s-i) + \sum_{v=0}^{j-1} A_v, \end{aligned}$$

where $A_v = \sum_{u=0}^{k-1} uk^v(\nu_{u+kv} - \xi_{u+kv})$. For fixed v in $\{0, \dots, j-1\}$, there are unique y and z in $\{0, \dots, k-1\}$ such that $\xi_{y+kv} = 1$ and $\nu_{z+kv} = 1$. Since $A_v = k^v(z-y)$ and y, z

are in $\{0,\ldots,k-1\}$, we shall have $A_v \in \{k^v(1-k),\ldots,k^v(k-1)\}$ and then

$$-k^j+1=\sum_{v=0}^{j-1}k^v(1-k)\leqslant\sum_{v=0}^{j-1}A_v\leqslant\sum_{v=0}^{j-1}k^v(k-1)=k^j-1.$$

However, since $s\neq i$, we get $(s-i)k^j\in\{(1-k)k^j,\ldots,-k^j\}\cup\{k^j,\ldots,(k-1)k^j\}$. This shows that $\nu_{kn}\neq\xi_{kn}$. □

In order to illustrate Theorem 10, let us consider following simple example.

Example 11. Let G be the triangle $(v_0v_1v_2)$ which can be seen as the complete graph G of three vertices v_0, v_1 and v_2. Recall that it is an undirected and unweighted graph in which every pair of distinct vertices is connected by a unique edge. We know that $n=3$ and since the graph G is complete, we shall have $k=3$. Let also S be the S-packing $(1,2,3,\ldots)$. In the polynomial ring $\mathbb{F}[x_0,\ldots,x_9]$, where $\mathbb{F}=\mathbb{Z}_{29}$, the extended ideal $\mathcal{J}_3'(S)$ is generated by the generators of $\mathcal{J}_3(S)$ and the polynomial

$$\begin{aligned} g &= x_9-\alpha_0x_0-\cdots-\alpha_8x_8 \\ &= x_9-x_1-2x_2-3x_4-6x_5-9x_7+11x_8 \end{aligned}$$

described in Theorem 10. According to the lexicographical ordering $x_0>\cdots>x_9$, the reduced Gröbner basis of $\mathcal{J}_3'(S)$ is formed by the following polynomials:

$$\begin{array}{ll} x_0-13x_9^5+9x_9^4-x_9^3-13x_9^2+6x_9+11, & x_1-10x_9^4-2x_9^3-6x_9^2-9x_9-4, \\ x_2+13x_9^5+x_9^4+3x_9^3-10x_9^2+3x_9-8, & x_3-2x_9^5-8x_9^4+6x_9^3+5x_9^2+4x_9+2, \\ x_4-14x_9^4+3x_9^3-6x_9^2+12x_9-4, & x_5+2x_9^5-7x_9^4-9x_9^3+x_9^2+13x_9+1, \\ x_6-14x_9^5-x_9^4-5x_9^3+8x_9^2-10x_9-14, & x_7-5x_9^4-x_9^3+12x_9^2-3x_9+7, \\ x_8+14x_9^5+6x_9^4+6x_9^3+9x_9^2+13x_9+6, & x_9^6+9x_9^5-5x_9^4+9x_9^3-7x_9^2+2x_9+1. \end{array}$$

The choice of $\mathbb{F}=\mathbb{Z}_{29}$ is motived by the constraint $p=29\geqslant k^n=27$ in Theorem 10 which ensures that the ideal $\mathcal{J}_3'(S)$ is in normal x_9-position. However, one can verify that this ideal is still in normal x_9-position for any choice of $\mathbb{F}=\mathbb{Z}_p$ for all prime numbers $p\geqslant 11$. The constraint $p\geqslant k^n$ will also be violated especially in Example 16 and in Example 17 once the ideal $\mathcal{J}_k'(S)$ is in normal x_{kn}-position.

Remark 12. Since the zeros of the ideal $\mathcal{J}_k(S)$ have components zero or one, they all lie in $\mathbb{F}^{kn}$. So, there is no need to use the algebraic closure of $\mathbb{F}$.

The following result establishes the connection between the zeros of the ideal $\mathcal{J}_k(S)$ and the S-packing k-colorings of the graph G.

Theorem 13. *The ideal $\mathcal{J}_k(S)$ is zero-dimensional if and only if there is an S-*packing *k-*coloring *of the graph G. Moreover, the correspondence between the zeros of the ideal $\mathcal{J}_k(S)$ and the S-packing k-colorings of the graph G is one-to-one.*

Proof. **The only if condition:** Suppose that c is an *S-packing k-coloring* of the graph G. To the mapping c, corresponds $\xi = (\xi_0, \ldots, \xi_{nk-1}) \in \mathbb{F}^{kn}$ defined for all $t \in \{0, \ldots, nk-1\}$ by:

$$\xi_t = \begin{cases} 1 & \text{if } c(v_j) = i, \\ 0 & \text{otherwise,} \end{cases}$$

such that $t = i + kj$, $0 \leqslant i \leqslant k-1$ and $0 \leqslant j \leqslant n-1$. Let p be a polynomial from (6). If $p = -1 + \sum_{q=0}^{k-1} x_{q+k\ell}$ for some $\ell \in \{0, \ldots, n-1\}$, we denote $s = c(v_\ell)$. Then

$$\begin{aligned} p(\xi) &= -1 + \sum_{q=0}^{k-1} \xi_{q+k\ell} \\ &= -1 + \xi_{s+k\ell} \\ &= 0. \end{aligned}$$

Suppose that $p = x_{i+kj} x_{q+kj}$ for some $0 \leqslant i < q \leqslant k-1$ and $0 \leqslant j \leqslant n-1$. It is clear that $p(\xi) = 0$ since the vertex v_j cannot be colored using two different colors i and q. Now suppose that $p = x_{i+kj} x_{i+k\ell}$ for some $0 \leqslant i \leqslant k-1$, $0 \leqslant j < \ell \leqslant n-1$ such that $\operatorname{dist}(v_j, v_\ell) \leqslant a_i$. According to the definition of the *S-packing k-coloring* of G and since $\operatorname{dist}(v_j, v_\ell) \leqslant a_i$, one should have $c(v_j) \neq i$ or $c(v_\ell) \neq i$, that is $\xi_{i+kj} = 0$ or $\xi_{i+k\ell} = 0$ and then $p(\xi) = 0$. The corresponding variety is then non empty and the ideal $\mathcal{J}_k(S)$ is zero dimensional.

The if condition: Suppose the ideal $\mathcal{J}_k(S)$ is zero dimensional and choose a zero $\xi = (\xi_0, \ldots, \xi_{nk-1}) \in \mathbb{F}^{kn}$ of the ideal $\mathcal{J}_k(S)$. We define the mapping c as follows: for j such that $0 \leqslant j \leqslant n-1$,

$$c(v_j) = \sum_{i=0}^{k-1} i \xi_{i+kj}$$

that is $c(v_j)$ is the unique s such that $\xi_{s+kj} = 1$. If now v_j and v_ℓ are two distinct vertices with the same color i, then $\xi_{i+kj} \xi_{i+k\ell} = 1$. Since ξ is a zero of the ideal $\mathcal{J}_k(S)$, $x_{i+kj} x_{i+k\ell} \notin \mathcal{J}_k(S)$ and we shall get $\operatorname{dist}(v_j, v_\ell) > a_i$. This shows that c is an *S-packing k-coloring* of the graph G. □

The Gröbner basis tool, with respect to any monomial ordering, can now be used to test whether the ideal $\mathcal{J}_k(S)$ is unit or again if the graph G has an *S-packing k-coloring*. The following algorithm computes the S-packing chromatic number of the graph G.

Algorithm: S-chromatic:

INPUT: A finite undirected and unweighted graph and an S-packing $(a_0, a_1, \ldots)$.

OUTPUT: The S-chromatic number of G.

1. Set $k = 1$,
2. Form the generators of $\mathcal{J}_k(S)$ from (6).

3. Compute a Gröbner basis B_k of the ideal $\mathcal{J}_k(S)$ with respect to some monomial ordering.
4. If $B_k \cap \mathbb{F} \neq \varnothing$, then $k = k+1$ and go to 2.
5. Return k.

The correctness of the algorithm S-chromatic is a consequence of Theorem 13 and its termination comes from the fact that the S-packing chromatic number of the graph G never exceeds n.

Example 14. Let G be the classical Petersen graph, Figure 1, and the S-packing $S = (1, 2, 2, 2, \ldots)$. We choose the field $\mathbb{F}$ to be $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ and the lexicographical ordering

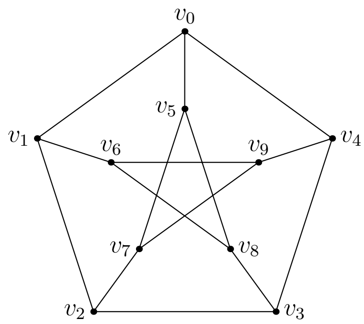

Figure 1: Petersen graph.

$x_0 > \cdots > x_{10k-1}$. For the values of $k = 1, 2, 3$ or 4, the corresponding reduced Gröbner basis B_k is unit; $1 \in B_k$. The reduced Gröbner basis for $k = 5$ is not unit and contains 602 polynomials. So, the S-chromatic number of the Petersen Graph is $k = 5$.

In order to have an idea about the complexity of the algorithm S-chromatic, let us introduce the ideal $\mathcal{I}_k(S)$ generated by the polynomials:

$$\begin{cases} x_t^2 - x_t & \text{for} \quad 0 \leqslant t \leqslant kn-1, \\ x_{i+kj}x_{q+kj} & \text{for} \quad 0 \leqslant i < q \leqslant k-1, 0 \leqslant j \leqslant n-1, \\ x_{i+kj}x_{i+k\ell} & \text{for} \quad 0 \leqslant i \leqslant k-1, 0 \leqslant j < \ell \leqslant n-1 \text{ and } \operatorname{dist}(v_j, v_\ell) \leqslant a_i. \end{cases} \tag{9}$$

Buchberger's criterion (see [7] for example) states that a basis G of an ideal $\mathcal{I}$ is a Gröbner basis of $\mathcal{I}$ if and only if for all pairs $f \neq g$ from G, the remainder on division of the S-polynomial $S(f, g)$ by G is zero. So, using this criterion, one can easily see that the polynomials in (9) form the reduced Gröbner basis G_0 of the ideal $\mathcal{I}_k(S)$ with respect to any monomial ordering.

The ideal $\mathcal{J}_k(S)$ is then obtained from the ideal $\mathcal{I}_k(S)$ by adding the polynomials:

$$P_j = -1 + \sum_{i=0}^{k-1} x_{i+kj} \quad \text{for} \quad 0 \leqslant j \leqslant n-1. \tag{10}$$

That is $\mathcal{J}_k(S) = \mathcal{I}_k(S) + \langle P_0, \ldots, P_{n-1}\rangle$. Let us now denote $\mathcal{I}_{k,j}(S)$ the ideal $\mathcal{I}_k(S) + \langle P_0, \ldots, P_j\rangle$. The step 3 in the algorithm S-chromatic can be performed by the following procedure:

Algorithm: S-Gröbner:

Input: A positive integer k, a finite undirected and unweighted graph and an S-packing $(a_0, a_1, \ldots)$.

Output: a Gröbner basis B_k of the ideal $\mathcal{J}_k(S)$.

1. Set $B_k = G_0$ (the set of polynomials described in (9))
2. For j from 0 to $n-1$ do:
   (a) compute a Gröbner basis B_k of the ideal $\mathcal{I}_{k,j}(S)$ by incrementing the already computed Gröbner basis B_k by P_j.
   (b) If $B_k = \{1\}$, then go to 3.
3. Return B_k.

Correctness and termination of this algorithm are obvious, we only insist on step 2a. For this, we refer to the following result:

Theorem 15. *[11] Let $\mathbb{F}$ be any field and suppose that $\mathcal{I}$ is a zero-dimensional ideal in $R = \mathbb{F}[x_1, \ldots, x_n]$. Let N be the number of common solutions of $\mathcal{I}$ over the algebraic closure of $\mathbb{F}$, counting multiplicities. Then, given a Gröbner basis for $\mathcal{I}$ (under any monomial ordering) and a polynomial $g \in R$, Gröbner bases for $\langle \mathcal{I}, g\rangle$ and $(\mathcal{I} : g)$ can be computed deterministically using $O\left((nN)^3\right)$ operations in $\mathbb{F}$.*

In [11], a method to compute effectively Gröbner bases for the ideals $\langle \mathcal{I}, g\rangle$ and $(\mathcal{I} : g)$ is described. This method can be applied in our situation since the ideal $\mathcal{I}_k(S)$ is zero-dimensional and the ideals $\mathcal{I}_{k,j}(S)$ are either zero-dimensional or unit. So according to Theorem 15, a Gröbner basis of the ideal $\mathcal{J}_k(S)$ can be computed deterministically using

$$O\left((nkN)^3\right) + O\left((nkN_0)^3\right) + \cdots + O\left((nkN_{n-2})^3\right)$$

operations in $\mathbb{F}$, where N is the number of common solutions of $\mathcal{I}_k(S)$ over the algebraic closure of $\mathbb{F}$ and N_j is the number of common solutions of $\mathcal{I}_{k,j}(S)$ over the algebraic closure of $\mathbb{F}$. These common solutions lie in $\mathbb{F}$ and there is no need to use its algebraic closure. Moreover, we have $N \geqslant N_0 \geqslant \cdots \geqslant N_{n-2}$ since $\mathcal{I}_k(S) \subset \mathcal{I}_{k,0}(S) \subset \cdots \subset \mathcal{J}_k(S)$. Then, a Gröbner basis of the ideal $\mathcal{J}_k(S)$ can be computed deterministically using $O\left((n^2kN)^3\right)$ operations in $\mathbb{F}$.

The method described in Remark 9 can also be used to test whether the ideal $\mathcal{J}_k(S)$ is unit and also to find the zeros of $\mathcal{J}_k(S)$. From these zeros, we know how to recover the corresponding S-packing k-colorings of the graph G. This is explained in the following algorithm S-color that returns the S-packing k-colorings of a given finite undirected and unweighted graph. The field $\mathbb{F}$ is supposed to be of characteristic zero or of characteristic $p \geqslant k^n$. The same argument as in the S-chromatic algorithm is used to ensure its termination. Theorem 5 and 10 are the arguments for its correctness.

Algorithm: S-color:

INPUT: A finite undirected and unweighted graph and an S-packing $(a_0, a_1, \dots)$.

OUTPUT: The list of the S-packing k-colorings of the graph G where k is the S-chromatic number of G.

1. Set $k = 1$,
2. Form the generators of $\mathcal{J}_k(S)$ from (6).
3. Compute the polynomial $P = x_{kn} - \sum_{t=0}^{kn-1} r(t)k^{q(t)}x_t$ described in Theorem 10.
4. Compute the Gröbner basis B_k of the ideal $\mathcal{J}_k(S) + \langle P\rangle$ with respect to the lexicographical ordering $x_0 > x_1 > \cdots > x_{kn-1} > x_{kn}$.
5. If $B_k \cap \mathbb{F} \neq \varnothing$, then $k = k + 1$ and go to 2.
6. Denote $z_1, \dots, z_d$ the roots of $g_{kn}(x_{kn})$ such that
$$B_k = (x_0 - g_0(x_{kn}), \dots, x_{kn-1} - g_{kn-1}(x_{kn}), g_{kn}(x_{kn})).$$
7. Return the list $c_1, \dots, c_d$ of the S-packing k-colorings of the graph G where, for any $j \in \{0, \dots, n-1\}$ and $\ell \in \{1, \dots, d\}$,
$$c_\ell(v_j) = \sum_{i=0}^{k-1} i g_{i+kj}(z_\ell).$$

In the examples below, an S-packing k-coloring of the graph G will be represented by the list:
$$[(i_0, 0), (i_1, 1), \dots, (i_{n-1}, n-1)].$$
By any pair (i_j, j) from the list above, we mean that the color i_j is assigned to the vertex v_j. We shall recall that such an S-packing k-coloring is obtained from a root z of $g_{kn}(x_{kn})$ and in this situation we have
$$i_j = \sum_{u=0}^{k-1} u g_{u+kj}(z). \tag{11}$$

Example 16. In this example, we are interested in a proper coloring of the graph G represented by the Figure 2. It is taken from [15] in which an algebraic characterization of a uniquely colorable graph is discussed. It represents a graph of 12 vertices without triangles that is 3-colorable in a unique way, up to permutation of the colors. The field $\mathbb{F}$ used in this example is $\mathbb{Z}_p$, the number of vertices is $n = 12$ and the number of colors is $k = 3$. The change of coordinates is the one proposed in Theorem 10, where p is an appropriate prime number. According to Theorem 10 the prime number p should be at

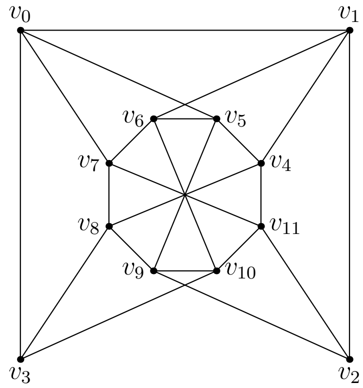

Figure 2: A uniquely 3-colorable graph.

least $k^n = 531441$. This last constraint is only a sufficient condition for the extended ideal $\mathcal{J}_k^{'}(S)$ to be in normal x_{kn}-position. However, we have verified that this ideal is still in normal x_{kn}-position for small prime numbers like 13, 19, 29, 31 or 37. We choose for example $p = 13$ and we add the polynomial:

$$\begin{aligned} x_{36} - \textstyle\sum_{t=1}^{35} \alpha_t x_t &= x_{36} - x_1 - 2x_2 - 3x_4 - 6x_5 + 4x_7 - 5x_8 - x_{10} - 2x_{11} \\ &\quad -3x_{13} - 6x_{14} + 4x_{16} - 5x_{17} - x_{19} - 2x_{20} - 3x_{22} - 6x_{23} \\ &\quad +4x_{25} - 5x_{26} - x_{28} - 2x_{29} - 3x_{31} - 6x_{32} + 4x_{34} - 5x_{35}. \end{aligned}$$

to the generators of $\mathcal{J}_3(S)$ according to the change of coordinates described in Theorem 10. The resulting Gröbner basis B_3 according to the lexicographical ordering $x_0 > \cdots > x_{36}$ has the desired form as in Remark 9 and contains the 36 polynomials

$$\begin{array}{ll} x_t - 4x_{36}^5 - 4x_{36}^4 - 3x_{36}^3 + 3x_{36}^2 + x_{36} - 6, & \text{for } t = 0, 18, 27, 33, \\ x_t - 5x_{36}^4 - 6x_{36}^2 - 2, & \text{for } t = 1, 19, 28, 34, \\ x_t + 4x_{36}^5 - 4x_{36}^4 + 3x_{36}^3 + 3x_{36}^2 - x_{36} - 6, & \text{for } t = 2, 20, 29, 35, \\ x_t - 2x_{36}^5 - 4x_{36}^4 + 5x_{36}^3 + 3x_{36} - 3, & \text{for } t = 3, 15, 24, 30, \\ x_t - 5x_{36}^4 + 5, & \text{for } t = 4, 16, 25, 31, \\ x_t + 2x_{36}^5 - 4x_{36}^4 - 5x_{36}^3 - 3x_{36} - 3, & \text{for } t = 5, 17, 26, 32, \\ x_t + 6x_{36}^5 - 5x_{36}^4 - 2x_{36}^3 - 3x_{36}^2 - 4x_{36} - 5, & \text{for } t = 6, 9, 12, 21, \\ x_t - 3x_{36}^4 + 6x_{36}^2 - 4, & \text{for } t = 7, 10, 13, 22, \\ x_t - 6x_{36}^5 - 5x_{36}^4 + 2x_{36}^3 - 3x_{36}^2 + 4x_{36} - 5, & \text{for } t = 8, 11, 14, 23, \end{array} \tag{12}$$

together with the polynomial

$$\begin{aligned} g_{36}(x_{36}) &= x_{36}^6 - 3x_{36}^4 - x_{36}^2 + 3 \\ &= (x_{36} - 1) \cdot (x_{36} + 1) \cdot (x_{36} + 4) \cdot (x_{36} + 5) \cdot (x_{36} - 5) \cdot (x_{36} - 4). \end{aligned}$$

Now, from the six roots of the polynomial $g_{36}(x_{36})$, we recover the corresponding proper 3-colorings of the graph G using the polynomials (12). For example, when we specialize the variable x_{36} to the root 4 of $g_{36}(x_{36})$, we get the zero

$$\begin{aligned} \xi &= (\xi_0, \ldots, \xi_{35}) \\ &= (1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0) \end{aligned}$$

of the ideal $\mathcal{J}_3(S)$, where ξ_t is the zero of the corresponding polynomial in (12) for any $t = 0, \ldots, 35$. To ξ corresponds the proper coloration

$$[(0,0),(1,1),(2,2),(2,3),(2,4),(1,5),(0,6),(2,7),(1,8),(0,9),(1,10),(0,11)]$$

where every pair (i, j) from this last coloration is computed by formula (11). To the remaining zeros -1, -4, -5 and 5 of the polynomial $g_{36}(x_{36})$ correspond respectively the proper colorations:

$$\begin{array}{l}[(2,0),(0,1),(1,2),(1,3),(1,4),(0,5),(2,6),(1,7),(0,8),(2,9),(0,10),(2,11)]\,,\\ [(0,0),(2,1),(1,2),(1,3),(1,4),(2,5),(0,6),(1,7),(2,8),(0,9),(2,10),(0,11)]\,,\\ [(2,0),(1,1),(0,2),(0,3),(0,4),(1,5),(2,6),(0,7),(1,8),(2,9),(1,10),(2,11)]\,,\\ [(1,0),(2,1),(0,2),(0,3),(0,4),(2,5),(1,6),(0,7),(2,8),(1,9),(2,10),(1,11)]\,,\\ [(1,0),(0,1),(2,2),(2,3),(2,4),(0,5),(1,6),(2,7),(0,8),(1,9),(0,10),(1,11)]\,.\end{array}$$

Example 17. Consider the simple graph G as the path of ten vertices $v_0, \ldots, v_9$ and the S-packing $S = (1, 2, 3, \ldots)$. The number of vertices is $n = 10$. We use the field $\mathbb{F} = \mathbb{Z}_{13}$ as in Example 16. The ideals $\mathcal{J}_1(S)$ and $\mathcal{J}_2(S)$ are unit which can be tested by computing Gröbner bases of such ideals with respect to any monomial ordering. Let now $k = 3$. As in Example 16, we add the polynomial:

$$\begin{array}{rcl} x_{30} - \sum_{t=1}^{29} \alpha_t x_t & = & x_{30} - x_1 - 2x_2 - 3x_4 - 6x_5 + 4x_7 - 5x_8 - x_{10} \\ & & -2x_{11} - 3x_{13} - 6x_{14} + 4x_{16} - 5x_{17} - x_{19} - 2x_{20} \\ & & -3x_{22} - 6x_{23} + 4x_{25} - 5x_{26} - x_{28} - 2x_{29}. \end{array}$$

to the generators of $\mathcal{J}_3(S)$ according to the change of coordinates described in Theorem 10 to get the extended ideal $\mathcal{J}_k'(S)$ which is in normal x_{30}-position. The resulting Gröbner basis B_3 of $\mathcal{J}_k'(S)$ according to the lexicographical ordering $x_0 > \cdots > x_{30}$ has the desired form as in Remark 9 and contains the 30 polynomials

$$\begin{array}{ll}
x_t + 2x_{30}^7 - 6x_{30}^5 + 4x_{30}^4 - 2x_{30}^3 + 5x_{30}^2 + 6x_{30} + 1, & \text{for } t = 2, 7, 14, 19, 26,\\
x_t - 2x_{30}^7 + 6x_{30}^6 - 4x_{30}^5 - 5x_{30}^4 - x_{30}^3 - 6x_{30}^2 - 3x_{30} - 2, & \text{for } t = 5, 10, 17, 22, 29,\\
x_t + 4x_{30}^7 - 3x_{30}^6 + 4x_{30}^5 - 3x_{30}^4 + x_{30}^3 - 4x_{30}^2 + x_{30} - 4, & \text{for } t = 8, 13, 20,\\
x_t - 4x_{30}^7 - 3x_{30}^6 + 6x_{30}^5 + 4x_{30}^4 + 2x_{30}^3 + 5x_{30}^2 - 4x_{30} + 4, & \text{for } t = 11, 16, 23,\\
x_t - 6x_{30}^7 + 3x_{30}^6 + 2x_{30}^5 - x_{30}^4 + x_{30}^3 - x_{30}^2 + 6x_{30} + 2, & \text{for } t = 6, 12, 18,\\
x_t + 6x_{30}^7 - 3x_{30}^6 - 2x_{30}^5 + x_{30}^4 - x_{30}^3 + x_{30}^2 - 6x_{30} - 3, & \text{for } t = 9, 15, 21,\\
x_0 + 4x_{30}^7 + 7x_{30}^6 - 4x_{30}^5 + 4x_{30}^4 + 5x_{30}^3 + 3x_{30}^2 + 4x_{30} - 1, & \\
x_1 - 6x_{30}^7 + 6x_{30}^6 - 3x_{30}^5 + 5x_{30}^4 - 3x_{30}^3 + 5x_{30}^2 + 3x_{30} - 1, & \\
x_3 + 5x_{30}^7 + 6x_{30}^6 + 3x_{30}^4 + 4x_{30}^3 + 4x_{30}^2 - 4x_{30} - 6, & \\
x_4 - 3x_{30}^7 + x_{30}^6 + 4x_{30}^5 + 2x_{30}^4 - 3x_{30}^3 + 2x_{30}^2 - 6x_{30} - 6, & \\
x_{24} + x_{30}^7 + 2x_{30}^6 + 6x_{30}^5 + 6x_{30}^4 - 4x_{30}^3 + 5x_{30}^2 + 4x_{30}, & \\
x_{25} - 3x_{30}^7 - 2x_{30}^6 + 3x_{30}^4 + 6x_{30}^3 + 3x_{30}^2 + 3x_{30} - 2, & \\
x_{27} - 4x_{30}^7 + 6x_{30}^6 + x_{30}^5 - 3x_{30}^4 - 2x_{30}^3 - 2x_{30}^2 + 6x_{30} - 6, & \\
x_{28} + 6x_{30}^7 + x_{30}^6 + 3x_{30}^5 - 5x_{30}^4 + 3x_{30}^3 - 5x_{30}^2 - 3x_{30} - 6, &
\end{array}$$

together with the polynomial

$$\begin{aligned} g_{30}(x_{30}) &= x_{30}^8 - 5x_{30}^7 + x_{30}^6 + 3x_{30}^5 + 3x_{30}^3 - x_{30}^2 - 5x_{30} - 1 \\ &= (x_{30}+2)\cdot(x_{30}+3)\cdot(x_{30}+4)\cdot(x_{30}+5) \\ &\quad \cdot(x_{30}+6)\cdot(x_{30}-5)\cdot(x_{30}-4)\cdot(x_{30}-3). \end{aligned}$$

As in Example 16, we recover the eight possible S-packing 3-colorings of the path graph G from the 30 polynomials above and from the roots -2, -3, -4, -5, -6, 5, 4 and 3 of the polynomial $g_{30}(x_{30})$. These S-packing 3-colorings are listed below in the same order of appearance of the corresponding roots:

$$\begin{array}{l} [(2,0),(0,1),(1,2),(0,3),(2,4),(0,5),(1,6),(0,7),(2,8),(1,9)]\,, \\ [(2,0),(0,1),(1,2),(0,3),(2,4),(0,5),(1,6),(0,7),(2,8),(0,9)]\,, \\ [(0,0),(1,1),(2,2),(0,3),(1,4),(0,5),(2,6),(0,7),(1,8),(0,9)]\,, \\ [(0,0),(1,1),(0,2),(2,3),(0,4),(1,5),(0,6),(2,7),(0,8),(1,9)]\,, \\ [(1,0),(0,1),(2,2),(0,3),(1,4),(0,5),(2,6),(0,7),(1,8),(0,9)]\,, \\ [(1,0),(2,1),(0,2),(1,3),(0,4),(2,5),(0,6),(1,7),(0,8),(2,9)]\,, \\ [(0,0),(2,1),(0,2),(1,3),(0,4),(2,5),(0,6),(1,7),(0,8),(2,9)]\,, \\ [(0,0),(1,1),(0,2),(2,3),(0,4),(1,5),(0,6),(2,7),(1,8),(0,9)]\,. \end{array}$$

According to Theorem 10 the prime number p should be at least $k^n = 59049$ to ensure that the extended ideal $\mathcal{J}_k^{'}(S)$ is in normal x_{30}-position. However, we have verified that this ideal is still in normal x_{30}-position for small prime numbers like 13, 29 or 31.

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