Tomaszewski's problem on randomly signed sums: breaking the 3/8 barrier

Ravi B. Boppana
Department of Mathematics
M. I. T.
Massachusetts, USA
rboppana@mit.edu

Ron Holzman
Department of Mathematics
Technion–Israel Institute of Technology
Israel
holzman@tx.technion.ac.il

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Abstract

Let $v_1, v_2, \ldots, v_n$ be real numbers whose squares add up to 1. Consider the 2^n signed sums of the form $S = \sum \pm v_i$. Holzman and Kleitman (1992) proved that at least $\frac{3}{8}$ of these sums satisfy $|S| \leqslant 1$. This $\frac{3}{8}$ bound seems to be the best their method can achieve. Using a different method, we improve the bound to $\frac{13}{32}$, thus breaking the $\frac{3}{8}$ barrier.

Keywords: combinatorial probability; probabilistic inequalities

1 Introduction

Let $v_1, v_2, \ldots, v_n$ be real numbers such that the sum of their squares is at most 1. Consider the 2^n signed sums of the form $S = \pm v_1 \pm v_2 \pm \cdots \pm v_n$. In 1986, B. Tomaszewski (see Guy [3]) asked the following question: is it always true that at least $\frac{1}{2}$ of these sums satisfy $|S| \leqslant 1$? Most examples with $n = 2$ and $v_1^2 + v_2^2 = 1$ show that $\frac{1}{2}$ can't be replaced with a bigger number.

Holzman and Kleitman [7] proved that at least $\frac{3}{8}$ of the sums satisfy $|S| \leqslant 1$. This result was an immediate consequence of their main result: at least $\frac{3}{8}$ of the sums satisfy the strict inequality $|S| < 1$, provided that each $|v_i|$ is strictly less than 1. This $\frac{3}{8}$ bound for $|S| < 1$ is best possible: consider the example with $n = 4$ and $v_1 = v_2 = v_3 = v_4 = \frac{1}{2}$. So $\frac{3}{8}$ seems to be a natural barrier to their method of proof.

Using a different method, we prove that more than $\frac{13}{32}$ of the sums satisfy $|S| \leqslant 1$. In other words, we break the $\frac{3}{8}$ barrier. Our method, roughly speaking, goes like this. We will let the first few $\pm$ signs be arbitrary. But once the partial sum becomes near 1 in absolute value, we will show that the final sum still has a decent chance of remaining at most 1 in absolute value.

We can actually improve the $\frac{13}{32}$ bound a tiny bit, to $\frac{13}{32} + 9 \times 10^{-6}$. Combining our method with other ideas, which could handle the tight cases for our analysis, may lead to further improvements of the bound. Still, the conjectured lower bound of $\frac{1}{2}$ currently appears to be out of reach.

Ten years after Holzman and Kleitman [7] but independently, Ben-Tal, Nemirovski, and Roos [1] proved that at least $\frac{1}{3}$ of the sums satisfy $|S| \leqslant 1$; they say that the proof is mainly due to P. van der Wal. Shnurnikov [9] refined the argument of [1] to prove a 36% bound. Even though these two bounds are weaker than that of Holzman and Kleitman, the methods used to prove them are noteworthy. In particular, we will use the conditioning argument of [1] and the fourth moment method of [9].

Let Tomaszewski's constant be the largest constant c such that the fraction of sums that satisfy $|S| \leqslant 1$ is always at least c. We now know that Tomaszewski's constant is between $\frac{13}{32}$ and $\frac{1}{2}$. Both [7] and [1] conjecture that Tomaszewski's constant is $\frac{1}{2}$. De, Diakonikolas, and Servedio [2] developed an algorithm to approximate Tomaszewski's constant. Specifically, given an $\epsilon > 0$, their algorithm will output a number that is within ϵ of Tomaszewski's constant. The running time of their algorithm is exponential in $1/\epsilon^3$, so it's not clear that we can run their algorithm in a reasonable amount of time to improve the known bounds on Tomaszewski's constant.

The conjectured lower bound of $\frac{1}{2}$ has been confirmed in some special cases. For example, von Heymann [6] and Hendriks and van Zuijlen [5] proved the conjecture when $n \leqslant 9$. Also, van Zuijlen [10] and von Heymann [6] proved the conjecture when all of the $|v_i|$ are equal.

We will use the language of probability. Let $\Pr[A]$ be the probability of an event A. Let $\mathbb{E}(X)$ be the expected value of a random variable X. A *random sign* is a random variable whose probability distribution is the uniform distribution on the set $\{-1, +1\}$. With this language, we can restate our main result.

Main Theorem. *Let v_1, v_2, ..., v_n be real numbers such that $\sum_{i=1}^n v_i^2$ is at most 1. Let a_1, a_2, ..., a_n be independent random signs. Let S be $\sum_{i=1}^n a_i v_i$. Then $\Pr[|S| \leqslant 1] > \frac{13}{32}$.*

In Section 2 of this paper, we will provide a short proof of a bound better than $\frac{3}{8}$. In Section 3, we will refine the analysis to improve the bound to $\frac{13}{32}$ and slightly beyond.

2 Beating the 3/8 bound

In this section, we will give the simplest proof we can of a bound better than $\frac{3}{8}$. Namely, we will prove a bound of $\frac{37}{98}$, which is a little more than 37.75%. In Section 3, we will improve the bound further.

We begin with a lemma. Roughly speaking, this lemma can be used to show that if a partial sum is a little less than 1, then the final sum has a decent chance of remaining less than 1 in absolute value.

Lemma 1. *Let x be a real number such that $|x| \leqslant 1$. Let v_1, v_2, ..., v_n be real numbers such that*

$$\sum_{i=1}^{n} v_i^2 \leqslant \frac{2}{7}(1+|x|)^2.$$

Let a_1, a_2, ..., a_n be independent random signs. Let Y be $\sum_{i=1}^{n} a_i v_i$. Then

$$\Pr[|x+Y| \leqslant 1] \geqslant \frac{37}{98}\,.$$

Proof. By symmetry, we may assume that $x \geqslant 0$. The fourth moment of Y is

$$\mathbb{E}(Y^4) = 3\Big(\sum_{i=1}^{n} v_i^2\Big)^2 - 2\sum_{i=1}^{n} v_i^4 \leqslant 3\Big(\sum_{i=1}^{n} v_i^2\Big)^2 \leqslant \frac{12}{49}(1+x)^4.$$

So, by the fourth moment version of Chebyshev's inequality[1],

$$\Pr[|Y| \geqslant 1+x] \leqslant \frac{\mathbb{E}(Y^4)}{(1+x)^4} \leqslant \frac{12}{49}\,.$$

Looking at the complement,

$$\Pr[|Y| < 1+x] \geqslant \frac{37}{49}\,.$$

Because Y has a symmetric distribution,

$$\Pr[-1-x < Y \leqslant 0] \geqslant \frac{1}{2}\Pr[|Y| < 1+x] \geqslant \frac{37}{98}.$$

Recall that $x \leqslant 1$. Hence if $-1-x < Y \leqslant 0$, then $|x+Y| \leqslant 1$. Therefore

$$\Pr[|x+Y| \leqslant 1] \geqslant \Pr[-1-x < Y \leqslant 0] \geqslant \frac{37}{98}\,. \qquad \square$$

Next we will use Lemma 1 to go beyond the $\frac{3}{8}$ bound.

Theorem 2. *Let v_1, v_2, ..., v_n be real numbers such that $\sum_{i=1}^{n} v_i^2$ is at most 1. Let a_1, a_2, ..., a_n be independent random signs. Let S be $\sum_{i=1}^{n} a_i v_i$. Then*

$$\Pr[|S| \leqslant 1] \geqslant \frac{37}{98}\,.$$

Proof. By inserting 0's, we may assume that $n \geqslant 4$. By permuting, we may assume that the four largest $|v_i|$ are $|v_n| \geqslant |v_1| \geqslant |v_{n-1}| \geqslant |v_2|$. By the quadratic mean inequality,

$$\frac{|v_1|+|v_2|+|v_{n-1}|+|v_n|}{4} \leqslant \sqrt{\frac{v_1^2+v_2^2+v_{n-1}^2+v_n^2}{4}} \leqslant \sqrt{\frac{1}{4}} = \frac{1}{2}\,.$$

[1]Shnurnikov [9] used the fourth moment in a similar situation.

So $|v_1| + |v_2| + |v_{n-1}| + |v_n| \leqslant 2$. Because of our ordering,

$$|v_1| + |v_2| \leqslant \frac{|v_1| + |v_n|}{2} + \frac{|v_2| + |v_{n-1}|}{2} \leqslant 1.$$

Given an integer t from 0 to n, let X_t be the partial sum $\sum_{i=1}^{t} a_i v_i$ and let Y_t be the remaining sum $\sum_{i=t+1}^{n} a_i v_i$. Let T be the smallest nonnegative integer t such that $t = n-1$ or $|X_t| > 1 - |v_{t+1}|$. In a stochastic process such as ours, T is called a stopping time, defined by the stopping rule in the previous sentence[2]. Note that $T \geqslant 2$, since $|v_1| + |v_2| \leqslant 1$. By the stopping rule, $|X_{T-1}| \leqslant 1 - |v_T|$. Hence by the triangle inequality,

$$|X_T| \leqslant |X_{T-1}| + |v_T| \leqslant 1 - |v_T| + |v_T| = 1.$$

Also by the stopping rule, if $T < n-1$, then $|X_T| > 1 - |v_{T+1}|$.

We will condition on T and X_T. We claim that

$$\Pr[|S| \leqslant 1 \mid T, X_T] \geqslant \frac{37}{98}\,.$$

By averaging over T and X_T, this claim implies the theorem. To prove the claim, we may assume by symmetry that $X_T \geqslant 0$. We will divide the proof of the claim into three cases, depending on T.

Case 1: $T = n-1$. In this case, $|Y_T| = |v_n| \leqslant 1$. Recall that $0 \leqslant X_T \leqslant 1$. Hence if $Y_T \leqslant 0$, then $|S| = |X_T + Y_T| \leqslant 1$. Therefore by symmetry,

$$\Pr[|S| \leqslant 1 \mid T, X_T] \geqslant \Pr[Y_T \leqslant 0 \mid T, X_T] \geqslant \frac{1}{2}\,.$$

Case 2: $T = n-2$. In this case,

$$|Y_T| \leqslant |v_{n-1}| + |v_n| \leqslant 2 - |v_1| \leqslant 2 - |v_{n-1}|\,.$$

Recall that $1 - |v_{n-1}| < X_T \leqslant 1$. Hence if $Y_T \leqslant 0$, then $|S| = |X_T + Y_T| \leqslant 1$. Therefore by symmetry,

$$\Pr[|S| \leqslant 1 \mid T, X_T] \geqslant \Pr[Y_T \leqslant 0 \mid T, X_T] \geqslant \frac{1}{2}\,.$$

Case 3: $T \leqslant n-3$. In this case, by the stopping rule,

$$\sum_{i=T+1}^{n} v_i^2 \leqslant 1 - \sum_{i=1}^{T} v_i^2 \leqslant 1 - v_1^2 - v_2^2 \leqslant 1 - 2v_{T+1}^2 < 1 - 2(1 - X_T)^2.$$

We can bound the final expression as follows:

$$1 - 2(1 - X_T)^2 = \frac{2}{7}(1 + X_T)^2 - \frac{1}{7}(4X_T - 3)^2 \leqslant \frac{2}{7}(1 + X_T)^2.$$

Hence the hypotheses of Lemma 1 are satisfied with $x = X_T$ and $Y = Y_T$. By Lemma 1, we conclude that

$$\Pr[|S| \leqslant 1 \mid T, X_T] = \Pr[|X_T + Y_T| \leqslant 1 \mid T, X_T] \geqslant \frac{37}{98}\,. \qquad \square$$

[2]A similar stopping rule was implicitly used by Ben-Tal *et al.* [1] and refined by Shnurnikov [9]. In addition, [9] pointed out the value of having $|v_1| + |v_2| \leqslant 1$.

3 Further improvement

In this section, we will improve the lower bound to $\frac{13}{32}$, which is 40.625%. At the end, we will sketch how to improve the bound further, to $\frac{13}{32} + 9 \times 10^{-6}$.

Let us examine where the proof of Theorem 2 is potentially tight. Looking at its Case 3, we see that the proof is potentially tight when $T = 2$ and $|v_1| = |v_2| = |v_3| = \frac{1}{4}$. But that scenario is impossible: if $T = 2$, then by the stopping rule, $|v_1| + |v_2| > 1 - |v_3|$. This suggests that we can sharpen the bound on $\sum_{i=T+1}^{n} v_i^2$ in terms of T and X_T.

Another idea is that our final bound on $\Pr[|S| \leqslant 1]$, instead of being the worst-case conditional bound, may be taken to be a weighted average of the conditional bounds, with weights corresponding to the distribution of T.

First, we state the following generalization of Lemma 1. Given a number c, define $F(c)$ by

$$F(c) = \frac{1}{2}(1 - 3c^2).$$

Lemma 3. *Let c be a nonnegative number. Let x be a real number such that $|x| \leqslant 1$. Let v_1, v_2, ..., v_n be real numbers such that*

$$\sum_{i=1}^{n} v_i^2 \leqslant c(1 + |x|)^2.$$

Let a_1, a_2, ..., a_n be independent random signs. Let Y be $\sum_{i=1}^{n} a_i v_i$. Then

$$\Pr[|x + Y| \leqslant 1] \geqslant F(c).$$

Proof. By symmetry, we may assume that $x \geqslant 0$. As in the proof of Lemma 1, the fourth moment of Y satisfies

$$\mathbb{E}(Y^4) \leqslant 3\Big(\sum_{i=1}^{n} v_i^2\Big)^2 \leqslant 3c^2(1 + x)^4.$$

So, by the fourth moment version of Chebyshev's inequality,

$$\Pr[|Y| \geqslant 1 + x] \leqslant \frac{\mathbb{E}(Y^4)}{(1 + x)^4} \leqslant 3c^2.$$

Following the proof of Lemma 1, by taking the complement and then using the symmetry of Y, we have

$$\Pr[|x + Y| \leqslant 1] \geqslant \frac{1}{2}(1 - 3c^2) = F(c).$$ □

Now we will use Lemma 3 to prove our $\frac{13}{32}$ lower bound.

Theorem 4. *Let v_1, v_2, ..., v_n be real numbers such that $\sum_{i=1}^{n} v_i^2$ is at most 1. Let a_1, a_2, ..., a_n be independent random signs. Let S be $\sum_{i=1}^{n} a_i v_i$. Then*

$$\Pr[|S| \leqslant 1] > \frac{13}{32}.$$

Proof. By inserting 0's, we may assume that $n \geqslant 4$. By symmetry, we may assume that each v_i is nonnegative. By permuting, we may assume that the v_i are ordered as follows:

$$v_n \geqslant v_1 \geqslant v_{n-1} \geqslant v_2 \geqslant v_3 \geqslant \ldots \geqslant v_{n-2}.$$

Except for the oddballs v_n and v_{n-1}, the order is decreasing. As in Theorem 2, we have $v_1 + v_2 + v_{n-1} + v_n \leqslant 2$ and $v_1 + v_2 \leqslant 1$.

Given an integer t from 0 to n, let M_t be the sum $\sum_{i=1}^{t} v_i$. Let K be the smallest nonnegative integer t such that $t = n-1$ or $M_t > 1 - v_{t+1}$. The parameter K measures how spread out the v_i are. Note that $K \geqslant 2$, since $v_1 + v_2 \leqslant 1$. By the definition of K, observe that $M_{K-1} \leqslant 1 - v_K$ and hence $M_K \leqslant 1$. Also, if $K < n-1$, then $M_K > 1 - v_{K+1}$ and hence $M_{K+1} > 1$.

Given an integer t from 0 to n, define the sums X_t and Y_t as in Theorem 2. Note that $|X_t| \leqslant M_t$. Following Theorem 2, let T be the smallest nonnegative integer t such that $t = n-1$ or $|X_t| > 1 - v_{t+1}$. Note that $T \geqslant K$. As before, we have $|X_{T-1}| \leqslant 1 - v_T$ and $|X_T| \leqslant 1$. Also, if $T < n-1$, then $|X_T| > 1 - v_{T+1}$.

We will bound from below the conditional probability $\Pr[|S| \leqslant 1 \mid T]$. Namely, we will prove the two-piece lower bound

$$\Pr[|S| \leqslant 1 \mid T] \geqslant \begin{cases} F\left(\frac{(K+1)^2 - T}{(2K+1)^2}\right) & \text{if } T \leqslant \frac{3K+2}{2}\,; \\ F\left(\frac{K}{4K+2}\right) & \text{if } T \geqslant \frac{3K+2}{2}\,. \end{cases}$$

We will actually prove the same lower bound on the refined conditional probability $\Pr[|S| \leqslant 1 \mid T, X_T]$. To prove this claim, we may assume by symmetry that $X_T \geqslant 0$. We will divide the proof of the claim into five cases, depending on T.

Case 1: $T = n-1$. The proof of this case is the same as Case 1 of Theorem 2, which yields the stronger bound $\Pr[|S| \leqslant 1 \mid T, X_T] \geqslant \frac{1}{2}$.

Case 2: $T = n-2$. The proof of this case is the same as Case 2 of Theorem 2, which yields the stronger bound $\Pr[|S| \leqslant 1 \mid T, X_T] \geqslant \frac{1}{2}$.

Case 3: $K + 1 \leqslant T \leqslant \frac{3K+2}{2}$ and $T \leqslant n-3$. By the quadratic mean inequality,

$$\sum_{i=1}^{K+1} v_i^2 \geqslant \frac{1}{K+1}\Big(\sum_{i=1}^{K+1} v_i\Big)^2 = \frac{1}{K+1} M_{K+1}^2 > \frac{1}{K+1}\,.$$

Hence, by splitting our sum into two parts, we get

$$\sum_{i=1}^{T} v_i^2 > \frac{1}{K+1} + (T-K-1)v_{T+1}^2 \geqslant \frac{1}{K+1} + (T-K-1)(1-X_T)^2.$$

As a simpler bound,

$$\sum_{i=1}^{T} v_i^2 \geqslant T v_{T+1}^2 > T(1-X_T)^2.$$

Multiplying the second-to-last inequality by $\frac{2T-K-1}{2K+1}$ and the last inequality by $\frac{3K+2-2T}{2K+1}$, both multipliers being nonnegative by the case assumption, we get

$$\sum_{i=1}^{T} v_i^2 \geqslant \frac{2T-K-1}{(K+1)(2K+1)} + \frac{(K+1)^2-T}{2K+1}(1-X_T)^2.$$

Therefore, looking at the complementary sum, we get

$$\sum_{i=T+1}^{n} v_i^2 \leqslant \frac{(K+1)^2-T}{(K+1)(2K+1)} \left[2-(K+1)(1-X_T)^2\right].$$

We can bound the bracketed expression as follows:

$$\begin{aligned} 2-(K+1)(1-X_T)^2 &= \frac{K+1}{2K+1}(1+X_T)^2 - \frac{2}{2K+1}\left[(K+1)X_T-K\right]^2 \\ &\leqslant \frac{K+1}{2K+1}(1+X_T)^2. \end{aligned}$$

Plugging this inequality back into the previous one, we get

$$\sum_{i=T+1}^{n} v_i^2 \leqslant \frac{(K+1)^2-T}{(2K+1)^2}(1+X_T)^2.$$

Hence the hypotheses of Lemma 3 are satisfied with $c = \frac{(K+1)^2-T}{(2K+1)^2}$, $x = X_T$, and $Y = Y_T$. By Lemma 3, we conclude that

$$\Pr[|S| \leqslant 1 \mid T, X_T] = \Pr[|X_T + Y_T| \leqslant 1 \mid T, X_T] \geqslant F\Big(\frac{(K+1)^2-T}{(2K+1)^2}\Big).$$

Case 4: $\frac{3K+2}{2} \leqslant T \leqslant n-3$. As in Case 3, we can bound $\sum_{i=1}^{T} v_i^2$ as follows:

$$\sum_{i=1}^{T} v_i^2 > \frac{1}{K+1} + (T-K-1)(1-X_T)^2.$$

Because $T \geqslant \frac{3K+2}{2}$, this inequality implies

$$\sum_{i=1}^{T} v_i^2 \geqslant \frac{1}{K+1} + \frac{K}{2}(1-X_T)^2.$$

Compare this bound with the combined bound from Case 3:

$$\sum_{i=1}^{T} v_i^2 \geqslant \frac{2T-K-1}{(K+1)(2K+1)} + \frac{(K+1)^2-T}{2K+1}(1-X_T)^2.$$

Note that our bound on $\sum_{i=1}^{T} v_i^2$ is the same as this bound from Case 3 when $T = \frac{3K+2}{2}$. So we can repeat the remainder of Case 3 to get the same lower bound on $\Pr[|S| \leqslant 1 \mid T, X_T]$ when $T = \frac{3K+2}{2}$. The bound on $\Pr[|S| \leqslant 1 \mid T, X_T]$ in Case 3 was

$$\Pr[|S| \leqslant 1 \mid T, X_T] \geqslant F\Big(\frac{(K+1)^2 - T}{(2K+1)^2}\Big).$$

When $T = \frac{3K+2}{2}$, this bound becomes

$$\Pr[|S| \leqslant 1 \mid T, X_T] \geqslant F\Big(\frac{K}{4K+2}\Big).$$

So we get the same bound in our current case.

Case 5: $T = K \leqslant n - 3$. By the quadratic mean inequality,

$$\sum_{i=1}^{T} v_i^2 \geqslant \frac{1}{T}\Big(\sum_{i=1}^{T} v_i\Big)^2 = \frac{1}{T} M_T^2 \geqslant \frac{1}{T} X_T^2 = \frac{1}{K} X_T^2.$$

We can bound the final expression as follows:

$$\begin{aligned}\frac{1}{K} X_T^2 &= \frac{1}{K+1} - (1 - X_T)^2 + \frac{1}{K(K+1)}\left[(K+1)X_T - K\right]^2 \\ &\geqslant \frac{1}{K+1} - (1 - X_T)^2.\end{aligned}$$

Plugging this inequality back into the previous one, we get

$$\sum_{i=1}^{T} v_i^2 \geqslant \frac{1}{K+1} - (1 - X_T)^2 = \frac{1}{K+1} + (T - K - 1)(1 - X_T)^2.$$

This is the same inequality we derived at the beginning of Case 3. So we can repeat the remainder of Case 3 to get the same lower bound:

$$\Pr[|S| \leqslant 1 \mid T, X_T] \geqslant F\Big(\frac{(K+1)^2 - T}{(2K+1)^2}\Big).$$

In summary, we have proved our claim on conditional probability:

$$\Pr[|S| \leqslant 1 \mid T] \geqslant \begin{cases} F\left(\frac{(K+1)^2 - T}{(2K+1)^2}\right) & \text{if } T \leqslant \frac{3K+2}{2}\,; \\ F\left(\frac{K}{4K+2}\right) & \text{if } T \geqslant \frac{3K+2}{2}\,. \end{cases}$$

Next, we will use this conditional bound to derive a lower bound on the unconditional probability $\Pr[|S| \leqslant 1]$.

As mentioned above, we always have $T \geqslant K$. In fact, assuming that $K \leqslant n - 4$, we have $T = K$ if the signs $a_1, \ldots, a_K$ are all equal, and otherwise $T \geqslant K + 2$. This follows from observing that if $a_1, \ldots, a_K$ are not all equal, then $|X_K| \leqslant \sum_{i=1}^{K-1} v_i - v_K \leqslant 1 - v_{K+1}$ and $|X_{K+1}| \leqslant \sum_{i=1}^{K-1} v_i - v_K + v_{K+1} \leqslant 1 - v_{K+2}$, by the definition of K and the ordering of the v_i.

This shows that for $K \leqslant n-4$ we have $\Pr[T = K] = \frac{1}{2^{K-1}}$ and $\Pr[T \geqslant K+2] = 1 - \frac{1}{2^{K-1}}$. Therefore

$$\begin{aligned}
&\Pr[\,|S| \leqslant 1] \\
&\quad = \frac{1}{2^{K-1}} \Pr[|S| \leqslant 1 \mid T = K] + \left(1 - \frac{1}{2^{K-1}}\right) \Pr[|S| \leqslant 1 \mid T \geqslant K+2] \\
&\quad \geqslant \frac{1}{2^{K-1}} F\left(\frac{(K+1)^2 - K}{(2K+1)^2}\right) + \left(1 - \frac{1}{2^{K-1}}\right) F\left(\frac{(K+1)^2 - (K+2)}{(2K+1)^2}\right).
\end{aligned}$$

Here we have used our conditional bounds, the fact that they are nondecreasing in T, and the inequality $K + 2 \leqslant \frac{3K+2}{2}$. Note that this lower bound on $\Pr[|S| \leqslant 1]$ remains valid without assuming that $K \leqslant n - 4$. Indeed, if $K = n - 3$ it is still true that $\Pr[T = K] = \frac{1}{2^{K-1}}$, and while $T = K + 1 = n - 2$ may occur in this case, it yields a conditional bound of $\frac{1}{2}$ as shown in Case 2 above, which is even better than our stated lower bound. The values $n - 2$ and $n - 1$ for K are of course covered by the conditional bound of $\frac{1}{2}$ in Cases 1 and 2 above.

Thus, to conclude our proof it suffices to show that

$$\frac{1}{2^{K-1}} F\left(\frac{(K+1)^2 - K}{(2K+1)^2}\right) + \left(1 - \frac{1}{2^{K-1}}\right) F\left(\frac{(K+1)^2 - (K+2)}{(2K+1)^2}\right) > \frac{13}{32}$$

holds for all $K \geqslant 2$. Substituting the relevant expressions into the formula for F and performing routine manipulations, the latter is shown to be equivalent to

$$64(K^2 + K) < 2^{K-1}(40K^2 + 40K - 15),$$

which indeed holds for $K \geqslant 2$. □

Can we improve this $\frac{13}{32}$ lower bound? Yes, a little. The idea is to replace the fourth moment with the more flexible pth moment, where p is a parameter to be optimized. To do so, we will need Khintchine's inequality. This inequality was first proved by Khintchine [8] in a weaker form and later proved by Haagerup [4] with the optimal constants. Namely, given $p \geqslant 2$, let B_p be the constant

$$B_p = \frac{2^{p/2}\Gamma(\frac{p+1}{2})}{\sqrt{\pi}},$$

where Γ is the gamma function. For example, $B_2 = 1$, $B_3 = 2\sqrt{2/\pi}$, and $B_4 = 3$.

Theorem 5 (Khintchine's inequality)**.** *Let p be a real number such that $p \geqslant 2$. Let v_1, v_2, ..., v_n be real numbers. Let a_1, a_2, ..., a_n be independent random signs. Let S be $\sum_{i=1}^{n} a_i v_i$. Then*

$$\mathbb{E}(|S|^p) \leqslant B_p \Big(\sum_{i=1}^{n} v_i^2\Big)^{p/2}.$$

For the improved lower bound, choose with foresight $p = 3.95937$. In Lemma 3, replace the fourth moment with the pth moment and apply Khintchine's inequality (with $S = Y$), which allows us to replace the function F with the function G defined by $G(c) = \frac{1}{2}(1 - B_p c^{p/2})$. Use this revised lemma in Theorem 4. The resulting lower bound is $G(\frac{1}{4})$, which is bigger than $\frac{13}{32} + 9 \times 10^{-6}$. We omit the details.

References

[1] A. Ben-Tal, A. Nemirovski, and C. Roos. Robust solutions of uncertain quadratic and conic-quadratic problems. *SIAM Journal on Optimization*, 13(2):535–560, 2002.

[2] A. De, I. Diakonikolas, and R. A. Servedio. A robust Khintchine inequality, and algorithms for computing optimal constants in Fourier analysis and high-dimensional geometry. *SIAM Journal on Discrete Mathematics*, 30(2):1058–1094, 2016.

[3] R. K. Guy. Any answers anent these analytical enigmas? *American Mathematical Monthly*, 93(4):279–281, 1986.

[4] U. Haagerup. The best constants in the Khintchine inequality. *Studia Mathematica*, 70(3):231–283, 1981.

[5] H. Hendriks and M. C. A. van Zuijlen. Linear combinations of Rademacher random variables. arXiv:1703.07251v1, 2017.

[6] F. von Heymann. Ideas for an old analytic enigma about the sphere that fail in intriguing ways. http://www.mi.uni-koeln.de/opt/wp-content/uploads/2017/02/Cube_sphere.pdf, 2012.

[7] R. Holzman and D. J. Kleitman. On the product of sign vectors and unit vectors. *Combinatorica*, 12(3):303–316, 1992.

[8] A. Khintchine. Über dyadische Brüche. *Math. Zeitschrift*, 18:109–116, 1923.

[9] I. Shnurnikov. On a sum of centered random variables with nonreducing variances. arXiv:1202.2990v2, 2012.

[10] M. C. A. van Zuijlen. On a conjecture concerning the sum of independent Rademacher random variables. arXiv:1112.4988v1, 2011.