Fair splitting of colored paths

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Abstract

This paper deals with two problems about splitting fairly a path with colored vertices, where "fairly" means that each part contains almost the same amount of vertices in each color.

Our first result states that it is possible to remove one vertex per color from a path with colored vertices so that the remaining vertices can be fairly split into two independent sets of the path. It implies in particular a conjecture of Ron Aharoni and coauthors. The proof uses the octahedral Tucker lemma.

Our second result is the proof of a particular case of a conjecture of Dömötör Pálvölgyi about fair splittings of necklaces for which one can decide which thieves are advantaged. The proof is based on a rounding technique introduced by Noga Alon and coauthors to prove the discrete splitting necklace theorem from the continuous one.

Keywords: Tucker's lemma; splitting necklace; independent sets

1 Introduction

This paper is about fair splittings of paths with colored vertices. "Fair" means throughout the paper that for each color j, the numbers of vertices of color j in each part differ by at most one.

Given a path P whose vertex set is partitioned into m subsets $V_1, \ldots, V_m$, Aharoni et al. [1, Conjecture 1.6] conjectured that there always exists an independent set S of P such that $|S \cap V_j| \geqslant |V_j|/2 - 1$, with strict inequality holding for at least $m/2$ subsets V_j. We prove that we can actually remove one vertex from each V_j so that the remaining vertices can be fairly split into two independent sets of P of almost same size.

Theorem 1. *Given a path P whose vertex set is partitioned into m subsets $V_1, \ldots, V_m$, there always exist two disjoint independent sets S_1 and S_2 covering all vertices but one in each V_j, with sizes differing by at most one, and satisfying for each $i \in \{1, 2\}$*

$$|S_i \cap V_j| \geqslant \frac{|V_j|}{2} - 1 \qquad \textit{for all } j \in [m].$$

Theorem 1 implies in particular that the conjecture by Aharoni et al. is true: the equality $|S_1 \cap V_j| + |S_2 \cap V_j| = |V_j| - 1$ holds for every j and thus one of S_1 or S_2 satisfies the inequality strictly for at least $m/2$ indices j.

The Borsuk-Ulam theorem was originally used for proving a special case of their conjecture (Theorem 1.7 of their paper). Here, we use the octahedral Tucker lemma, which is a combinatorial version of the Borsuk-Ulam theorem.

Another result in combinatorics deals with the fair splitting of a path whose vertices are colored and is a consequence of the Borsuk-Ulam theorem: the splitting necklace theorem. Consider an open necklace of n beads, each having a color $j \in [m]$. We denote by a_j the number of beads of color j. A *fair q-splitting* is the partition of the beads into q parts, each containing $\lfloor a_j/q \rfloor$ or $\lceil a_j/q \rceil$ beads of color j. The picturesque motivation is the division of the necklace among q thieves after its robbery. A theorem of Alon [2] states that such a partition is always achievable with no more than $(q-1)m$ cuts when a_j is divisible by q. Using a flow-based rounding argument, Alon, Moshkovitz, and Safra [3] were able to show that such a partition is achievable without this assumption. In a paper in which he proved the splitting necklace theorem when $q = 2$ via the octahedral Tucker lemma, Pálvölgyi [6] conjectured that for each j such that a_j is not divisible by q, it is possible to decide which thieves get $\lfloor a_j/q \rfloor$ and which get $\lceil a_j/q \rceil$ and to still have a fair q-splitting not requiring more than $(q-1)m$ cuts. The conjecture is known to be true when $q = 2$ (see [6]), when $a_j \leqslant q$ for all j (by a greedy assignment), and when $m = 2$ (see [5]). With a simple trick inspired by the argument of Alon, Moshkovitz, and Safra, we show that the conjecture is true when the remainder in the euclidian division of a_j by q is 0, 1, or $q-1$ for all j. This result implies in particular the conjecture for $q = 3$.

2 Fair splitting by independent sets of a path

2.1 Proof

The combinatorial counterpart of the Borsuk-Ulam theorem is Tucker's lemma. Our main tool is a special case of this counterpart when the triangulation is the first barycentric subdivision of the cross-polytope. It turns out that in this case, Tucker's lemma can be directly expressed in combinatorial terms. This kind of formulation goes back to Matoušek [4] and Ziegler [8].

As in oriented matroid theory, we define $\preceq$ to be the following partial order on $\{+, -, 0\}$:

$$0 \preceq +, \quad 0 \preceq -, \quad + \text{ and } - \text{ are not comparable.}$$

We extend it for sign vectors by simply taking the product order: for $\boldsymbol{x}, \boldsymbol{y} \in \{+,-,0\}^n$, we have $\boldsymbol{x} \preceq \boldsymbol{y}$ if the following implication holds for every $i \in [n]$

$$x_i \neq 0 \Longrightarrow x_i = y_i.$$

Lemma 2 (Octahedral Tucker lemma)**.** *Let s and n be positive integers. If there exists a map $\lambda : \{+,-,0\}^n \setminus \{\mathbf{0}\} \to \{\pm 1, \ldots, \pm s\}$ satisfying $\lambda(-\boldsymbol{x}) = -\lambda(\boldsymbol{x})$ for all $\boldsymbol{x}$ and $\lambda(\boldsymbol{x}) + \lambda(\boldsymbol{y}) \neq 0$ when $\boldsymbol{x} \preceq \boldsymbol{y}$, then $s \geqslant n$.*

In the proof, we use the following notations for $\boldsymbol{x} \in \{+,-,0\}^n$:

$$\boldsymbol{x}^+ = \{i \in [n] \colon x_i = +\} \qquad \text{and} \qquad \boldsymbol{x}^- = \{i \in [n] \colon x_i = -\}.$$

Note that $\boldsymbol{x} \preceq \boldsymbol{y}$ if and only if simultaneously $\boldsymbol{x}^+ \subseteq \boldsymbol{y}^+$ and $\boldsymbol{x}^- \subseteq \boldsymbol{y}^-$. We also use the notion of alternating sequences. A sequence of elements in $\{+,-,0\}^n$ is *alternating* if all terms are nonzero and any two consecutive terms are different. Given an $\boldsymbol{x} = (x_1, \ldots, x_n) \in \{+,-,0\}^n$, we denote by $\operatorname{alt}(\boldsymbol{x})$ the maximum length of an alternating subsequence of $x_1, \ldots, x_n$.

Proof of Theorem 1. The proof consists in applying Lemma 2 on a map λ we define now. Let n be the number of vertices of P. Without loss of generality, we assume that the vertices of P are $1, 2, \ldots, n$ in this order when one goes from one endpoint to the other. In the definition of λ, we use the quantity $t = \max\left\{ \operatorname{alt}(\boldsymbol{x}) \colon \boldsymbol{x} \in \{+,-,0\}^n \text{ s.t. } J(\boldsymbol{x}) = \varnothing \right\}$, where

$$J(\boldsymbol{x}) = \Big\{ j \in [m] \colon |\boldsymbol{x}^+ \cap V_j| = |\boldsymbol{x}^- \cap V_j| = |V_j|/2 \quad \text{or} \quad \max(|\boldsymbol{x}^+ \cap V_j|, |\boldsymbol{x}^- \cap V_j|) > |V_j|/2 \Big\}.$$

Note that $t \geqslant 0$.

Consider a vector $\boldsymbol{x} \in \{+,-,0\}^n \setminus \{\mathbf{0}\}$. We distinguish two cases. In the case where $J(\boldsymbol{x}) \neq \varnothing$, we set $\lambda(\boldsymbol{x}) = \pm(t + j')$, where j' is the maximum element in $J(\boldsymbol{x})$ and where the sign is defined as follows. When $|\boldsymbol{x}^+ \cap V_{j'}| = |\boldsymbol{x}^- \cap V_{j'}| = |V_{j'}|/2$, the sign is $+$ if $\min(\boldsymbol{x}^+ \cap V_{j'}) < \min(\boldsymbol{x}^- \cap V_{j'})$ and $-$ otherwise. When $\max\left(|\boldsymbol{x}^+ \cap V_{j'}|, |\boldsymbol{x}^- \cap V_{j'}|\right) > |V_{j'}|/2$, the sign is $+$ if $|\boldsymbol{x}^+ \cap V_{j'}| > |V_{j'}|/2$, and $-$ otherwise. In the case where $J(\boldsymbol{x}) = \varnothing$, we set $\lambda(\boldsymbol{x}) = \pm \operatorname{alt}(\boldsymbol{x})$, where the sign is the first nonzero element of $\boldsymbol{x}$.

Let us check that the map λ satisfies the condition of Lemma 2. Consider $\boldsymbol{x} \in \{+,-,0\}^n \setminus \{\mathbf{0}\}$. The relation $J(-\boldsymbol{x}) = J(\boldsymbol{x})$ immediately implies $\lambda(-\boldsymbol{x}) = -\lambda(\boldsymbol{x})$. Now, consider $\boldsymbol{x}, \boldsymbol{y} \in \{+,-,0\}^n \setminus \{\mathbf{0}\}$ such that $\boldsymbol{x} \preceq \boldsymbol{y}$ and $|\lambda(\boldsymbol{x})| = |\lambda(\boldsymbol{y})|$. We cannot have simultaneously $J(\boldsymbol{x}) = \varnothing$ and $J(\boldsymbol{y}) \neq \varnothing$ since then $|\lambda(\boldsymbol{x})| \leqslant t$ and $|\lambda(\boldsymbol{y})| > t$. Suppose first that $J(\boldsymbol{x}) \neq \varnothing$. Since $\boldsymbol{x}^+ \subseteq \boldsymbol{y}^+$ and $\boldsymbol{x}^- \subseteq \boldsymbol{y}^-$, the signs of $\lambda(\boldsymbol{x})$ and $\lambda(\boldsymbol{y})$ are the same. Suppose now $J(\boldsymbol{y}) = \varnothing$. Then $J(\boldsymbol{x}) = \varnothing$. In this case we have $\operatorname{alt}(\boldsymbol{x}) = \operatorname{alt}(\boldsymbol{y})$, and it is simple to check (and well-known) that the first nonzero coordinates of $\boldsymbol{x}$ and $\boldsymbol{y}$ have the same value.

We can thus apply Lemma 2 with $s = t + m$. It gives $t + m \geqslant n$, which implies that there exists $\boldsymbol{z}' \in \{+,-,0\}^n$ such that $J(\boldsymbol{z}') = \varnothing$ and $\operatorname{alt}(\boldsymbol{z}') \geqslant n - m$, which in turn implies that there exists $\boldsymbol{z} \in \{+,-,0\}^n$ such that $J(\boldsymbol{z}) = \varnothing$ and $\operatorname{alt}(\boldsymbol{z}) = |\boldsymbol{z}^+| + |\boldsymbol{z}^-| = n - m$.

Let $S_1 = \boldsymbol{z}^+$ and $S_2 = \boldsymbol{z}^-$. They are both independent sets of P and their sizes differ by at most one. Because $J(\boldsymbol{z}) = \varnothing$, we have $|S_1 \cap V_j| + |S_2 \cap V_j| \leqslant |V_j| - 1$ for all j. The fact that $|S_1| + |S_2| = n - m$ leads then to $|S_1 \cap V_j| + |S_2 \cap V_j| = |V_j| - 1$ for all j. Now, using again $J(\boldsymbol{z}) = \varnothing$, we have each of $|S_1 \cap V_j|$ and $|S_2 \cap V_j|$ non-larger than $|V_j|/2$, which leads directly to the inequality of the statement. □

The proof shows that S_1 and S_2 alternate along the path P. Theorem 1 combined with this remark leads to the following corollary, which improves Theorem 1.8 in the aforementioned paper by Aharoni et al. In particular, if m and the number of vertices of P have the same parity, replacing "path" by "cycle" in the statement of Theorem 1 does not change the conclusion.

Corollary 3. *Given an n-cycle C whose vertex set is partitioned into m subsets $V_1, \ldots, V_m$, there always exist two disjoint sets S_1 and S_2 covering all vertices but one in each V_j, with sizes differing by at most one, and satisfying for each $i \in \{1, 2\}$*

$$|S_i \cap V_j| \geqslant \frac{|V_j|}{2} - 1 \qquad \text{for all } j \in [m],$$

where one of the S_i's is an independent set of size $\left\lfloor \frac{n-m}{2} \right\rfloor$ and the other induces at most $\left\lceil \frac{n-m}{2} \right\rceil - \left\lfloor \frac{n-m}{2} \right\rfloor$ edge.

Proof. Remove an arbitrary edge of the cycle C and apply Theorem 1 to the path P we obtain this way. When $n - m$ is even, the fact that S_1 and S_2 alternate implies that they are independent sets of C as well, with the same property as for the case of a path. When $n - m$ is odd, one of S_1 and S_2 is independent and of size $\left\lfloor \frac{n-m}{2} \right\rfloor$ and the other is of size $\left\lceil \frac{n-m}{2} \right\rceil$ and may contain the two endpoints of P, but the other properties are kept. □

2.2 Extension to arbitrary numbers of independent sets

There is a generalization of the octahedral Tucker lemma – the $\mathbb{Z}_p$-Tucker lemma [8] – which deals with an arbitrary prime number p of signs, instead of simply two signs '+' and '−'. Using this generalization in place of the octahedral Tucker lemma in the proof of Theorem 1 leads quite easily to a generalization of Theorem 1 involving a number p of 2-stable sets. However, it seems that there is no simple way to controle their sizes. We conjecture actually that something stronger holds. In a graph, a subset of vertices is *q-stable* if no two of them are at distance less than q, where the distance is counted in terms of edges. In particular, the 2-stable sets of a graph are precisely its independent sets.

Conjecture 4. Given a positive integer q and a path P whose vertex set is partitioned into m subsets $V_1, \ldots, V_m$ of sizes at least $q - 1$, there always exist pairwise disjoint q-stable sets $S_1, \ldots, S_q$ covering all vertices but $q - 1$ in each V_j, with sizes differing by at most one, and satisfying

$$|S_i \cap V_j| \geqslant \left\lfloor \frac{|V_j| + 1}{q} \right\rfloor - 1$$

for all $i \in [q]$ and all $j \in [m]$.

This conjecture is obviously true for $q = 1$ and Theorem 1 is the special case where $q = 2$. The way Theorem 1 is written might suggest a lower bound of the form $|V_j|/q - 1$, but there are simple counterexamples. Consider for instance a case with $q = 3$ and $|V_1| = 7$. If there were pairwise disjoint 3-stable sets S_1, S_2, S_3 covering all vertices of V_1 but two, with each $|S_i \cap V_1|$ of size at least $7/3 - 1 = 1.33$, we would have $|V_1| - 2 = |S_1 \cap V_1| + |S_2 \cap V_1| + |S_3 \cap V_1| \geqslant 6$, a contradiction.

The independent sets S_i of Theorem 1 satisfy automatically the additional inequality $|S_i \cap V_j| \leqslant |V_j|/2$ for all j, and it is actually used in the proof itself of that theorem. We believe that the stronger version of Conjecture 4 with an upper bound of $|V_j|/q$ on $|S_i \cap V_j|$ for all i and j is true.

Since Conjecture 4 is true for $q \in \{1, 2\}$, the following proposition implies that it is true for any power of two. The other cases remain open.

Proposition 5. *If Conjecture 4 holds for both q' and q'', then it holds also for $q = q'q''$.*

The proof uses extensively the relations

$$\left\lfloor \frac{1}{c} \left\lfloor \frac{a}{b} \right\rfloor \right\rfloor = \left\lfloor \frac{a}{bc} \right\rfloor \qquad \text{and} \qquad \left\lceil \frac{1}{c} \left\lceil \frac{a}{b} \right\rceil \right\rceil = \left\lceil \frac{a}{bc} \right\rceil \tag{1}$$

that hold for any $a, b, c \in \mathbb{Z}$ (actually, only $c \in \mathbb{Z}$ is required for these relations to hold).

Let us prove them. We have $\left\lfloor \frac{1}{c} \left\lfloor \frac{a}{b} \right\rfloor \right\rfloor \leqslant \frac{1}{c} \left\lfloor \frac{a}{b} \right\rfloor \leqslant \frac{a}{bc}$, and thus $\left\lfloor \frac{1}{c} \left\lfloor \frac{a}{b} \right\rfloor \right\rfloor \leqslant \left\lfloor \frac{a}{bc} \right\rfloor$. We also have $\frac{a}{bc} \geqslant \left\lfloor \frac{a}{bc} \right\rfloor$ and thus $\frac{a}{b} \geqslant c \left\lfloor \frac{a}{bc} \right\rfloor$, which implies since $c \in \mathbb{Z}$ that $\left\lfloor \frac{a}{b} \right\rfloor \geqslant c \left\lfloor \frac{a}{bc} \right\rfloor$. Therefore $\left\lfloor \frac{1}{c} \left\lfloor \frac{a}{b} \right\rfloor \right\rfloor \geqslant \left\lfloor \frac{a}{bc} \right\rfloor$, which implies the left equality. The right one is then immediate since $-\lfloor -x \rfloor = \lceil x \rceil$ for all $x \in \mathbb{R}$.

Proof of Proposition 5. Consider a path P whose vertices are partitioned into m subsets $V_1, \ldots, V_m$ of sizes at least $q - 1$. We assume that the conjecture is true for q' and q''. We aim at proving that there exist pairwise disjoint q-stable sets $S_1, \ldots, S_q$ satisfying the conclusion of Conjecture 4.

Since the conjecture is assumed to be true for q', there exist pairwise disjoint q'-stable sets $T_1, \ldots, T_{q'}$ of P, covering all vertices but $q' - 1$ of each V_j and such that for each $i \in [q']$

$$|T_i \cap V_j| \geqslant \left\lfloor \frac{|V_j| + 1}{q'} \right\rfloor - 1 \quad \text{for all } j \in [m], \quad \text{and}$$

$$\left\lceil \frac{n - (q' - 1)m}{q'} \right\rceil \geqslant |T_i| \geqslant \left\lfloor \frac{n - (q' - 1)m}{q'} \right\rfloor,$$

where n is the number of vertices of P. The conjecture being also assumed to be true for q'', we apply it on the path Q_i whose vertices are the elements of T_i (in the relative positions they have on P), for each $i \in [q']$. Note that $|T_i \cap V_j| \geqslant q'' - 1$ for every j. Therefore, for each i, there are pairwise disjoint q''-stable sets $S_{i1}, \ldots, S_{iq''}$ of Q_i covering all vertices but $q''-1$ of each $T_i \cap V_j$ and such that for each $k \in [q'']$

$$|S_{ik} \cap V_j| \geqslant \left\lfloor \frac{|T_i \cap V_j|+1}{q''} \right\rfloor - 1 \quad \text{for all } j \in [m], \quad \text{and}$$

$$\left\lceil \frac{|T_i|-(q''-1)m}{q''} \right\rceil \geqslant |S_{ik}| \geqslant \left\lfloor \frac{|T_i|-(q''-1)m}{q''} \right\rfloor.$$

Using the relation (1), we get directly that each of the $q = q'q''$ subsets S_{ik} satisfies

$$|S_{ik} \cap V_j| \geqslant \left\lfloor \frac{|V_j|+1}{q} \right\rfloor - 1 \quad \text{for all } j \in [m], \quad \text{and}$$

$$\left\lceil \frac{n-(q-1)m}{q} \right\rceil \geqslant |S_{ik}| \geqslant \left\lfloor \frac{n-(q-1)m}{q} \right\rfloor.$$

For each V_j, the number of uncovered vertices is exactly $q'-1+q'(q''-1) = q-1$. Moreover, each T_i is q'-stable for P and each S_{ik} is q''-stable for Q_i. Thus each S_{ik} is q-stable for P and this finishes the proof. □

The proof of Proposition 5 can be adapted in a straightforward way to get that it is also true for the aforementioned version of Conjecture 4 with the upper bounds on the $|S_i \cap V_j|$'s. We get thus that this stronger conjecture is also true for any power of two.

3 Fair splitting of the necklace with advantages

The result we prove in this section is the following one. We denote by r_j the remainder of the euclidian division of a_j by q.

Theorem 6. *When $r_j \in \{0, 1, q-1\}$ for all j, then it is possible to choose for each j such that $r_j \neq 0$, the thieves who get an additional bead of color j and still have a fair q-splitting not requiring more than $(q-1)m$ cuts.*

The following corollary is an immediate consequence of the previous theorem (and answered positively what was identified as a first interesting question by Pálvölgyi).

Corollary 7. *When there are three thieves, it is possible to choose for each j such that $r_j \neq 0$, the thief (if $r_j = 1$) or the two thieves (if $r_j = 2$) who get an additional bead and still have a fair 3-splitting not requiring more than $2m$ cuts.*

It is proved by rounding in an appropriate way a fair splitting obtained by a continuous version of the splitting necklace theorem.

While in the splitting necklace theorem the cuts have to take place between the beads, this condition is relaxed in the "continuous version" of the splitting necklace theorem, also proved by Alon [2]. In this latter version, cuts are allowed to be located on beads, and thieves may then receive fractions of beads. In this case, there always exists a *continuous* *fair q-splitting* for which each thief receives an amount of exactly a_j/q beads of color j, with no more than $(q-1)m$ cuts.

Consider a continuous fair q-splitting and denote by B_j the beads of color j that are split between two or more thieves. Our result is obtained by showing that we can move the cuts located on beads in the B_j's so that we reach a "discrete" fair q-splitting with the desired allocation. If $B_j = \varnothing$, then we already have a discrete fair splitting for the beads of color j. For each j with $B_j \neq \varnothing$, we build a bipartite graph $G_j = (U_j, E_j)$ with the thieves on one side and the beads in B_j on the other side. We put an edge between a thief t and a bead k if t receives a part of k. For an edge $e = tk \in E_j$, let $u_e \in (0,1)$ be the amount of bead k received by thief t. We have for all $k \in B_j$ and all $t \in [q]$ (we identify the thieves with the integers in $[q]$)

$$\sum_{e \in E_j} u_e = |B_j|, \qquad \sum_{e \in \delta(k)} u_e = 1, \quad \text{and} \quad \sum_{e \in \delta(t)} u_e = \alpha_{tj} + \frac{r_j}{q} \quad \text{for some integer } \alpha_{tj} \geqslant 0, \tag{2}$$

where $\delta(v)$ is the set of edges incident to a vertex v. Note that the degree of each thief-vertex t in G_j is at least $\alpha_{tj}+1$ and the degree of each bead-vertex is at least 2.

Changing the values of the u_e's, while keeping them nonnegative and while satisfying the equalities (2), leads to another continuous fair q-splitting with at most $(q-1)m$ cuts. The u_e's form a flow. It is thus always possible to choose the continuous fair q-splitting in such a way that G_j has no cycle for every j (basic properties of flows). In the proofs below, G_j will therefore always be assumed to be without cycle. To get our result, we are going to select a subset F of E_j such that each bead-vertex is incident to exactly one edge in F. This subset of edges will encode an assignment of the beads in B_j compatible with the already assigned beads (which does not increase the number of cuts), and leading to the desired allocation.

For the proof of the case $r_j = 1$, such a subset of edges is obtained as a special object of graph theory that we describe now. Let $H = (V, E)$ be a bipartite graph and let $b : V \to \mathbb{Z}_+$. A *b-factor* is a subset $F \subseteq E$ such that each vertex $v \in V$ is incident to exactly $b(v)$ edges of F. There exists a b-factor if and only if each subset X of V spans at least $\sum_{v \in X} b(v) - \frac{1}{2}\sum_{v \in V} b(v)$ edges, see [7, Corollary 21.4a].

Lemma 8. *When $r_j = 1$, it is possible to move the cuts located on the beads of color j and to get a discrete fair q-splitting for which we choose the thief getting the additional bead of color j.*

Proof. We have in this case $|B_j| = \sum_{t \in [q]} \alpha_{tj} + 1$ (using (2)). For a thief t, define $b(t) = \alpha_{tj}$, except when t is the thief chosen for the additional bead, in which case define $b(t) = \alpha_{tj}+1$. For each bead k, define $b(k) = 1$. Consider a subset X of vertices of G_j. Denote by T the thief-vertices in X and by K the bead-vertices in X. The edges spanned by X is $\delta(T) \setminus E[T : B_j \setminus K]$. We have $|\delta(T)| \geqslant \sum_{t \in T}(\alpha_{tj}+1)$ and $|E[T : B_j \setminus K]| \leqslant |T| + |B_j| - |K| - 1$. To get this latter inequality, we use the fact that G_j has no cycle. The number of edges spanned by X is thus at least $|K| - |B_j| + 1 + \sum_{t \in T} \alpha_{tj}$.

The quantity $\sum_{v \in X} b(v) - \frac{1}{2}\sum_{v \in U_j} b(v)$ is at most $1 + \sum_{t \in T} \alpha_{tj} + |K| - |B_j|$. According to the above mentioned result, there exists a b-factor in G_j. □

Lemma 9. *When $r_j = q-1$, it is possible to move the cuts located on the beads of color j and to get a discrete fair q-splitting for which we choose the thief getting one bead of color j less than the other thieves.*

Proof. Since G_j is without cycle, its number of edges is at most $q-1+|B_j|$. The degree of each vertex in B_j being at least 2, it implies that $q-1+|B_j| \geqslant 2|B_j|$, and thus $|B_j| \leqslant q-1$. The fact that $r_j = q-1$ leads finally to $|B_j| = q-1$ (using (2)), which implies that $\alpha_{tj} = 0$ for all $t \in [q]$. For any proper subset $T \subset [q]$ of distinct thieves, we have thus $\sum_{t \in T} \sum_{e \in \delta(t)} u_e = |T| - \frac{|T|}{q}$, which means that the size of the neighborhood of T in G_j is at least $\left\lceil |T| - \frac{|T|}{q} \right\rceil = |T|$. This latter equality holds because $|T| \leqslant q-1$. Hall's theorem ensures then that we can assign the $q-1$ beads in B_j to any choice of $q-1$ thieves. □

Proof of Theorem 6. For the colors j such that $r_j = 0$, the original rounding procedure introduced by Alon, Moshkovitz, and Safra makes the job. For the colors j such that $r_j \in \{1, q-1\}$, Lemmas 8 and 9 allow to conclude. □

Note that this approach may fail already when $q = 4$ and $r_j = 2$: if thieves a and b receive each half of a bead and thieves c and d receive each half of another bead, it is impossible to move the cuts so that both a and b are advantaged.

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