Nice restrictions of reflection arrangements

Tilman Möller
Fakultät für Mathematik
Ruhr-Universität Bochum
D-44780 Bochum, Germany
tilman.moeller@rub.de

Gerhard Röhrle *
Fakultät für Mathematik
Ruhr-Universität Bochum
D-44780 Bochum, Germany
gerhard.roehrle@rub.de

Submitted: Sep 22, 2016; Accepted: Aug 28, 2017; Published: Sep 8, 2017
Mathematics Subject Classifications: 20F55, 52B30, 52C35, 14N20

Abstract

In [5], Hoge and the second author classified all nice and all inductively factored reflection arrangements. In this note we extend this classification by determining all nice and all inductively factored restrictions of reflection arrangements.

Keywords: hyperplane arrangements, complex reflection groups, restricted arrangements, nice arrangement, inductively factored arrangement

1 Introduction

The notion of a nice (or factored) arrangement is due to Terao [18]. This class generalizes the class of supersolvable arrangements, [12] (cf. [13, Thm. 3.81]). There is an inductive version of this notion, so called inductively factored arrangements, see Definition 2.12. This inductive class (properly) contains the class of supersolvable arrangements and is (properly) contained in the class of inductively free arrangements, see [4, Rem. 3.33].

For an overview on properties of nice and inductively factored arrangements, and for their connection with the Orlik-Solomon algebra, see [13, §3], [7], and [4]. In [4], Hoge and the second author proved an addition-deletion theorem for nice arrangements, see Theorem 2.11 below. This is an analogue of Terao's celebrated addition-deletion theorem [17] for free arrangements for the class of nice arrangements.

In [5], Hoge and the second author classified all nice and all inductively factored reflection arrangements. Extending this earlier work, in this note we classify all nice and all inductively factored restrictions $\mathscr{A}^X$, for $\mathscr{A}$ a reflection arrangement and X in the intersection lattice $L(\mathscr{A})$ of $\mathscr{A}$, see Theorems 1.5 and 1.6. If $\mathscr{A}^X$ is inductively factored for every $X \in L(\mathscr{A})$, then $\mathscr{A}$ is called hereditarily inductively factored, see Definition 2.17.

*Supported by DFG-priority program SPP1489 "Algorithmic and Experimental Methods in Algebra, Geometry, and Number Theory".

In order to state our main results, we need a bit more notation: For fixed $r, \ell \geqslant 2$ and $0 \leqslant k \leqslant \ell$ we denote by $\mathscr{A}_\ell^k(r)$ the intermediate arrangements, defined by Orlik and Solomon in [11, §2] (see also [13, §6.4]), that interpolate between the reflection arrangements $\mathscr{A}(G(r,r,\ell)) = \mathscr{A}_\ell^0(r)$ and $\mathscr{A}(G(r,1,\ell)) = \mathscr{A}_\ell^\ell(r)$, of the monomial groups $G(r,r,\ell)$ and $G(r,1,\ell)$, respectively. The arrangements $\mathscr{A}_\ell^k(r)$ are relevant for us, as they occur as restrictions of $\mathscr{A}(G(r,r,\ell))$, [11, Prop. 2.14] (cf. [13, Prop. 6.84]). For $k \neq 0, \ell$, these are not reflection arrangements themselves. See Section 3 for further details.

Suppose that W is a finite, unitary reflection group acting on the complex vector space V. Let $\mathscr{A}(W) = (\mathscr{A}(W), V)$ be the associated hyperplane arrangement of W. We refer to $\mathscr{A}(W)$ as a *reflection arrangement*. Thanks to Proposition 2.10, the question whether $\mathscr{A}$ is nice reduces to the case when $\mathscr{A}$ is irreducible. Therefore, we may assume that W is irreducible. First we recall the classification results from [5].

Theorem 1.1 ([5, Thm. 1.3, Thm. 1.5]). *Let W be a finite, irreducible, complex reflection group with reflection arrangement $\mathscr{A} = \mathscr{A}(W)$. Then we have the following:*

(i) $\mathscr{A}$ is nice if and only if either $\mathscr{A}$ is supersolvable or $W = G(r,r,3)$ for $r \geqslant 3$.

(ii) $\mathscr{A}$ is factored if and only if $\mathscr{A}$ is hereditarily factored.

Thanks to Proposition 2.13, the question whether $\mathscr{A}$ is inductively factored reduces to the case when $\mathscr{A}$ is irreducible.

Theorem 1.2 ([5, Cor. 1.4, Cor. 1.6]). *Let W be a finite, irreducible, complex reflection group with reflection arrangement $\mathscr{A} = \mathscr{A}(W)$. Then we have the following:*

(i) $\mathscr{A}$ is inductively factored if and only if it is supersolvable.

(ii) $\mathscr{A}$ is inductively factored if and only if $\mathscr{A}$ is hereditarily inductively factored.

Terao [18] showed that every supersolvable arrangement is factored. Indeed, every supersolvable arrangement is inductively factored, see Proposition 2.14. Moreover, Jambu and Paris showed that each inductively factored arrangement is inductively free, see Proposition 2.15. Each of these classes of arrangements is properly contained in the other, see [4, Rem. 3.33].

In view of these proper containments, we first recall the classifications of the inductively free and the supersolvable restrictions of reflection arrangements, from [1] and [2], respectively, as they give an indication of the kind of results to be expected. Here and later on we use the classification and labeling of the irreducible unitary reflection groups due to Shephard and Todd, [14].

Theorem 1.3 ([1, Thm. 1.2]). *Let W be a finite, irreducible, complex reflection group with reflection arrangement $\mathscr{A} = \mathscr{A}(W)$ and let $X \in L(\mathscr{A})$. The restricted arrangement $\mathscr{A}^X$ is inductively free if and only if one of the following holds:*

(i) $\mathscr{A}$ is inductively free;

(ii) $W = G(r, r, \ell)$ *and* $\mathscr{A}^X \cong \mathscr{A}_p^k(r)$, *where* $p = \dim X$ *and* $p - 2 \leqslant k \leqslant p$;

(iii) W *is one of* $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}$, *or* G_{34} *and* $X \in L(\mathscr{A}) \setminus \{V\}$ *with* $\dim X \leqslant 3$.

Note that Proposition 2.15 and Theorem 1.3(iii) imply that for W an irreducible, complex reflection group of exceptional type, for W as in Theorem 1.3(iii), for $\mathscr{A} = \mathscr{A}(W)$ and $X \in L(\mathscr{A})$ with $\dim X \geqslant 4$, the restricted arrangement $\mathscr{A}^X$ is not inductively factored.

Theorem 1.4 ([2, Thm. 1.3])**.** *Let W be a finite, irreducible, complex reflection group with reflection arrangement $\mathscr{A} = \mathscr{A}(W)$ and let $X \in L(\mathscr{A})$ with $\dim X \geqslant 3$. Then the restricted arrangement $\mathscr{A}^X$ is supersolvable if and only if one of the following holds:*

(i) $\mathscr{A}$ *is supersolvable;*

(ii) $W = G(r, r, \ell)$ *and* $\mathscr{A}^X \cong \mathscr{A}_p^p(r)$ *or* $\mathscr{A}_p^{p-1}(r)$, *where* $p = \dim X$;

(iii) $\mathscr{A}^X$ *is* (E_6, A_3), (E_7, D_4), (E_7, A_2^2), *or* (E_8, A_5).

In part (iii) of the theorem and later on we use the convention to label the W-orbit of $X \in L(\mathscr{A})$ by the type T which is the Shephard-Todd label [14] of the complex reflection group W_X. We then denote the restriction $\mathscr{A}^X$ simply by the pair (W, T).

Note that thanks to Proposition 2.14, every supersolvable restriction from Theorem 1.4 is also inductively factored.

Thanks to the compatibility of nice arrangements and inductively factored arrangements with the product construction for arrangements, see Propositions 2.10 and 2.13, as well as by the product rule (2.2) for restrictions in products, the question whether the restrictions $\mathscr{A}^X$ are nice or inductively factored reduces readily to the case when $\mathscr{A}$ is irreducible. Thus we may assume that W is irreducible. We can formulate our classification as follows:

Theorem 1.5. *Let W be a finite, irreducible, complex reflection group with reflection arrangement $\mathscr{A} = \mathscr{A}(W)$ and let $X \in L(\mathscr{A}) \setminus \{V\}$. The restricted arrangement $\mathscr{A}^X$ is nice if and only if one of the following holds:*

(i) $\mathscr{A}^X$ *is supersolvable;*

(ii) $\mathscr{A}$ *is nice;*

(iii) $W = G(r, r, \ell)$, $\ell \geqslant 4$ *and* $\mathscr{A}^X \cong \mathscr{A}_p^{p-2}(r)$, *where* $p = \dim X$;

(iv) $\mathscr{A}^X$ *is one of* $(E_6, A_1A_2), (E_7, A_4)$, *or* $(E_7, (A_1A_3)'')$.

Note that (E_6, A_1A_2) and (E_7, A_4) in (iv) above are isomorphic, see Lemma 4.1(iv).

In contrast to the situation for the full reflection arrangements (Theorems 1.1 and 1.2), the notions of niceness and inductive factoredness coincide for their restricted counterparts.

Theorem 1.6. *Let W be a finite, irreducible, complex reflection group with reflection arrangement $\mathscr{A} = \mathscr{A}(W)$ and let $X \in L(\mathscr{A}) \setminus \{V\}$. The restricted arrangement $\mathscr{A}^X$ is inductively factored if and only if it is factored.*

We also extend both theorems to the corresponding hereditary subclasses.

Theorem 1.7. *Let W be a finite, irreducible, complex reflection group with reflection arrangement $\mathscr{A} = \mathscr{A}(W)$ and let $X \in L(\mathscr{A}) \setminus \{V\}$. Then the restricted arrangement $\mathscr{A}^X$ is (inductively) factored if and only if it is hereditarily (inductively) factored.*

While Theorem 1.2(i) shows that the class of inductively factored reflection arrangements coincides with the class of supersolvable reflection arrangements, in contrast, Theorems 1.4, 1.5, and 1.6 show that the class of inductively factored restrictions of reflection arrangements properly contains the class consisting of supersolvable restrictions of reflection arrangements.

The paper is organized as follows. In the next section we recall the required notions and relevant properties of free, inductively free, supersolvable and nice arrangements mostly taken from [13], [18] and [4].

In Section 3 we classify all nice and all inductively factored cases among the intermediate arrangements $\mathscr{A}_\ell^k(r)$, and complete the proofs of Theorems 1.5 – 1.7 in Section 4.

For general information about arrangements we refer the reader to [13].

2 Recollections and Preliminaries

2.1 Hyperplane arrangements

Let $\mathbb{K}$ be a field and let $V = \mathbb{K}^\ell$ be an ℓ-dimensional $\mathbb{K}$-vector space. A *(central) hyperplane arrangement* $\mathscr{A}$ in V is a finite collection of hyperplanes in V each containing the origin of V. We also use the term ℓ-arrangement for $\mathscr{A}$. The empty ℓ-arrangement is denoted by Φ_ℓ.

The *lattice* $L(\mathscr{A})$ of $\mathscr{A}$ is the set of subspaces of V of the form $H_1 \cap \cdots \cap H_i$ where $\{H_1, \ldots, H_i\}$ is a subset of $\mathscr{A}$. For $X \in L(\mathscr{A})$, we have two associated arrangements, firstly $\mathscr{A}_X := \{H \in \mathscr{A} \mid X \subseteq H\} \subseteq \mathscr{A}$, the *localization of $\mathscr{A}$ at X*, and secondly, the *restriction of $\mathscr{A}$ to X*, $(\mathscr{A}^X, X)$, where $\mathscr{A}^X := \{X \cap H \mid H \in \mathscr{A} \setminus \mathscr{A}_X\}$. Note that V belongs to $L(\mathscr{A})$ as the intersection of the empty collection of hyperplanes and $\mathscr{A}^V = \mathscr{A}$. The lattice $L(\mathscr{A})$ is a partially ordered set by reverse inclusion: $X \leqslant Y$ provided $Y \subseteq X$ for $X, Y \in L(\mathscr{A})$.

If $0 \in H$ for each H in $\mathscr{A}$, then $\mathscr{A}$ is called *central.* If $\mathscr{A}$ is central, then the *center* $T_{\mathscr{A}} := \cap_{H \in \mathscr{A}} H$ of $\mathscr{A}$ is the unique maximal element in $L(\mathscr{A})$ with respect to the partial order. Throughout, we only consider central arrangements.

The *product* $\mathscr{A} = (\mathscr{A}_1 \times \mathscr{A}_2, V_1 \oplus V_2)$ of two arrangements $(\mathscr{A}_1, V_1), (\mathscr{A}_2, V_2)$ is defined by

$$\mathscr{A} := \mathscr{A}_1 \times \mathscr{A}_2 = \{H_1 \oplus V_2 \mid H_1 \in \mathscr{A}_1\} \cup \{V_1 \oplus H_2 \mid H_2 \in \mathscr{A}_2\}, \tag{2.1}$$

see [13, Def. 2.13].

An arrangement $\mathscr{A}$ is called *reducible*, if it is of the form $\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2$, where $\mathscr{A}_i \neq \Phi_0$ for $i = 1, 2$, else $\mathscr{A}$ is *irreducible*, [13, Def. 2.15].

If $\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2$ is a product, then by [13, Prop. 2.14] there is a lattice isomorphism

$$L(\mathscr{A}_1) \times L(\mathscr{A}_2) \cong L(\mathscr{A}) \quad \text{by} \quad (X_1, X_2) \mapsto X_1 \oplus X_2.$$

With (2.1), it is easy to see that for $X = X_1 \oplus X_2 \in L(\mathscr{A})$, we have $\mathscr{A}_X = (\mathscr{A}_1)_{X_1} \times (\mathscr{A}_2)_{X_2}$ and

$$\mathscr{A}^X = \mathscr{A}_1^{X_1} \times \mathscr{A}_2^{X_2}. \tag{2.2}$$

2.2 Free hyperplane arrangements

Let $S = S(V^*)$ be the symmetric algebra of the dual space V^* of V. Let $\mathrm{Der}(S)$ be the S-module of $\mathbb{K}$-derivations of S. Since S is graded, $\mathrm{Der}(S)$ is a graded S-module.

Let $\mathscr{A}$ be an arrangement in V. Then for $H \in \mathscr{A}$ we fix $\alpha_H \in V^*$ with $H = \ker \alpha_H$. The *defining polynomial* $Q(\mathscr{A})$ of $\mathscr{A}$ is given by $Q(\mathscr{A}) := \prod_{H \in \mathscr{A}} \alpha_H \in S$. The *module of $\mathscr{A}$-derivations* of $\mathscr{A}$ is defined by

$$D(\mathscr{A}) := \{\theta \in \mathrm{Der}(S) \mid \theta(Q(\mathscr{A})) \in Q(\mathscr{A})S\}.$$

We say that $\mathscr{A}$ is *free* if $D(\mathscr{A})$ is a free S-module, cf. [13, §4].

If $\mathscr{A}$ is a free arrangement, then the S-module $D(\mathscr{A})$ admits a basis of n homogeneous derivations, say $\theta_1, \ldots, \theta_n$, [13, Prop. 4.18]. While the θ_i's are not unique, their polynomial degrees $\mathrm{pdeg}\, \theta_i$ are unique (up to ordering). This multiset is the set of *exponents* of the free arrangement $\mathscr{A}$ and is denoted by $\exp \mathscr{A}$.

Terao's celebrated *Addition-Deletion Theorem* plays a pivotal role in the study of free arrangements, [17], cf. [13, Thm. 4.51]. In turn this theorem motivates the stronger notion of *inductively free* arrangements, see [17] or [13, Def. 4.53].

2.3 Supersolvable arrangements

Let $\mathscr{A}$ be an arrangement. We say that $X \in L(\mathscr{A})$ is *modular* provided $X + Y \in L(\mathscr{A})$ for every $Y \in L(\mathscr{A})$, [13, Cor. 2.26].

Definition 2.3 ([15])**.** Let $\mathscr{A}$ be a central arrangement of rank r. We say that $\mathscr{A}$ is *supersolvable* provided there is a maximal chain

$$V = X_0 < X_1 < \ldots < X_{r-1} < X_r = T_{\mathscr{A}}$$

of modular elements X_i in $L(\mathscr{A})$, cf. [13, Def. 2.32].

The connection of this notion with freeness is due to Jambu and Terao.

Theorem 2.4 ([8, Thm. 4.2])**.** *A supersolvable arrangement is inductively free.*

2.4 Nice and inductively factored arrangements

The notion of a *nice* or *factored* arrangement goes back to Terao [18]. It generalizes the concept of a supersolvable arrangement, see [12, Thm. 5.3] and [13, Prop. 2.67, Thm. 3.81]. Terao's main motivation was to give a general combinatorial framework to deduce factorizations of the underlying Orlik-Solomon algebra, see also [13, §3.3]. We recall the relevant notions from [18] (cf. [13, §2.3]):

Definition 2.5. Let $\pi = (\pi_1, \dots, \pi_s)$ be a partition of $\mathscr{A}$.

(a) π is called *independent*, provided for any choice $H_i \in \pi_i$ for $1 \leqslant i \leqslant s$, the resulting s hyperplanes are linearly independent, i.e. $r(H_1 \cap \ldots \cap H_s) = s$.

(b) Let $X \in L(\mathscr{A})$. The *induced partition* π_X of $\mathscr{A}_X$ is given by the non-empty blocks of the form $\pi_i \cap \mathscr{A}_X$.

(c) π is *nice* for $\mathscr{A}$ or a *factorization* of $\mathscr{A}$ provided

   (i) π is independent, and

   (ii) for each $X \in L(\mathscr{A}) \setminus \{V\}$, the induced partition π_X admits a block which is a singleton.

If $\mathscr{A}$ admits a factorization, then we also say that $\mathscr{A}$ is *factored* or *nice*.

Remark 2.6. The class of nice arrangements is closed under taking localizations. For, if $\mathscr{A}$ is non-empty and π is a nice partition of $\mathscr{A}$, then the non-empty parts of the induced partition π_X form a nice partition of $\mathscr{A}_X$ for each $X \in L(\mathscr{A}) \setminus \{V\}$; cf. the proof of [18, Cor. 2.11].

The main motivation in [18] to introduce the notion of a nice partition was that it allows for a combinatorial characterization of tensor factorizations as a graded $\mathbb{K}$-algebra of the Orlik-Solomon algebra of an arrangement. We record a set of consequences of this result that are relevant for our purposes, see [18] (cf. [13, §3.3]).

Corollary 2.7. *Let $\pi = (\pi_1, \dots, \pi_s)$ be a nice partition of $\mathscr{A}$. Then the following hold:*

(i) $s = r = r(\mathscr{A})$ *and*

$$\operatorname{Poin}(\mathscr{A}, t) = \prod_{i=1}^{r} (1 + |\pi_i| t);$$

(ii) the multiset $\{|\pi_1|, \dots, |\pi_r|\}$ *only depends on* $\mathscr{A}$*;*

(iii) for any $X \in L(\mathscr{A})$*, we have*

$$r(X) = |\{i \mid \pi_i \cap \mathscr{A}_X \neq \varnothing\}|.$$

Remark 2.8. Suppose that $\mathscr{A}$ is free of rank r. Then $\mathscr{A} = \Phi_{\ell-r} \times \mathscr{A}_0$, where $\mathscr{A}_0$ is an essential, free r-arrangement (cf. [13, §3.2]), and so, $\exp \mathscr{A} = \{0^{\ell-r}, \exp \mathscr{A}_0\}$. Suppose that $\pi = (\pi_1, \dots, \pi_r)$ is a nice partition of $\mathscr{A}$. Then by the factorization properties of the Poincaré polynomials for free and factored arrangements, we have

$$\exp \mathscr{A} = \{0^{\ell-r}, |\pi_1|, \dots, |\pi_r|\}.$$

In particular, if $\mathscr{A}$ is essential, then

$$\exp \mathscr{A} = \{|\pi_1|, \dots, |\pi_\ell|\}.$$

Finally, we record [18, Ex. 2.4], which shows that nice arrangements generalize supersolvable ones (cf. [12, Thm. 5.3], [6, Prop. 3.2.2], [13, Prop. 2.67, Thm. 3.81]).

Proposition 2.9. *Let $\mathscr{A}$ be a central, supersolvable arrangement of rank r. Let*

$$V = X_0 < X_1 < \ldots < X_{r-1} < X_r = T_{\mathscr{A}}$$

be a maximal chain of modular elements in $L(\mathscr{A})$. Define $\pi_i = \mathscr{A}_{X_i} \setminus \mathscr{A}_{X_{i-1}}$ for $1 \leqslant i \leqslant r$. Then $\pi = (\pi_1, \dots, \pi_r)$ is a nice partition of $\mathscr{A}$.

In [4, Prop. 3.29], it was shown that the product construction behaves well with factorizations.

Proposition 2.10. *Let $\mathscr{A}_1, \mathscr{A}_2$ be two arrangements. Then $\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2$ is nice if and only if both $\mathscr{A}_1$ and $\mathscr{A}_2$ are nice.*

Following Jambu and Paris [7], we introduce further notation. Suppose $\mathscr{A}$ is not empty. Let $\pi = (\pi_1, \dots, \pi_s)$ be a partition of $\mathscr{A}$. Let $H_0 \in \pi_1$ and let $(\mathscr{A}, \mathscr{A}', \mathscr{A}'')$ be the triple associated with H_0. Then π induces a partition π' of $\mathscr{A}'$, i.e. the non-empty subsets $\pi_i \cap \mathscr{A}'$. Note that since $H_0 \in \pi_1$, we have $\pi_i \cap \mathscr{A}' = \pi_i$ for $i = 2, \dots, s$. Also, associated with π and H_0, we define the *restriction map*

$$\varrho := \varrho_{\pi, H_0} : \mathscr{A} \setminus \pi_1 \to \mathscr{A}'' \text{ given by } H \mapsto H \cap H_0$$

and set

$$\pi_i'' := \varrho(\pi_i) = \{H \cap H_0 \mid H \in \pi_i\} \text{ for } 2 \leqslant i \leqslant s.$$

In general, ϱ need not be surjective nor injective. However, since we are only concerned with cases when $\pi'' = (\pi_2'', \dots, \pi_s'')$ is a partition of $\mathscr{A}''$, ϱ has to be onto and $\varrho(\pi_i) \cap \varrho(\pi_j) = \varnothing$ for $i \neq j$.

The following analogue of Terao's Addition-Deletion Theorem for free arrangements for the class of nice arrangements is proved in [4, Thm. 3.5].

Theorem 2.11. *Suppose that $\mathscr{A} \neq \Phi_\ell$. Let $\pi = (\pi_1, \dots, \pi_s)$ be a partition of $\mathscr{A}$. Let $H_0 \in \pi_1$ and let $(\mathscr{A}, \mathscr{A}', \mathscr{A}'')$ be the triple associated with H_0. Then any two of the following statements imply the third:*

(i) π *is nice for* $\mathscr{A}$*;*

(ii) π' *is nice for* $\mathscr{A}'$*;*

(iii) $\varrho : \mathscr{A} \setminus \pi_1 \to \mathscr{A}''$ *is bijective and* π'' *is nice for* $\mathscr{A}''$*.*

Note the bijectivity condition on ϱ in Theorem 2.11 is necessary, cf. [4, Ex. 3.3]. Theorem 2.11 motivates the following stronger notion of factorization, cf. [7], [4, Def. 3.8].

Definition 2.12. The class $\mathcal{IFAC}$ of *inductively factored* arrangements is the smallest class of pairs $(\mathscr{A}, \pi)$ of arrangements $\mathscr{A}$ together with a partition π subject to

(i) $(\Phi_\ell, (\varnothing)) \in \mathcal{IFAC}$ for each $\ell \geqslant 0$;

(ii) if there exists a partition π of $\mathscr{A}$ and a hyperplane $H_0 \in \pi_1$ such that for the triple $(\mathscr{A}, \mathscr{A}', \mathscr{A}'')$ associated with H_0 the restriction map $\varrho = \varrho_{\pi,H_0} : \mathscr{A} \setminus \pi_1 \to \mathscr{A}''$ is bijective and for the induced partitions π' of $\mathscr{A}'$ and π'' of $\mathscr{A}''$ both $(\mathscr{A}', \pi')$ and $(\mathscr{A}'', \pi'')$ belong to $\mathcal{IFAC}$, then $(\mathscr{A}, \pi)$ also belongs to $\mathcal{IFAC}$.

If $(\mathscr{A}, \pi)$ is in $\mathcal{IFAC}$, then we say that $\mathscr{A}$ is *inductively factored with respect to* π, or else that π is an *inductive factorization* of $\mathscr{A}$. Frequently, we simply say $\mathscr{A}$ is *inductively factored* without reference to a specific inductive factorization of $\mathscr{A}$.

In [4, Prop. 3.30], Proposition 2.10 was strengthened further by showing that the compatibility with products restricts to the class of inductively factored arrangements.

Proposition 2.13. *Let* $\mathscr{A}_1, \mathscr{A}_2$ *be two arrangements. Then* $\mathscr{A} = \mathscr{A}_1 \times \mathscr{A}_2$ *is inductively factored if and only if both* $\mathscr{A}_1$ *and* $\mathscr{A}_2$ *are inductively factored and in that case the multiset of exponents of* $\mathscr{A}$ *is given by* $\exp \mathscr{A} = \{\exp \mathscr{A}_1, \exp \mathscr{A}_2\}$.

The connection with the previous notions is as follows.

Proposition 2.14 ([4, Prop. 3.11])**.** *If* $\mathscr{A}$ *is supersolvable, then* $\mathscr{A}$ *is inductively factored.*

Proposition 2.15 ([7, Prop. 2.2], [4, Prop. 3.14])**.** *Let* $\pi = (\pi_1, \dots, \pi_r)$ *be an inductive factorization of* $\mathscr{A}$*. Then* $\mathscr{A}$ *is inductively free with* $\exp \mathscr{A} = \{0^{\ell-r}, |\pi_1|, \dots, |\pi_r|\}$.

Remark 2.16. In analogy to inductively free arrangements, for inductively factored arrangements one can present a so called induction table of factorizations, cf. [4, Rem. 3.16].

If $\mathscr{A}$ is inductively factored, then $\mathscr{A}$ is inductively free, by Proposition 2.15. The latter can be described by a so called *induction table*, cf. [13, §4.3, p. 119]. In this process we start with an inductively free arrangement and add hyperplanes successively ensuring that [13, Def. 4.53(ii)] is satisfied. This process is referred to as *induction of hyperplanes*. This procedure amounts to choosing a total order on $\mathscr{A}$, say $\mathscr{A} = \{H_1, \dots, H_n\}$, so that each of the subarrangements $\mathscr{A}_0 := \Phi_\ell$, $\mathscr{A}_i := \{H_1, \dots, H_i\}$ and each of the restrictions $\mathscr{A}_i^{H_i}$ is inductively free for $i = 1, \dots, n$. In the associated induction table we record in the i-th row the information of the i-th step of this process, by listing $\exp \mathscr{A}_i' = \exp \mathscr{A}_{i-1}$, the defining form α_{H_i} of H_i, as well as $\exp \mathscr{A}_i'' = \exp \mathscr{A}_i^{H_i}$, for $i = 1, \dots, n$.

The proof of Proposition 2.15 shows that if π is an inductive factorization of $\mathscr{A}$ and $H_0 \in \mathscr{A}$ is distinguished with respect to π, then the triple $(\mathscr{A}, \mathscr{A}', \mathscr{A}'')$ with respect to H_0 is a triple of inductively free arrangements. Thus an induction table of $\mathscr{A}$ can be constructed, compatible with suitable inductive factorizations of the subarrangements $\mathscr{A}_i$.

Let $\mathscr{A} = \{H_1, \ldots, H_n\}$ be a choice of a total order on $\mathscr{A}$. Then, starting with the empty partition for Φ_ℓ, we can attempt to build inductive factorizations π_i of $\mathscr{A}_i$ consecutively, resulting in an inductive factorization $\pi = \pi_n$ of $\mathscr{A} = \mathscr{A}_n$. This is achieved by invoking Theorem 2.11 repeatedly in order to derive that each π_i is an inductive factorization of $\mathscr{A}_i$.

We then add the inductive factorizations π_i of $\mathscr{A}_i$ as additional data into an induction table for $\mathscr{A}$ (or else record to which part of π_{i-1} the new hyperplane H_i is appended to). The data in such an extended induction table together with the "Addition" part of Theorem 2.11 then proves that $\mathscr{A}$ is inductively factored. We refer to this technique as *induction of factorizations* and the corresponding table as an *induction table of factorizations* for $\mathscr{A}$. See Table 1 for an example.

Definition 2.17. In analogy to hereditary freeness and hereditary inductive freeness, [13, Def. 4.140, p. 253], we say that $\mathscr{A}$ is *hereditarily factored* provided $\mathscr{A}^X$ is factored for every $X \in L(\mathscr{A})$ and that $\mathscr{A}$ is *hereditarily inductively factored* provided $\mathscr{A}^X$ is inductively factored for every $X \in L(\mathscr{A})$.

We recall a useful fact from [4, Lem. 3.27].

Lemma 2.18. *Suppose that $\ell = 3$. Then $\mathscr{A}$ is (inductively) factored if and only if it is hereditarily (inductively) factored.*

By Remark 2.6, the class of factored arrangements is closed under taking localizations. This feature descends to the class of inductively factored arrangements and its hereditary subclass.

Theorem 2.19 ([9, Thm. 1, Rem. 4])**.** *The class of (hereditarily) inductively factored arrangements is closed under taking localizations.*

3 The intermediate arrangements $\mathscr{A}_\ell^k(r)$

In this section we discuss the intermediate arrangements $\mathscr{A}_\ell^k(r)$ from [11, §2] (cf. [13, §6.4]) in more detail, as they occur as restrictions of $\mathscr{A}(G(r,r,\ell))$, [11, Prop. 2.14] (cf. [13, Prop. 6.84]). They interpolate between the reflection arrangements of $G(r,r,\ell)$ and $G(r,1,\ell)$. For $\ell \geqslant 2$ and $0 \leqslant k \leqslant \ell$ the defining polynomial of $\mathscr{A}_\ell^k(r)$ is given by

$$Q(\mathscr{A}_\ell^k(r)) = x_1 \cdots x_k \prod_{\substack{1 \leqslant i < j \leqslant \ell \\ 0 \leqslant n < r}} (x_i - \zeta^n x_j),$$

where ζ is a primitive r^{th} root of unity, so that $\mathscr{A}_\ell^\ell(r) = \mathscr{A}(G(r,1,\ell))$ and $\mathscr{A}_\ell^0(r) = \mathscr{A}(G(r,r,\ell))$. Note that for $0 < k < \ell$, $\mathscr{A}_\ell^k(r)$ is not a reflection arrangement. Thanks to

[11, Props. 2.11, 2.13], each of these arrangements is free with

$$\exp(\mathscr{A}_\ell^k(r)) = \{1, r+1, 2r+1, \ldots, (\ell-2)r+1, (\ell-1)r+k-\ell+1\}$$

(cf. [13, Props. 6.82, 6.85]). The supersolvable and inductively free instances among the $\mathscr{A}_\ell^k(r)$ are classified in Theorems 1.3(ii) and 1.4(ii).

We abbreviate the hyperplanes in $\mathscr{A}_\ell^k(r)$ as follows. For $1 \leqslant a < b \leqslant \ell, 0 \leqslant n < r$ and $1 \leqslant c \leqslant k$, let

$$H_{a,b}^n := \ker(x_a - \zeta^n x_b) \quad \text{and} \quad H_c := \ker(x_c).$$

Lemma 3.1. *Let $\mathscr{A} = \mathscr{A}_\ell^{\ell-3}(r)$ for $r \geqslant 2, \ell \geqslant 4$. Then $\mathscr{A}$ is not nice.*

Proof. It suffices to show the result for $\mathscr{A}_4^1(r)$. For, let $\mathscr{A} = \mathscr{A}_\ell^{\ell-3}(r)$ for $r \geqslant 2, \ell \geqslant 5$. Then $\mathscr{A}_4^1(r)$ is realized as a localization of $\mathscr{A}$ as follows. Setting

$$X := \bigcap_{\substack{\ell-3 \leqslant a < b \leqslant \ell \\ 0 \leqslant n < r}} H_{a,b}^n$$

one readily checks that

$$\mathscr{A}_X \cong \mathscr{A}_4^1(r).$$

Consequently, if $\mathscr{A}_4^1(r)$ is not nice, then neither is $\mathscr{A}$, thanks to Remark 2.6.

Next we show that $\mathscr{A} = \mathscr{A}_4^1(r)$ fails to be nice. We are going to use Corollary 2.7 repeatedly on lattice elements of rank 2, to show that no partition of $\mathscr{A}$ can be nice. For that suppose $\pi = (\pi_1, \pi_2, \pi_3, \pi_4)$ is a nice partition of $\mathscr{A}$. Since $\mathscr{A}$ is free with $\exp \mathscr{A} = \{1, r+1, 2r+1, 3r-2\}$, the cardinalities of the parts of π coincide with the exponents of $\mathscr{A}$, by Remark 2.8.

First note that for $\{a, b, c, d\} = \{1, 2, 3, 4\}$ and any m, n with $0 \leqslant m, n < r$, we have $\{H_{a,b}^m, H_{c,d}^n\} \in L(\mathscr{A})$ of rank 2. So by Corollary 2.7(iii), there can't be hyperplanes of this kind in the same π_i unless they have a coordinate in common. Since there are 6 different pairs of distinct coordinates (a, b) for $H_{a,b}$ and each such excludes exactly one other such pair, there can be at most 3 different pairs of coordinates showing up among the members within the same π_i. Likewise, the coordinate hyperplane H_1 can't be in the same part as any of the $H_{a,b}$ with $a \neq 1$.

If two hyperplanes have only one coordinate in common, then their intersection is contained in a third hyperplane. That is

$$\{H_{a,b}^m, H_{a,c}^n, H_{b,c}^p\} \in L(\mathscr{A}) \text{ for } 1 \leqslant a < b < c \leqslant 4,\ 0 \leqslant m, n < r \text{ and } p \equiv n - m \mod r. \tag{3.2}$$

The intersection of two hyperplanes with the same pair of coordinates (a, b) is contained in every hyperplane associated with the pair (a, b), and if $a = 1$, is also contained in H_1, of course. Thus, by Corollary 2.7(iii) again, each of the two sets of hyperplanes $\{H_{1,b}^z \mid 0 \leqslant z < r\} \cup \{H_1\}$ and $\{H_{a,b}^z \mid 0 \leqslant z < r\}$ for $a \neq 1$ has to split into two parts of π.

Now let $|\pi_1| = 3r - 2$. Since $|\{H_{a,b}^z \mid 0 \leqslant z < r\}| = r$ for any $1 \leqslant a < b \leqslant 4$ and because of the previous restrictions discussed above, if $r \geqslant 3$, then π_1 has to contain hyperplanes of three different coordinate pairs. In case $r = 2$, it is sufficient to consider the case $\pi_1 = \{H^0_{1,2}, H^1_{1,2}, H^0_{1,3}, H^1_{1,3}\}$. The other possibilities are equivalent to this one. Suppose that $H^0_{3,4} \in \pi_2$ and $H^1_{3,4} \in \pi_3$, since they have to be in separate parts. By (3.2), we have $H^0_{1,4} \in \pi_2 \cap \pi_3$, which is absurd. So let $r \geqslant 2$ and suppose that π_1 contains hyperplanes of three different coordinate pairs. The possibilities are as follows:

(i) $H_{1,2}, H_{1,3}, H_{1,4} \in \pi_1$;

(ii) $H_{1,2}, H_{2,3}, H_{2,4} \in \pi_1$;

(iii) $H_{1,3}, H_{2,3}, H_{3,4} \in \pi_1$;

(iv) $H_{1,4}, H_{2,4}, H_{3,4} \in \pi_1$.

The cases (ii), (iii) and (iv) are equivalent, by symmetry, and only in case (i) we might have $H_1 \in \pi_1$.

First consider case (ii). Since $|\pi_1| = 3r - 2$ there are exactly two hyperplanes with the same set of coordinates that are not in π_1. More precisely, this must be one with pair $(2, 3)$ and one with the pair $(2, 4)$. Again, because of symmetry, it is sufficient to consider the case when $H^0_{2,3}, H^0_{2,4} \notin \pi_1$. Suppose that $H^0_{2,3} \in \pi_2$. It follows from (3.2) that $H^0_{1,3}, H^1_{3,4} \in \pi_2$. Then it further follows that $H^0_{2,4} \in \pi_2$. But this is a contradiction, as two hyperplanes in π_2 without a common coordinate are disallowed.

Now consider case (i). Independent whether or not H_1 belongs to π_1, there are at least two hyperplanes of the form as in (i) that do not belong to π_1. Because of symmetry, it suffices to consider the case $H^0_{1,2}, H^0_{1,3} \notin \pi_1$. Suppose $H^0_{1,2} \in \pi_2$. It follows from (3.2) that $H^1_{2,3} \in \pi_2$. It then follows that also $H^0_{1,3} \in \pi_2$. However, since there must be a hyperplane $H^k_{1,4}$ in π_1, we also get $H^k_{2,4} \in \pi_2$, which is a contradiction, again because having two hyperplanes in π_2 without a common coordinate is not possible.

Ultimately, we see that there is no nice partition of $\mathscr{A}^1_4(r)$. □

Lemma 3.3. *Let $\mathscr{A} = \mathscr{A}^{\ell-2}_\ell(r)$ for $r, \ell \geqslant 2$. Then $\mathscr{A}$ is inductively factored.*

Proof. The idea is to start with $\mathscr{A}^{\ell-2}_{\ell-1}(r) \times \Phi_1$ and give an inductive chain of factorizations up to $\mathscr{A}^{\ell-2}_\ell(r)$. By [2, Lem. 3.1], $\mathscr{A}^{\ell-2}_{\ell-1}(r)$ is supersolvable, and therefore it is inductively factored, by Proposition 2.14. Thus $\mathscr{A}^{\ell-2}_{\ell-1}(r) \times \Phi_1$ is inductively factored, by Proposition 2.13. Let $\pi = (\pi_1, \ldots, \pi_{\ell-1})$ be an inductive factorization of $\mathscr{A}^{\ell-2}_{\ell-1}(r)$ and set $\pi_\ell = \emptyset$. Then, with Proposition 2.13, $\pi^{(0)} := (\pi_1, \ldots, \pi_\ell)$ is an inductive factorization of $\mathscr{A}_0 := \mathscr{A}^{\ell-2}_{\ell-1}(r) \times \Phi_1$.
To obtain $\mathscr{A}^{\ell-2}_\ell(r)$ from $\mathscr{A}_0$, we have to add each hyperplane of the set $\{H^z_{i,\ell} \mid 0 \leqslant i < \ell, 0 \leqslant z < r\}$, which contains $(\ell-1)r$ different hyperplanes, to $\mathscr{A}_0$. Comparing the exponents of $\mathscr{A}^{\ell-2}_{\ell-1}$ to those of $\mathscr{A}^{\ell-2}_\ell$, we see that $(\ell-1)r - 1$ of these hyperplanes have to be added to π_ℓ and one to $\pi_{\ell-1}$. These remaining hyperplanes $\{\tilde{H}_1, \ldots, \tilde{H}_{(\ell-1)r}\}$ are ordered as indicated in Table 1. Define $\mathscr{A}_i := \mathscr{A}_{i-1} \cup \{\tilde{H}_i\}$. Furthermore let $\pi^{(i)}$ be the partition of $\mathscr{A}_i$ obtained from $\pi^{(i-1)}$ after $\tilde{H}_i$ is added. The induction of hyperplanes is given in

Table 1 below. In each step $i = 1, \dots, (\ell-1)r$, we have to show that $(\mathscr{A}_i'', (\pi^{(i)})'') \in \mathcal{IFAC}$ as well as

$$\exp(\mathscr{A}_i'') = \{|\pi_1^{(i)}|, \dots, |\pi_{j-1}^{(i)}|, |\pi_{j+1}^{(i)}|, \dots, |\pi_\ell^{(i)}|\} \text{ with } \tilde{H}_i \in \pi_j^{(i)}.$$

$(\pi^{(i)})'$	$\exp \mathscr{A}_i'$	$\tilde{H}_i$	$\exp \mathscr{A}_i''$
$\pi, \{\}$	$\exp(\mathscr{A}_{\ell-1}^{\ell-2}(r)), 0$	$H_{1,\ell}^0$	$\exp(\mathscr{A}_{\ell-1}^{\ell-2}(r))$
$\pi, \{H_{1,\ell}^0\}$	$\exp(\mathscr{A}_{\ell-1}^{\ell-2}(r)), 1$	$H_{1,\ell}^1$	$\exp(\mathscr{A}_{\ell-1}^{\ell-2}(r))$
$\pi, \{H_{1,\ell}^0, H_{1,\ell}^1\}$	$\exp(\mathscr{A}_{\ell-1}^{\ell-2}(r)), 2$	$H_{1,\ell}^2$	$\exp(\mathscr{A}_{\ell-1}^{\ell-2}(r))$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\pi, \{H_{1,\ell}^0, \dots, H_{1,\ell}^{r-1}\}$	$\exp(\mathscr{A}_{\ell-1}^{\ell-2}(r)), r$	$H_{2,\ell}^0$	$\exp(\mathscr{A}_{\ell-1}^{\ell-2}(r))$
$\pi, \{H_{1,\ell}^0, \dots, H_{1,\ell}^{r-1}, H_{2,\ell}^0\}$	$\exp(\mathscr{A}_{\ell-1}^{\ell-2}(r)), r$	$H_{2,\ell}^1$	$\exp(\mathscr{A}_{\ell-1}^{\ell-2}(r))$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\pi, \{H_{1,\ell}^0, \dots, H_{\ell-3,\ell}^{r-1}\}$	$\exp(\mathscr{A}_{\ell-1}^{\ell-2}(r)), (\ell-3)r$	$H_{\ell-2,\ell}^0$	$\exp(\mathscr{A}_{\ell-1}^{\ell-2}(r))$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\pi, \{H_{1,\ell}^0, \dots, H_{\ell-2,\ell}^{r-2}\}$	$\exp(\mathscr{A}_{\ell-1}^{\ell-2}(r)), (\ell-2)r-1$	$H_{\ell-2,\ell}^{r-1}$	$\exp(\mathscr{A}_{\ell-1}^{\ell-2}(r))$
$\pi, \{H_{1,\ell}^0, \dots, H_{\ell-2,\ell}^{r-1}\}$	$\exp(\mathscr{A}_{\ell-1}^{\ell-2}(r)), (\ell-2)r$	$H_{\ell-1,\ell}^0$	$\exp(\mathscr{A}_{\ell-1}^{\ell-2}(r))$
$\pi_1, \dots, \pi_{\ell-1} \cup \{H_{\ell-1,\ell}^0\}, \{H_{1,\ell}^0, \dots, H_{\ell-2,\ell}^{r-1}\}$	$\exp(\mathscr{A}_{\ell-1}^{\ell-1}(r)), (\ell-2)r$	$H_{\ell-1,\ell}^1$	$\exp(\mathscr{A}_{\ell-1}^{\ell-1}(r))$
$\pi_1, \dots, \pi_{\ell-1} \cup \{H_{\ell-1,\ell}^0\}, \{H_{1,\ell}^0, \dots, H_{\ell-2,\ell}^{r-1}, H_{\ell-1,\ell}^1\}$	$\exp(\mathscr{A}_{\ell-1}^{\ell-1}(r)), (\ell-2)r+1$	$H_{\ell-1,\ell}^2$	$\exp(\mathscr{A}_{\ell-1}^{\ell-1}(r))$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\pi_1, \dots, \pi_{\ell-1} \cup \{H_{\ell-1,\ell}^0\}, \{H_{1,\ell}^0, \dots, H_{\ell-1,\ell}^{r-2}\} \setminus \{H_{\ell-1,\ell}^0\}$	$\exp(\mathscr{A}_{\ell-1}^{\ell-1}(r)), (\ell-1)r-2$	$H_{\ell-1,\ell}^{r-1}$	$\exp(\mathscr{A}_{\ell-1}^{\ell-1}(r))$
$\pi_1, \dots, \pi_{\ell-1} \cup \{H_{\ell-1,\ell}^0\}, \{H_{1,\ell}^0, \dots, H_{\ell-1,\ell}^{r-1}\} \setminus \{H_{\ell-1,\ell}^0\}$	$\exp(\mathscr{A}_{\ell}^{\ell-2}(r))$		

Table 1: Induction Table of Factorizations for $\mathscr{A}_\ell^{\ell-2}(r)$

To prove that this table is indeed an inductive factorization table, and therefore that ultimately $\mathscr{A}_\ell^{\ell-2}(r)$ is inductively factored, we are going to proceed in 3 steps.

First let $\tilde{H}_i \in \{H_{1,\ell}^0, \dots, H_{\ell-2,\ell}^{r-1}\}$. Since only $\pi_\ell^{(i)}$ contains hyperplanes that involve the coordinate x_ℓ, one can easily verify that $\mathscr{A}_i'' \cong \mathscr{A}_{\ell-1}^{\ell-2}(r)$ and $(\pi^{(i)})'' \cong \pi$, so it is inductively factored by assumption.

For the next step we first need to specify the the inductive factorization $(\pi_1, \dots, \pi_{\ell-1})$ of $\mathscr{A}_{\ell-1}^{\ell-2}(r)$ we intend to start with. Let $\pi_{\ell-1} = \{H_{j,\ell-1}^n \mid 1 \leqslant j < \ell-1, 0 \leqslant n < r\}$. This is indeed possible. By starting with an arbitrary inductive factorization $(\pi_1, \dots, \pi_{\ell-2}, \emptyset)$ for $\mathscr{A}_{\ell-2}^{\ell-2} \times \Phi_1$ and then inductively adding the remaining hyperplanes $\{H_{j,\ell-1}^z \mid 1 \leqslant j < \ell-1, 0 \leqslant z < r\}$ to the empty part of the partition, one can easily verify that this gives an inductive factorization chain.

Now assume that $\tilde{H}_i = H_{\ell-1,\ell}^0$ and let $\pi_{\ell-1}$ be like before. It is easy to verify that $\pi_\ell^{(i-1)} \cong \pi_{\ell-1}$ and that $\mathscr{A}_i'' \cong \mathscr{A}_{\ell-1}^{\ell-2}(r)$, so by assumption we get

$$\left(\mathscr{A}_i'', (\pi^{(i)})''\right) \cong \left(\mathscr{A}_{\ell-1}^{\ell-2}(r), \pi\right) \in \mathcal{IFAC}.$$

Finally let $\tilde{H}_i \in \{H_{\ell-1,\ell}^1, \dots, H_{\ell-1,\ell}^{r-1}\}$. Then, since $H_{\ell-1,\ell}^0 \in \pi_{\ell-1}^{(i)}$ and $H_{\ell-1,\ell}^0 \cap \tilde{H}_i = H_{\ell-1} \cap \tilde{H}_i$ we get a new coordinate hyperplane in the restriction and therefore $\mathscr{A}_i'' \cong \mathscr{A}_{\ell-1}^{\ell-1}(r)$. Furthermore, we have $(\pi^{(i)})'' \cong (\pi_1, \dots, \pi_{\ell-2}, \pi_{\ell-1} \cup \{H_{\ell-1}\})$, so we have to show that this

is indeed an inductive factorization of $\mathscr{A}_{\ell-1}^{\ell-1}(r)$.
Starting with $(\mathscr{A}_{\ell-1}^{\ell-2}(r),\pi)$ and adding $H_{\ell-1}$ to $\pi_{\ell-1}$, we get

$$\left((\mathscr{A}_{\ell-1}^{\ell-2}(r))'',\pi''\right)=\left(\mathscr{A}_{\ell-2}^{\ell-2}(r),(\pi_1,\dots,\pi_{\ell-2})\right).$$

This is inductively factored due to the previous construction. Consequently,

$$\left(\mathscr{A}_i'',(\pi^{(i)})''\right)\cong\left(\mathscr{A}_{\ell-1}^{\ell-1}(r),(\pi_1,\dots,\pi_{\ell-2},\pi_{\ell-1}\cup\{H_{\ell-1}\})\right)\in\mathcal{IFAC}$$

and for $i=(\ell-1)r$ we get that $\mathscr{A}_i=\mathscr{A}_\ell^{\ell-2}(r)$ is indeed inductively factored, as desired. □

We are now able to classify all nice and all inductively factored instances among the $\mathscr{A}_\ell^k(r)$.

Theorem 3.4. *Let $\mathscr{A}=\mathscr{A}_\ell^k(r)$ for $r,\ell\geqslant 2$ and $0\leqslant k\leqslant\ell$.*

(i) If $\ell=2$, then $\mathscr{A}$ is inductively factored.

(ii) If $\ell=3$, then $\mathscr{A}$ is nice.

(iii) For $\ell\geqslant 3$, $\mathscr{A}$ is inductively factored if and only if $\ell-2\leqslant k\leqslant\ell$.

(iv) For $\ell\geqslant 4$, $\mathscr{A}$ is nice if and only if $\mathscr{A}$ is inductively factored.

Proof. If $\ell=2$, then $\mathscr{A}$ is supersolvable, so part (i) follows from Proposition 2.14. So suppose that $\ell\geqslant 3$. For $k=0,\ell$ the result follows from Theorem 1.1. By Theorem 1.4(ii), $\mathscr{A}_\ell^{\ell-1}(r)$ is also supersolvable, and so the result follows in this case again from Proposition 2.14.

Let $\ell\geqslant 4$. It follows from [9, Ex. 1] that $\mathscr{A}_\ell^k(r)$ is not nice for $1\leqslant k\leqslant\ell-4$. For completeness, we include the easy argument. So let $1\leqslant k\leqslant\ell-4$. Define

$$X:=\bigcap_{\substack{k+1\leqslant a<b\leqslant\ell\\0\leqslant n<r}}H_{a,b}^n.$$

Then one checks that

$$\mathscr{A}_X\cong\mathscr{A}_{\ell-k}^0(r)=\mathscr{A}(G(r,r,\ell-k)).$$

For $1\leqslant k\leqslant\ell-4$, it follows from Theorem 1.1 that $\mathscr{A}(G(r,r,\ell-k))$ is not nice. Consequently, neither is $\mathscr{A}_\ell^k(r)$, by Remark 2.6. For $k=\ell-3$, it follows from Lemma 3.1 that $\mathscr{A}_\ell^{\ell-3}(r)$ is not nice either.

Finally, for $r,\ell\geqslant 3$, Lemma 3.3 shows that $\mathscr{A}_\ell^{\ell-2}(r)$ is inductively factored. This completes the proof of the theorem. □

Theorem 3.4 readily extends to the hereditary subclasses.

Corollary 3.5. *Let $\mathscr{A}=\mathscr{A}_\ell^k(r)$ for $r,\ell\geqslant 2$ and $0\leqslant k\leqslant\ell$. Then $\mathscr{A}$ is (inductively) factored if and only if $\mathscr{A}$ is hereditarily (inductively) factored.*

Proof. The result follows immediately from Theorem 3.4 and the pattern of restrictions to hyperplanes among the $\mathscr{A}_\ell^k(r)$ from [11, Props. 2.11, 2.13] (cf. [13, Props. 6.82]). □

4 Nice Restrictions of Reflection Arrangements

Throughout this section, let W be an irreducible unitary reflection group and let $\mathscr{A} = \mathscr{A}(W)$ be its reflection arrangement.

We begin with a proof of Theorem 1.5. It follows from Theorem 1.1(ii) that if $\mathscr{A}$ is nice, then so is $\mathscr{A}^X$ for any $X \in L(\mathscr{A})$. If $W = G(r, r, \ell)$ for $\ell \geqslant 3$, then the result follows from Theorem 3.4.

This leaves the instances when W is of exceptional type. If $\dim X \leqslant 2$, then $\mathscr{A}^X$ is supersolvable, and so is nice, by Proposition 2.14. So we concentrate on those instances when $X \in L(\mathscr{A})$ with $\dim X \geqslant 3$. Using the tables [13, App. C, D], we first address each restriction $\mathscr{A}^X$ for W of exceptional type and $\dim X = 3$ and some instances for $\dim X = 4$. The failure to admit a nice partition is determined computationally. Then each higher rank restriction can be analyzed by a suitable localization, making use of the fact that $L(\mathscr{A}(W_Y)^X) = L(\mathscr{A}(W)^X)_Y)$, see [13, Lem. 2.11, Cor. 6.28]. It then follows that $\mathscr{A}(W)^X$ is not nice because the localization $(\mathscr{A}(W)^X)_Y$ fails to be nice, cf. Remark 2.6. We readily find a suitable 3-dimensional localization which is not nice using the tables [13, App. C].

The following result is due to Orlik and Terao, [13, App. D].

Lemma 4.1. *We have the following lattice isomorphisms of* 3*-dimensional restrictions:*

(i) $(E_6, A_3) \cong \mathscr{A}_3^2(2)$;

(ii) $(E_7, D_4) \cong \mathscr{A}_3^3(2)$;

(iii) $(F_4, A_1) \cong (F_4, \tilde{A}_1) \cong (E_7, A_1^4) \cong (E_7, (A_1A_3)') \cong (E_8, A_1D_4) \cong (E_8, D_5)$;

(iv) $(E_6, A_1A_2) \cong (E_7, A_4)$;

(v) $(E_7, A_2^2) \cong (E_8, A_5)$;

(vi) $(G_{26}, A_0) \cong (G_{32}, C(3)) \cong (G_{34}, G(3, 3, 3))$.

We note that the isomorphism between $(E_8, A_1^2A_3)$ and (E_8, A_2A_3) claimed in [13, App. D] is not correct (cf. [10]). Thus we need to treat both restrictions separately. It follows from Theorem 1.4 that the cases listed in Lemma 4.1(i), (ii), and (v) are all supersolvable and thus are inductively factored, and so are nice. Moreover, it follows from Theorem 1.1(i) and [3, Thm. 1.2] that the cases listed in Lemma 4.1(vi) are not nice. For each of the other kinds it suffices to consider only one of the isomorphic restrictions. In our next result, we determine all nice restrictions of dimension at least 3 for an ambient irreducible, non-supersolvable reflection arrangement of exceptional type.

Lemma 4.2. *Let $\mathscr{A} = \mathscr{A}(W)$ be an irreducible, non-supersolvable reflection arrangement of exceptional type. Let $X \in L(\mathscr{A})$ with $\dim X \geqslant 3$. Then $\mathscr{A}^X$ is nice if and only if $\mathscr{A}^X$ is supersolvable or one of $(E_6, A_1A_2) \cong (E_7, A_4), (E_7, (A_1A_3)'')$.*

Proof. Using our discussion above together with Theorems 1.1 and 1.4 along with Lemma 4.1, we still have to analyze the rank 3 restrictions in Table 2 along with some rank 4 restrictions in Table 3. Using the tables of all orbit types for the irreducible reflection groups of exceptional type in [11, App.] (see also [13, App. C]) as well as the restrictions imposed on factorizations in Remark 2.6, we determine which of the remaining 3-dimensional restrictions admit a nice partition (Table 2) and also determine that some 4-dimensional restrictions $\mathscr{A}(W)^X$ do not admit a nice partition (Table 3).

To illustrate the argument, we indicate this for (E_6, A_1^3) which turns out to be not nice. The arguments for the other non-nice restrictions in Tables 2 and 3 are quite similar and are left to the reader. We emphasize that we have verified all instances of both tables with the aid of a computer by means of the computer algebra system SAGE, [16]. The program in question uses an intelligent brute force routine which we outline next. For the singleton part of a partition π of $\mathscr{A}$ it chooses a representative H among the orbits of the automorphism group of $L(\mathscr{A})$. The rest of π is then constructed as follows. Consider the set of all localizations $\mathscr{A}_X$ so that $X \subset H$ and $r(X) = 2$. For each such $\mathscr{A}_X$, it follows from Corollary 2.7(iii) that $\mathscr{A}_X \setminus \{H\}$ belongs to a single part of π. For each such resulting partition π, the algorithm then tests whether the conditions in Definition 2.5 are fulfilled.

Let $\mathscr{B}$ be the restriction (E_6, A_1^3). Then we have

$$\mathscr{B} = \left\{ \begin{array}{lll} H_0 = \{2x - y - z = 0\}, & H_1 = \{2x - y + z = 0\}, & H_2 = \{x + y = 0\}, \\ H_3 = \{2x + y + z = 0\}, & H_4 = \{2x + y - z = 0\}, & H_5 = \{x - y = 0\}, \\ H_6 = \{y + z = 0\}, & H_7 = \{y - z = 0\}, & H_8 = \{y = 0\}, \\ H_9 = \{x = 0\} & & \end{array} \right\}.$$

We claim that $\mathscr{B}$ does not admit a nice partition. By way of contradiction, suppose that $\pi = (\pi_1, \pi_2, \pi_3)$ is a factorization of $\mathscr{B}$ and let $\exp \mathscr{B} = \{1, 4, 5\} = \{|\pi_1|, |\pi_2|, |\pi_3|\}$. Suppose $X \in L(\mathscr{B})$ is a flat of rank 2. Then $|X| \in \{2, 3, 4\}$ and one readily checks that

- if $|X| = 2$, then $X = \{H_i, H_j\}$ with $(i, j) \subset \{0, 1, 2\}$ or $(i, j) \subset \{3, 4, 5\}$;
- if $|X| = 3$, then $X = \{H_6, H_7, H_8\}$ or $X = \{H_i, H_j, H_k\}$ with $(i, j, k) \in \{0, 1, 2\} \times \{3, 4, 5\} \times \{6, 7, 8\}$ but $(i, j, k) \notin \{(0, 3, 6), (1, 4, 7), (2, 5, 8)\}$;
- if $|X| = 4$, then $X = \{H_i, H_j, H_k, H_9\}$ with $(i, j, k) \in \{(0, 3, 6), (1, 4, 7), (2, 5, 8)\}$.

Now consider a rank 2 flat X with $|X| = 4$. Removing one hyperplane from X and adding to it a different one results in a subset Z of $\mathscr{B}$ which is not a member of $L(\mathscr{B})$. Then there is always a rank 2 flat Y in $L(\mathscr{B})$ satisfying $Y \subsetneq Z$. Since every hyperplane of $\mathscr{B}$ is part of such a set X, it follows from Corollary 2.7 that there is no candidate for the singleton part π_1, because the remaining three hyperplanes can't be joined with a fourth one to build a part of π. Thus π is not a nice partition of $\mathscr{B}$, so (E_6, A_1^3) is not nice.

Next we are going to explicitly determine inductive factorizations for the two restrictions (E_6, A_1A_2) and $(E_7, (A_1A_3)'')$.

Let $\mathscr{C}$ be the restriction (E_6, A_1A_2). Then $\exp \mathscr{C} = \{1, 4, 5\}$ and one checks that

$$Q(\mathscr{C}) = x(x \pm z)(x \pm y)(y \pm z)(2x \pm (y - z))(3y - z) \in \mathbb{R}[x, y, z].$$

We claim that $\pi = (\pi_1, \pi_2, \pi_3)$ with

$$\begin{aligned}
\pi_1 &= \{\{x=0\}\}\,, \\
\pi_2 &= \left\{ \begin{array}{l} \{2x+y-z=0\}, \{2x-y-z=0\}, \\ \{y-z=0\}, \{3y-z=0\} \end{array} \right\} \text{ and} \\
\pi_3 &= \left\{ \begin{array}{l} \{x+z=0\}, \{x-z=0\}, \{y+z=0\}, \\ \{x+y=0\}, \{x-y=0\} \end{array} \right\}
\end{aligned}$$

is an inductive factorization of $\mathscr{C}$. The niceness of π is easily verified by checking the conditions in Definition 2.5.

Let $H_0 = \{3y-z=0\}$, then $\mathscr{C}' = \mathscr{C} \setminus \{H_0\}$ is supersolvable and

$$\{x=0\} \leqslant \{x=0\} \cap \{2x+y-z=0\} \cap \{2x-y-z=0\} \cap \{y-z=0\} \leqslant T_{\mathscr{C}'}$$

is a maximal chain of modular elements in $L(\mathscr{C}')$ that induces the factorization π' by Proposition 2.9, so π indeed is an inductive factorization.

Let $\mathscr{D}$ be the restriction $(E_7, (A_1A_3)'')$. Then $\exp \mathscr{D} = \{1, 5, 5\}$ and one checks that

$$Q(\mathscr{D}) = xyz(x \pm y)(x-z)(x-2z)(y \pm 2z)(x \pm y - 2z) \in \mathbb{R}[x, y, z].$$

We claim that $\pi = (\pi_1, \pi_2, \pi_3)$ with

$$\begin{aligned}
\pi_1 &= \{\{x-z=0\}\}\,, \\
\pi_2 &= \left\{ \begin{array}{l} \{x=0\}, \{z=0\}, \{x-2z=0\} \\ \{y+2z=0\}, \{y-2z=0\} \end{array} \right\} \text{ and} \\
\pi_3 &= \left\{ \begin{array}{l} \{y=0\}, \{x+y-2z=0\}, \{x-y-2z=0\}, \\ \{x+y=0\}, \{x-y=0\} \end{array} \right\}
\end{aligned}$$

is an inductive factorization for $\mathscr{D}$. Again, the conditions from Definition 2.5 are easy to verify, so π is a nice partition. With $H_0 = \{y-2z=0\}$, the map

$$\varrho : \mathscr{D} \setminus \pi_2 \to \mathscr{D}'', H \mapsto H \cap H_0$$

is a bijection, so with Theorem 2.11, $(\mathscr{D}', \pi')$ is nice as well. Now let $H_1 = \{y+2z=0\}$, then the arrangement $\mathscr{D} \setminus \{H_0, H_1\}$ is supersolvable and

$$\{x-z=0\} \leqslant \{x-z=0\} \cap \{x=0\} \cap \{z=0\} \cap \{x-2z=0\} \leqslant T_{\mathscr{D} \setminus \{H_0, H_1\}}$$

is a maximal chain of modular elements in $L(\mathscr{D} \setminus \{H_0, H_1\})$ that induces the factorization $(\pi')'$ by Proposition 2.9, so π is an inductive factorization for $\mathscr{D}$.

$\mathscr{A}^X$	$\exp \mathscr{A}^X$	nice
(F_4, A_1)	1,5,7	false
(G_{29}, A_1)	1,9,11	false
(H_4, A_1)	1,11,19	false
(G_{31}, A_1)	1,13,17	false
(G_{33}, A_1^2)	1,7,9	false
(G_{33}, A_2)	1,6,7	false
(G_{34}, A_1^3)	1,13,19	false
(G_{34}, A_1A_2)	1,13,16	false
(G_{34}, A_3)	1,11,13	false
(E_6, A_1^3)	1,4,5	false
(E_6, A_1A_2)	1,4,5	**true**
$(E_7, A_1^2A_2)$	1,5,7	false
$(E_7, (A_1A_3)'')$	1,5,5	**true**
$(E_8, A_1^3A_2)$	1,7,11	false
$(E_8, A_1A_2^2)$	1,7,11	false
$(E_8, A_1^2A_3)$	1,7,9	false
(E_8, A_2A_3)	1,7,9	false
(E_8, A_1A_4)	1,7,8	false

Table 2: Rank 3 restrictions of the exceptional groups

$\mathscr{A}^X$	$\exp \mathscr{A}^X$	nice
(G_{33}, A_1)	1,7,9,11	false
(E_6, A_1^2)	1,4,5,7	false
(E_6, A_2)	1,4,5,5	false
(E_7, A_1A_2)	1,5,7,8	false
(E_7, A_3)	1,5,5,7	false
(E_8, A_2^2)	1,7,11,11	false
(E_8, A_4)	1,7,8,9	false
(E_8, D_4)	1,5,7,11	false

Table 3: Some rank 4 restrictions of exceptional groups

For the remaining higher rank restrictions, we utilize a suitable localization of rank 3 from Table 2, of rank 4 from Table 3, or else one which appears earlier in the same table and which is not nice.

$\mathscr{A}^X$	non-nice localization
(G_{34}, A_1)	$(D_5, A_1) = A_4^1(2)$
(G_{34}, A_1^2)	(G_{33}, A_1^2)
(G_{34}, A_2)	(G_{33}, A_2)

(E_6, A_1)	$(D_5, A_1) = A_4^1(2)$
(E_7, A_1)	$(D_5, A_1) = A_4^1(2)$
(E_7, A_1^2)	(E_6, A_1^2)
(E_7, A_2)	(E_6, A_2)
$(E_7, (A_1^3)')$	(E_6, A_1^3)
$(E_7, (A_1^3)'')$	(E_6, A_1^3)
(E_8, A_1)	$(D_5, A_1) = A_4^1(2)$
(E_8, A_1^2)	(E_7, A_1^2)
(E_8, A_2)	(E_7, A_2)
(E_8, A_1^3)	(E_6, A_1^3)
(E_8, A_1A_2)	(E_7, A_1A_2)
(E_8, A_3)	(E_7, A_3)
(E_8, A_1^4)	(E_7, A_1^4)
$(E_8, A_1^2A_2)$	$(E_7, A_1^2A_2)$
(E_8, A_1A_3)	$(E_7, (A_1A_3)')$

Table 4: Higher rank restrictions in exceptional groups

The result now follows from Table 4, Remark 2.6, the data in Table 2, as well as Lemma 4.1 and our discussion above. □

Next we prove Theorem 1.6. It follows from Theorem 1.1(ii) that if $\mathscr{A}$ is inductively factored, then so is $\mathscr{A}^X$ for any $X \in L(\mathscr{A})$. If $W = G(r, r, \ell)$ for $\ell \geqslant 3$, then the result follows from Theorem 3.4.

It follows from Theorem 1.3 and Proposition 2.15 that for $\dim X \geqslant 4$, $\mathscr{A}^X$ is not inductively factored.

If W is exceptional, then the result follows from the next lemma. Let $\mathscr{A} = \mathscr{A}(W)$. Recall that if $\dim X \leqslant 2$, then $\mathscr{A}^X$ is supersolvable and so is inductively factored. On the other hand, by Proposition 2.15 and Theorem 1.3(iii), the restricted arrangement $\mathscr{A}^X$ is not inductively factored, for $X \in L(\mathscr{A})$ with $\dim X \geqslant 4$.

Lemma 4.3. *Let W be irreducible of exceptional type with reflection arrangement $\mathscr{A} = \mathscr{A}(W)$. Let $X \in L(\mathscr{A})$. Then $\mathscr{A}^X$ is nice if and only if it is inductively factored.*

Proof. It follows from Lemma 4.2 that up to isomorphism there are only two non-supersolvable, nice restrictions. One checks that the calculated nice partitions are inductively factored. □

Finally, Theorem 1.7 follows from Theorems 1.6, 3.4, along with [13, Prop. 6.82] and Lemma 2.18.

References

[1] N. Amend, T. Hoge and G. Röhrle, *On inductively free restrictions of reflection arrangements*, J. Algebra, **418** (2014), 197–212.

[2] N. Amend, T. Hoge and G. Röhrle, *Supersolvable restrictions of reflection arrangements*, J. Combin. Theory Ser. A, **127** (2014), 336–352.

[3] T. Hoge and G. Röhrle, *On supersolvable reflection arrangements*, Proc. AMS, **142** (2014), no. 11, 3787–3799.

[4] T. Hoge and G. Röhrle, *Addition-Deletion Theorems for Factorizations of Orlik-Solomon Algebras and nice Arrangements*, European J. Combin. **55** (2016), 20–40.

[5] T. Hoge and G. Röhrle, *Nice Reflection Arrangements*, Electron. J. Combin. **23(2)**, (2016), #P2.9.

[6] M. Jambu, *Fiber-type arrangements and factorization properties.* Adv. Math. **80** (1990), no. 1, 1–21.

[7] M. Jambu and L. Paris, *Combinatorics of Inductively Factored Arrangements*, Europ. J. Combinatorics **16** (1995), 267–292.

[8] M. Jambu and H. Terao, *Free arrangements of hyperplanes and supersolvable lattices*, Adv. in Math. **52** (1984), no. 3, 248–258.

[9] T. Möller and G. Röhrle, *Localizations of inductively factored arrangements*, in: Algorithmic and Experimental Methods in Algebra, Geometry, and Number Theory, Springer Verlag, 2017.

[10] T. Möller and G. Röhrle, *Counting chambers in restricted Coxeter arrangements*, arXiv:1706.09649

[11] P. Orlik and L. Solomon, *Arrangements Defined by Unitary Reflection Groups*, Math. Ann. **261**, (1982), 339–357.

[12] P. Orlik, L. Solomon, and H. Terao, *Arrangements of hyperplanes and differential forms.* Combinatorics and algebra (Boulder, Colo., 1983), 29–65, Contemp. Math., **34**, Amer. Math. Soc., Providence, RI, 1984.

[13] P. Orlik and H. Terao, *Arrangements of hyperplanes*, Springer-Verlag, 1992.

[14] G.C. Shephard and J.A. Todd, *Finite unitary reflection groups.* Canadian J. Math. **6**, (1954), 274–304.

[15] R. P. Stanley, *Supersolvable lattices*, Algebra Universalis **2** (1972), 197–217.

[16] W. A. Stein et al., *Sage Mathematics Software*, The Sage Development Team, 2009.

[17] H. Terao, *Arrangements of hyperplanes and their freeness I, II*, J. Fac. Sci. Univ. Tokyo **27** (1980), 293–320.

[18] H. Terao, *Factorizations of the Orlik-Solomon Algebras*, Adv. in Math. **92**, (1992), 45–53.