Vertex-addition strategy for domination-like invariants

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Abstract

In [J. Graph Theory **13** (1989) 749–762], McCuaig and Shepherd gave an upper bound of the domination number for connected graphs with minimum degree at least two. In this paper, we propose a simple strategy which, together with the McCuaig-Shepherd theorem, gives a sharp upper bound of the domination number via the number of leaves. We also apply the same strategy to other domination-like invariants, and find a relationship between such invariants and the number of leaves.

Keywords: domination; total domination; Roman domination

1 Introduction

1.1 Domination concept and our strategy

All graphs considered in this paper are finite, simple, and undirected. Let G be a graph. For $u \in V(G)$, we let $d_G(u)$, $N_G(u)$ and $N_G[u]$ denote the *degree*, the *open neighborhood* and the *closed neighborhood* of u, respectively; thus $d_G(u) = |N_G(u)|$ and $N_G[u] = N_G(u) \cup \{u\}$. We let $\delta(G)$ and $\Delta(G)$ denote the *minimum degree* and the *maximum degree* of G, respectively. A vertex $u \in V(G)$ is a *leaf* of G if the degree of u in G is exactly one. We let $L(G)$ denote the set of leaves of G. An edge of G is called

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a *pendant edge* if the edge is incident with a leaf of G. For two subsets X, Y of $V(G)$, we say that X *dominates* Y if $Y \subseteq \bigcup_{u \in X} N_G[u]$. A subset of $V(G)$ which dominates $V(G)$ is called a *dominating set* of G. The minimum cardinality of a dominating set of G is called the *domination number* of G, and is denoted by $\gamma(G)$.

The domination number is one of the important invariants in Graph Theory, and it can be widely applied to real problems, for example, school bus routing problem, social network theory and the location of radio stations (see [10, 11]). To meet various additional requirements for above problems, many domination-like concepts were defined and studied.

We first introduce an orthodox flow of research of domination-like concepts by citing the original result on domination. The following is a well-known result given by Ore.

Theorem A (Ore [17]) *Let G be a connected graph of order n ($\geqslant 2$). Then $\gamma(G) \leqslant \frac{n}{2}$.*

The upper bound of $\gamma(G)$ in Theorem A is best possible. Furthermore, the connected graphs G attaining the equality in Theorem A were characterized by Fink, Jacobson, Kinch and Roberts [7] and Payan and Xuong [18] as follows (here the *corona* of a graph H is the graph obtained from H by adding a pendant edge to each vertex of H).

Theorem B (Fink et al. [7]; Payan and Xuong [18]) *Let G be a connected graph of order n. Then $\gamma(G) = \frac{n}{2}$ if and only if G is either C_4 or the corona of a connected graph.*

In particular, any connected graphs G with $\gamma(G) = \frac{|V(G)|}{2}$ except for C_4 have some leaves. Thus one may suspect that the domination number of many connected graphs G with $\delta(G) \geqslant 2$ is much less than the half of $|V(G)|$. For connected graphs with minimum degree at least two, McCuaig and Shepherd [16] showed the following theorem (here $\mathcal{B}$ is the set consisting of graphs depicted in Figure 1).

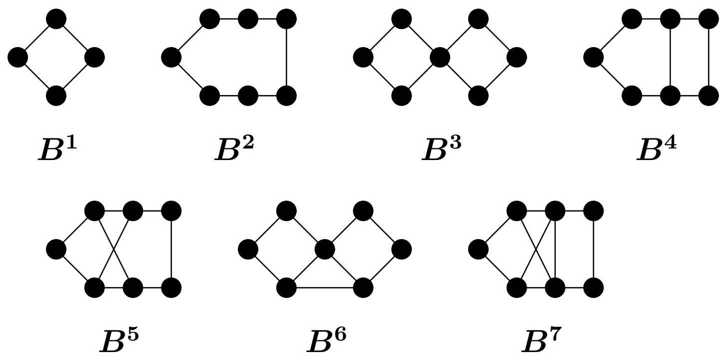


Figure 1: The graphs belonging to $\mathcal{B}$

Theorem C (McCuaig and Shepherd [16]) *Let G be a connected graph of order n with $\delta(G) \geqslant 2$. Then either $G \in \mathcal{B}$ or $\gamma(G) \leqslant \frac{2n}{5}$.*

Considering Theorem C, we know that the existence of leaves is a cause of an increase of the domination number. On the other hand, it seems that Theorem C gives no insight to general graphs G (which do not require the condition $\delta(G) \geqslant 2$). Thus one problem naturally arises: Find a relationship between the domination number and the number of leaves. As a related result to the problem, for example, Favaron [6] gave the following theorem.

Theorem D (Favaron [6]) *Let $l \geqslant 2$ be an integer, and let T be a tree of order n having exactly l leaves. Then $\gamma(T) \leqslant \frac{n+l}{3}$.*

However, there exist infinitely many connected (non-tree) graphs G with n vertices, l leaves and $\gamma(G) > \frac{n+l}{3}$ (see Theorem 2.1 in Section 2). Thus, when we study a relationship between domination and leaves in general graphs, it is insufficient to only consider trees.

In order to get a desired relation, we propose a simple vertex-addition strategy as follows: For a given connected graph G,

(S1) we construct a new graph H with $\delta(H) \geqslant 2$ which is obtained from G by adding a special graph to each leaf of G,

(S2) give a small dominating set S of H which is assured by Theorem C, and

(S3) reduce S to a dominating set of G.

In Section 2, we show the following theorem by the above vertex-addition strategy.

Theorem 1.1 *Let $l \geqslant 0$ be an integer, and let G be a connected graph of order n $(\geqslant 3)$ having exactly l leaves. Then either $l = 0$ and $G \in \mathfrak{B}$ or*

$$\gamma(G) \leqslant \begin{cases} \frac{2n+l}{5} & (0 \leqslant l \leqslant \frac{n}{2}) \\ n - l & (\frac{n+1}{2} \leqslant l \leqslant n-1). \end{cases}$$

Note that Theorem 1.1 is a common generalization of Theorems A and C. In Section 2, we also give a generalization of Theorem B by using the proof technique of Theorem C.

We return to general domination-like concepts. For many domination-like invariants μ, the same upper bounds were found:

(D1) A sharp upper bound of $\mu(G)$ for every connected graph G (of large order).

(D2) A sharp upper bound of $\mu(G)$ for every connected graph G with $\delta(G) \geqslant 2$ (with finite exceptions).

In many cases, the following results are also given.

(D1′) A characterization of connected graphs G attaining the equality of the bound in (D1).

(D2′) A characterization of connected graphs G with $\delta(G) \geqslant 2$ attaining the equality of the bound in (D2).

Our main aim in this paper is to give the following new steps for an invariant μ by using the vertex-addition strategy.

(D3) A sharp upper bound of $\mu(G)$ for every connected graph G with l leaves (i.e., a common generalization of (D1) and (D2)).

(D3′) A generalization of (D1′) and (D2′) (if (D1′) and (D2′) are known).

In general, the results such as (D2) and (D2′) tend to be independently shown from (D1) and (D1′). Thus our strategy might give an alternative proof of the results such as (D1) and (D1′).

As we mentioned above, domination-like invariants were widely defined. Thus it is difficult to deal with all of them. In this paper, as typical domination-like invariants, we deal with two especially famous invariants, namely, total domination and Roman domination in Sections 3 and 4, respectively. Our main results for such invariants are Theorems 3.2 and 4.2.

1.2 Definitions

Our notation and terminology are standard, and mostly taken from [5]. Exceptions are as follows.

For $n \geqslant 3$, we let P_n and C_n denote the *path* and the *cycle* of order n, respectively. A vertex u of a connected graph H is a *central vertex* if for any $v \in V(H)$, the distance from u to v is at most the radius of H. Note that a path of odd order has exactly one central vertex, and a path of even order has exactly two central vertices.

A graph G is l*-leaf minimally connected* if

(L1) G is connected and $|V(G)| \geqslant 2$,

(L2) G has exactly l leaves, and

(L3) for each $e \in E(G)$, either e is a bridge of G or $G - e$ has at least $l + 1$ leaves.

Note that a graph G is 0-leaf minimally connected if and only if G is a connected graph with $\delta(G) \geqslant 2$ and $\delta(G - e) = 1$ for any non-bridge edge $e \in E(G)$. The following fact clearly holds.

Fact 1.1 *Let $l \geqslant 0$ be an integer, and let G be a connected graph of order n ($\geqslant 2$) having exactly l leaves. Then G has a spanning l-leaf minimally connected subgraph.*

The following lemma will be used in the proof of our main results.

Lemma 1.2 *Let G be an l-leaf minimally connected graph, and let H be a graph obtained from G by adding for each $v \in L(G)$, a new graph C_v and some edges between $N_G[v]$ and $V(C_v)$. Let $e \in E(G)$ be an edge incident with no leaves of G. Then e is a bridge of H or $|L(H)| < |L(H - e)|$.*

Proof. We may assume that e is not a bridge of H. By the construction of H, e is not a bridge of G. Since G is l-leaf minimally connected, it follows that $G - e$ has at least $l + 1$ leaves. In particular, e is incident with a vertex x with $d_G(x) = 2$. Write $e = xx'$ and $N_G(x) = \{x', y\}$. Suppose that $d_H(x) \geqslant 3$. Since $d_G(x) = 2$ and e is incident with no leaves of G, this implies that y is a leaf of G, and hence e is a bridge of G, which is a contradiction. Thus $d_H(x) = 2$, and so x is a leaf of $H - e$. Consequently we have $|L(H)| < |L(H - e)|$. □

Let C be a connected graph, and let $v \in V(C)$. The (C, v)-*corona* of a graph G is the graph obtained from G by adding a copy C_u of C to each vertex $u \in V(G)$ and identifying u and the vertex of C_u corresponding with v (see Figure 2). Note that if C

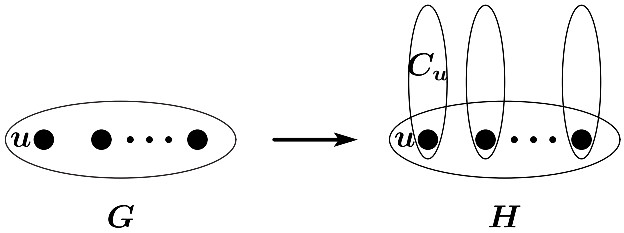


Figure 2: The (C, v)-corona H of G

is a path of order 2, the (C, v)-corona of a graph G is exactly the corona of G (defined in the paragraph preceding Theorem B in Subsection 1.1).

Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a set of vertex-disjoint graphs with some *attachment vertices*. (For example, we will use a set of some copies of U^i in the first paragraph of Section 2.) A graph G is *minimally connected* with respect to $\mathcal{A}$ if

(M1) G is obtained from $\bigcup_{1 \leqslant i \leqslant m} A_i$ by adding $m - 1$ edges $e_1, \ldots, e_{m-1}$,

(M2) each edge e_j joins an attachment vertex of A_i and an attachment vertex of $A_{i'}$ for some i, i' $(i \neq i')$, and

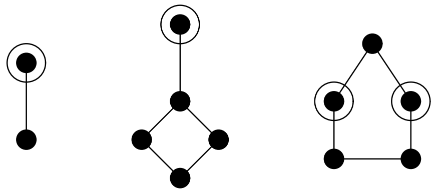
(M3) G is connected.

Note that if $\mathcal{A}$ is a set of copies of a graph A with exactly one attachment vertex v, then a minimally connected graph with respect to $\mathcal{A}$ is the (A, v)-corona of a tree.

2 Domination

Let U^0, U^1 and U^2 be the graphs depicted in Figure 3. We define the *attachment vertices* of U^i as the vertices of U^i enclosed with a circle.

Let $l \geqslant 0$ be an integer. A set $\mathcal{A}$ of vertex-disjoint graphs is (γ, l)-*optimal* if



U^0 U^1 U^2

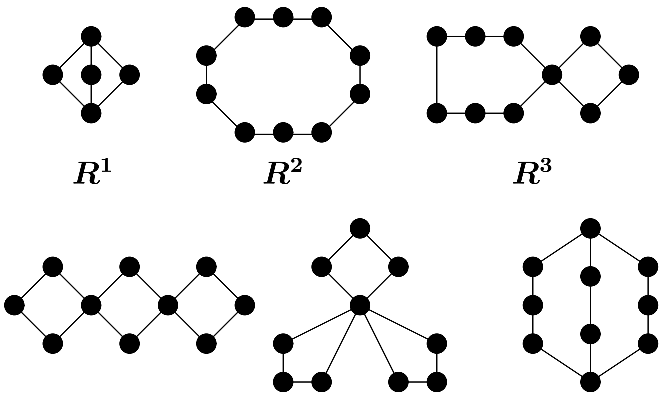
Figure 3: The graphs U^0, U^1 and U^2

(O1) each graph in $\mathcal{A}$ is a copy of one of U^0, U^1 and U^2,

(O2) $|\{A \in \mathcal{A} : A \simeq U^0\}| = l$, and

(O3) if $\mathcal{A}$ contains a copy of U^0 or U^1, then $|\mathcal{A}| \geqslant 2$.

Let $\mathfrak{F}^l$ be the set of minimal-connected graphs with respect to a (γ, l)-optimal set. Then we can easily check that every graph in $\mathfrak{F}^l$ is l-leaf minimally connected. Let $\mathfrak{R}$ be the set of graphs depicted in Figure 4.



R^1 R^2 R^3

R^4 R^5 R^6

Figure 4: The graphs belonging to $\mathfrak{R}$

In the proof of Theorem C, McCuaig and Shepherd [16] showed the following theorem.

Theorem E (McCuaig and Shepherd [16]) *Let G be a 0-leaf minimally connected graph of order n. Then $\gamma(G) \geqslant \frac{2n}{5}$ if and only if $G \in \{B^1, B^2, B^3\} \cup \mathfrak{F}^0 \cup \mathfrak{R}$. In particular, either $G \in \{B^1, B^2, B^3\}$ or $\gamma(G) \leqslant \frac{2n}{5}$.*

Now we give a natural generalization of Theorem E (by using Theorem E).

Theorem 2.1 *Let $l \geqslant 0$ be an integer, and let G be an l-leaf minimally connected graph of order n $(\geqslant 3)$. Then $\gamma(G) \geqslant \frac{2n+l}{5}$ if and only if either $l = 0$ and $G \in \{B^1, B^2, B^3\} \cup \mathfrak{R}$ or $G \in \mathfrak{F}^l$. In particular, either $l = 0$ and $G \in \{B^1, B^2, B^3\}$ or $\gamma(G) \leqslant \frac{2n+l}{5}$.*

Proof. If $l = 0$, then by Theorem E, the desired result holds. Thus we may assume that $l \geqslant 1$. If $G \in \mathcal{F}^l$, then we can easily check that $\gamma(G) = \frac{2n+l}{5}$. Thus it suffices to show that if $\gamma(G) \geqslant \frac{2n+l}{5}$, then $G \in \mathcal{F}^l$. For $v \in L(G)$, let u_v be the unique neighbor of v. Let H be the graph obtained from G by adding for each $v \in L(G)$, a new path $Q_v = x_v y_v z_v$ and edges vx_v, vz_v (see Figure 5). Then $\delta(H) \geqslant 2$ and H has a bridge

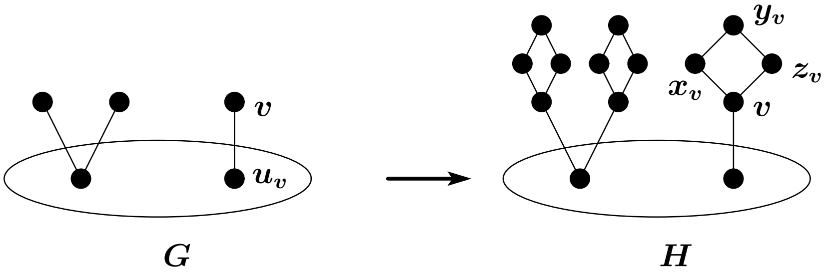


Figure 5: Construction of H

vu_v where v is a leaf of G. In particular, $H \notin \{B^1, B^2, B^3\} \cup \mathcal{R}$. Furthermore, by Lemma 1.2 and the construction of H, we see that H is 0-leaf minimally connected. This together with Theorem E leads to $\gamma(H) \leqslant \frac{2|V(H)|}{5}$. Let S be a dominating set of H with $|S| = \gamma(H)$, and let $S_0 = (S - (\bigcup_{v \in L(G)}(\{v\} \cup V(Q_v)))) \cup \{u_v : v \in L(G)\}$. Then by the construction of H, S_0 is a dominating set of G. For $v \in L(G)$, since $\{v\} \cup V(Q_v)$ cannot be dominated by one vertex of H, we have $|S \cap (\{v, u_v\} \cup V(Q_v))| \geqslant 2$. This implies that $|S_0| \leqslant |S| - l$. Since $|V(H)| = n + 3l$, we have

$$\begin{aligned} \gamma(G) &\leqslant |S_0| \\ &\leqslant |S| - l \\ &\leqslant \frac{2|V(H)|}{5} - l \\ &= \frac{2(n+3l)}{5} - l \\ &= \frac{2n+l}{5}. \end{aligned} \tag{2.1}$$

Since $\gamma(G) \geqslant \frac{2n+l}{5}$, the equality of (2.1) holds. In particular, $\gamma(H) = \frac{2|V(H)|}{5}$. Since $H \notin \{B^1, B^2, B^3\} \cup \mathcal{R}$, it follows from Theorem E that $H \in \mathcal{F}^0$. By the construction of H, there exist l disjoint cycles of H having 4 vertices such that G is obtained from H by deleting 3 consecutive vertices of each of those cycles. It follows that $G \in \mathcal{F}^l$.

This completes the proof of Theorem 2.1. □

Proof of Theorem 1.1. Let G be a connected graph of order n ($\geqslant 3$) having exactly l leaves. Since every leaf of G is adjacent to a vertex in $V(G) - L(G)$, $V(G) - L(G)$ is a dominating set of G. In particular,

$$\gamma(G) \leqslant n - l. \tag{2.2}$$

By Fact 1.1, G has a spanning l-leaf minimally connected subgraph H. Since deleting an edge cannot decrease the domination number, we have $\gamma(G) \leqslant \gamma(H)$. This together with Theorem 2.1 implies that either $l = 0$ and $H \in \{B^1, B^2, B^3\}$ or $\gamma(G) \leqslant \gamma(H) \leqslant \frac{2n+l}{5}$. Since $\mathcal{B}$ is the set of graphs B containing one of B^1, B^2 and B^3 as a spanning subgraph and satisfying $\gamma(B) > \frac{2|V(B)|}{5}$, this together with (2.2) implies that either $l = 0$ and $H \in \mathcal{B}$ or $\gamma(G) \leqslant \min\{\frac{2n+l}{5}, n-l\}$. Consequently Theorem 1.1 holds. $\square$

Remark 1 In the proof of Theorem 1.1, we further assume that $\gamma(G) \geqslant \frac{n}{2}$. *Then it follows from Theorem 2.1 that* $l = \frac{n}{2}$ *and* $H \in \mathcal{F}^{\frac{n}{2}}$, *and so* H *is the corona of a tree. By tedious arguments, this implies that* G *is either* C_4 *or the corona of a connected graph. Thus we also get Theorem B as a corollary of Theorem 2.1.*

Now we argue a sharpness of Theorem 1.1. Fix an integer $l \geqslant 0$. Let n be an integer with $n \geqslant \max\{2l, 3\}$ and $n + 3l \equiv 0 \pmod 5$, and let $\mathcal{A}$ be a (γ, l)-optimal set with $|\mathcal{A}| = \frac{n+3l}{5}$. Then every minimal-connected graph G with respect to $\mathcal{A}$ has n vertices and l leaves, and satisfies $\gamma(G) = \frac{2n+l}{5}$. Thus Theorem 1.1 for the case where $0 \leqslant l \leqslant \frac{n}{2}$ is best possible.

Let n be an integer with $\max\{l + 1, 3\} \leqslant n \leqslant 2l - 1$. Let L_1 be a star having exactly $2l-n+1$ leaves, and for each $2 \leqslant i \leqslant n-l$, let L_i be a star of order 2. For each i $(1 \leqslant i \leqslant n - l)$, we define the attachment vertex of L_i as one of the central vertices of L_i. Then every minimal-connected graph G with respect to $\{L_i : 1 \leqslant i \leqslant n - l\}$ has n vertices and l leaves, and satisfies $\gamma(G) = n - l$. Thus Theorem 1.1 for the case where $\frac{n+1}{2} \leqslant l \leqslant n - 1$ is best possible.

3 Total domination

In this section, we find a relationship between total domination and the number of leaves.

3.1 Definition and known results

Let G be a graph without isolated vertices. For two subsets X, Y of $V(G)$, we say that X *totally dominates* Y if $Y \subseteq \bigcup_{u\in X} N_G(u)$. A subset of $V(G)$ which totally dominates $V(G)$ is called a *total dominating set* of G. The minimum cardinality of a total dominating set of G is called the *total domination number* of G, and is denoted by $\gamma_t(G)$. The concept of total domination was introduced in [3], and has been actively studied (see a book [13]).

A study of upper bounds of the total domination number derives from the following theorem proved by Cockayne, Dawes and Hedetniemi [3].

Theorem F (Cockayne et al. [3]) *Let* G *be a connected graph of order* n $(\geqslant 3)$. *Then* $\gamma_t(G) \leqslant \frac{2n}{3}$.

Brigham, Carrington and Vitray [1] characterized the graphs attaining the equality of Theorem F.

Theorem G (Brigham et al. [1]) *Let G be a connected graph of order n. Then $\gamma_t(G) = \frac{2n}{3}$ if and only if G is either C_3 or C_6 or the (P_3, v)-corona of a connected graph where v is an endvertex of P_3.*

For graphs with minimum degree at least two, Henning [12] gave a sharp upper bound of the total domination number as follows (here $\mathcal{B}_t$ is the set consisting of graphs depicted in Figure 6).

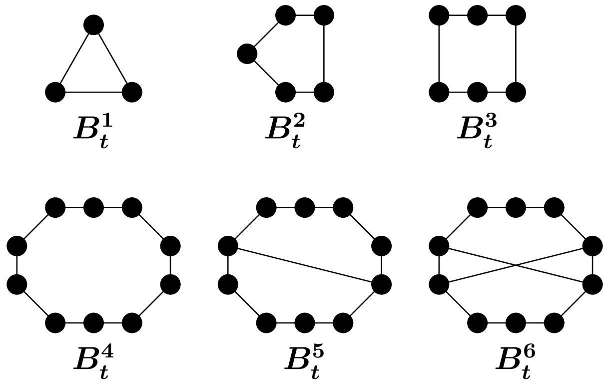


Figure 6: The graphs belonging to $\mathcal{B}_t$

Theorem H (Henning [12]) *Let G be a connected graph of order n with $\delta(G) \geqslant 2$. Then either $G \in \mathcal{B}_t$ or $\gamma_t(G) \leqslant \frac{4n}{7}$.*

Indeed, he showed a stronger theorem than Theorem H. In order to state his result, we give a further definition.

Let U_t^0 and U_t^1 be the graphs depicted in Figure 7. We define the *attachment vertex* of U_t^i as the vertex of U_t^i enclosed with a circle.

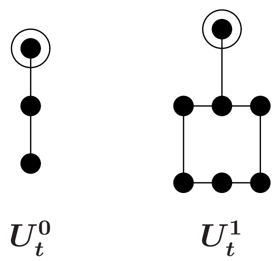


Figure 7: The graphs U_t^0 and U_t^1

Let $l \geqslant 0$ be an integer. A set $\mathcal{A}$ of vertex-disjoint graphs is (γ_t, l)-*optimal* if

(O1′) each graph in $\mathcal{A}$ is a copy of one of U_t^0 and U_t^1,

(O2′) $|\{A \in \mathcal{A} : A \simeq U_t^0\}| = l$, and

(O3′) $|\mathcal{A}| \geqslant 2$.

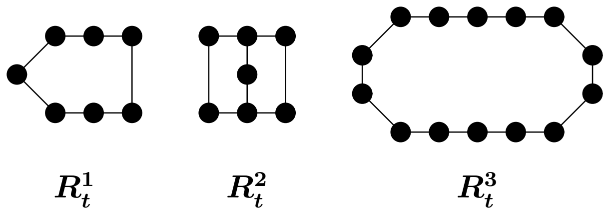


Figure 8: The graphs belonging to $\mathcal{R}_t$

Let $\mathcal{F}_t^l$ be the set of minimal-connected graphs with respect to a (γ_t, l)-optimal set. Then we can easily check that every graph in $\mathcal{F}_t^l$ is l-leaf minimally connected. Let $\mathcal{R}_t$ be the set of graphs depicted in Figure 8.

Henning [12] showed the following theorem, which gives Theorem H as a corollary.

Theorem I (Henning [12]) *Let G be a 0-leaf minimally connected graph of order n. Then $\gamma_t(G) \geqslant \frac{4n}{7}$ if and only if $G \in \{B_t^1, B_t^2, B_t^3, B_t^4\} \cup \mathcal{F}_t^0 \cup \mathcal{R}_t$. In particular, either $G \in \{B_t^1, B_t^2, B_t^3, B_t^4\}$ or $\gamma_t(G) \leqslant \frac{4n}{7}$.*

3.2 Main result for total domination

The main result in this section is the following.

Theorem 3.1 *Let $l \geqslant 0$ be an integer, and let G be an l-leaf minimally connected graph of order n $(\geqslant 3)$. Then $\gamma_t(G) \geqslant \frac{4n+2l}{7}$ if and only if either $l = 0$ and $G \in \{B_t^1, B_t^2, B_t^3, B_t^4\} \cup \mathcal{R}_t$ or $G \in \mathcal{F}_t^l$. In particular, either $l = 0$ and $G \in \{B_t^1, B_t^2, B_t^3, B_t^4\}$ or $\gamma_t(G) \leqslant \frac{4n+2l}{7}$.*

Proof. If $l = 0$, then by Theorem I, the desired result holds. Thus we may assume that $l \geqslant 1$. If $G \in \mathcal{F}_t^l$, then we can easily check that $\gamma_t(G) = \frac{4n+2l}{7}$. Thus it suffices to show that if $\gamma_t(G) \geqslant \frac{4n+2l}{7}$, then $G \in \mathcal{F}_t^l$. For $v \in L(G)$, let u_v be the unique neighbor of v. Let H be the graph obtained from G by adding for each $v \in L(G)$, a new path $Q_v = x_v y_v z_v w_v$ and edges $vx_v, u_v w_v$ (see Figure 9). Then $\delta(H) \geqslant 2$ and H

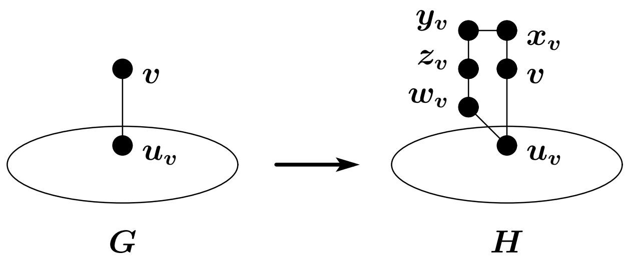


Figure 9: Construction of H

has a cutvertex u_v where v is a leaf of G. In particular, $H \notin \{B_t^1, B_t^2, B_t^3, B_t^4\} \cup \mathcal{R}_t$. Furthermore, by Lemma 1.2 and the construction of H, we see that H is 0-leaf minimally connected. This together with Theorem I leads to $\gamma_t(H) \leqslant \frac{4|V(H)|}{7}$.

Let S be a total dominating set of H with $|S| = \gamma_t(H)$. Then the following claim holds.

Claim 3.1 *Let $v \in L(G)$, and suppose that $|S \cap (\{v, u_v\} \cup V(Q_v))| \leqslant 3$. Then $|S \cap (\{v, u_v\} \cup V(Q_v))| = 3$, $S \cap \{v, w_v\} = \varnothing$ and $S \cap (N_H(u_v) - \{v, w_v\}) \neq \varnothing$.*

Proof. Since $S \cap (\{v, u_v\} \cup V(Q_v))$ totally dominates $\{v\} \cup V(Q_v)$ in H, we have $|S \cap (\{v, u_v\} \cup V(Q_v))| = 3$. Since $\gamma_t(C_6) = 4$ and $\{v, u_v\} \cup V(Q_v)$ induces C_6 in H, if $S \cap (\{v, u_v\} \cup V(Q_v))$ totally dominates $\{v, u_v\} \cup V(Q_v)$, then $|S \cap (\{v\} \cup V(Q_v))| \geqslant 4$, which is a contradiction. Thus $S \cap (\{v, u_v\} \cup V(Q_v))$ cannot totally dominate $\{u_v\}$ in H. This leads to the desired conclusion. ■

By Claim 3.1, $|S \cap (\{v, u_v\} \cup V(Q_v))| \geqslant 3$ for every $v \in L(G)$. For $v \in L(G)$, if $|S \cap (\{v, u_v\} \cup V(Q_v))| = 3$, let $T_v = \{u_v\}$; if $|S \cap (\{v, u_v\} \cup V(Q_v))| \geqslant 4$, let $T_v = \{v, u_v\}$. Let $S_0 = (S - (\bigcup_{v \in L(G)}(\{v, u_v\} \cup V(Q_v)))) \cup (\bigcup_{v \in L(G)} T_v)$. For $v \in L(G)$, it follows from Claim 3.1 and the definition of T_v that if $v \notin S_0$, then $v \notin S$. Thus we have

$$S \cap V(G) \subseteq S_0. \tag{3.1}$$

We show that S_0 is a total dominating set of G. Since S is a total dominating set of H and $\{u_v : v \in L(G)\} \subseteq S_0$, it follows that S_0 totally dominates $V(G) - \{u_v : v \in L(G)\}$ in G. Thus it suffices to show that S_0 totally dominates $\{u_v\}$ for each $v \in L(G)$. If $|S \cap (\{v, u_v\} \cup V(Q_v))| \geqslant 4$, then $v \in S_0$, and hence S_0 totally dominates $\{u_v\}$ in G; if $N_G(u_v) \cap S \cap V(G) \neq \varnothing$, then it follows from (3.1) that S_0 totally dominates $\{u_v\}$ in G. Thus we may assume that $|S \cap (\{v, u_v\} \cup V(Q_v))| = 3$ and $N_G(u_v) \cap S \cap V(G) = \varnothing$. Since $S \cap (N_H(u_v) - \{v, w_v\}) \neq \varnothing$ by Claim 3.1, this implies that there exists $v' \in L(G)$ with $v' \neq v$ such that $u_{v'} = u_v$ (i.e., u_v is adjacent to two leaves v and v' of G) and $S \cap \{v', w_{v'}\} \neq \varnothing$. Then by Claim 3.1, $|S \cap (\{v', u_{v'}\} \cup V(Q_{v'}))| \geqslant 4$, and hence $v' \in S_0$. Consequently S_0 totally dominates $\{u_v\}$ in G.

Claim 3.2 *We have $|S_0| \leqslant |S| - 2l$.*

Proof. Let $u \in \{u_v : v \in L(G)\}$, and set $X_u = N_G(u) \cap L(G)$. We first show that

$$|S_0 \cap X_u| + 1 \leqslant \sum_{v \in X_u} |S \cap (\{v\} \cup V(Q_v))| + |S \cap \{u\}| - 2|X_u|. \tag{3.2}$$

Fix a vertex $v_0 \in X_u$. Then

$$\begin{aligned} |S_0 \cap \{v_0\}| + 1 &= |T_{v_0}| \\ &\leqslant |S \cap (\{v_0, u_{v_0}\} \cup V(Q_{v_0}))| - 2 \\ &= |S \cap (\{v_0\} \cup V(Q_{v_0}))| + |S \cap \{u\}| - 2. \end{aligned} \tag{3.3}$$

Furthermore, by the definition of T_v,

$$|S_0 \cap \{v\}| \leqslant |S \cap (\{v\} \cup V(Q_v))| - 2 \text{ for every } v \in X_u - \{v_0\}. \tag{3.4}$$

It follows from (3.3) and (3.4) that (3.2) holds.

Since $u \in \{u_v : v \in L(G)\}$ is arbitrary, we have

$$\begin{aligned}
&\big|S_0 \cap \{v, u_v : v \in L(G)\}\big| \\
&\quad = \sum_{u \in \{u_v : v \in L(G)\}} (|S_0 \cap X_u| + 1) \\
&\quad \leqslant \sum_{u \in \{u_v : v \in L(G)\}} \left(\sum_{v \in X_u} |S \cap (\{v\} \cup V(Q_v))| + |S \cap \{u\}| - 2|X_u| \right) \\
&\quad = \sum_{v \in L(G)} |S \cap (\{v\} \cup V(Q_v))| + |S \cap \{u_v : v \in L(G)\}| - 2l \\
&\quad = \sum_{v \in L(G)} |S \cap (\{v, u_v\} \cup V(Q_v))| - 2l.
\end{aligned}$$

This together with the fact that $S_0 \cap (V(G) - \{v, u_v : v \in L(G)\}) = S \cap (V(G) - \{v, u_v : v \in L(G)\})$ leads to the desired conclusion. ■

Since $|V(H)| = n + 4l$, it follows from Claim 3.2 that

$$\begin{aligned}
\gamma_t(G) &\leqslant |S_0| \\
&\leqslant |S| - 2l \\
&\leqslant \frac{4|V(H)|}{7} - 2l \\
&= \frac{4(n+4l)}{7} - 2l \\
&= \frac{4n+2l}{7}.
\end{aligned} \tag{3.5}$$

Since $\gamma_t(G) \geqslant \frac{4n+2l}{7}$, the equality of (3.5) holds. In particular, $\gamma_t(H) = \frac{4|V(H)|}{7}$. Since $H \notin \{B_t^1, B_t^2, B_t^3, B_t^4\} \cup \mathcal{R}_t$, it follows from Theorem I that $H \in \mathcal{F}_t^0$. By the construction of H, there exist l disjoint cycles of H having 6 vertices such that G is obtained from H by deleting 4 consecutive vertices of each of those cycles. It follows that $G \in \mathcal{F}_t^l$. This completes the proof of Theorem 3.1. □

As a corollary of Theorem 3.1, we get the following theorem.

Theorem 3.2 *Let $l \geqslant 0$ be an integer, and let G be a connected graph of order n $(\geqslant 3)$ having exactly l leaves. Then either $l = 0$ and $G \in \mathcal{B}_t$ or*

$$\gamma_t(G) \leqslant \begin{cases} \frac{4n+2l}{7} & (0 \leqslant l \leqslant \frac{n}{3}) \\ n - l & (\frac{n+1}{3} \leqslant l \leqslant n-2) \\ 2 & (l = n-1). \end{cases}$$

Proof. If $l = n-1$, then G is a star, and hence $\gamma_t(G) = 2$, as desired. Thus we may assume that $l \leqslant n-2$. Then $|V(G) - L(G)| \geqslant 2$ and each vertex in $V(G)$ is adjacent to a vertex in $V(G) - L(G)$. In particular, $V(G) - L(G)$ is a total dominating set of G, and so

$$\gamma_t(G) \leqslant n - l. \tag{3.6}$$

By Fact 1.1, G has a spanning l-leaf minimally connected subgraph H. Since deleting an edge cannot decrease the total domination number, we have $\gamma_t(G) \leqslant \gamma_t(H)$. This together with Theorem 3.1 implies that either $l = 0$ and $H \in \{B_t^1, B_t^2, B_t^3, B_t^4\}$ or $\gamma_t(G) \leqslant \gamma_t(H) \leqslant \frac{4n+2l}{7}$. Since $\mathcal{B}_t$ is the set of graphs B containing one of B_t^1, B_t^2, B_t^3 and B_t^4 as a spanning subgraph and satisfying $\gamma_t(B) > \frac{4|V(B)|}{7}$, this together with (3.6) implies that either $l = 0$ and $H \in \mathcal{B}_t$ or $\gamma_t(G) \leqslant \min\{\frac{4n+2l}{7}, n-l\}$. Consequently Theorem 3.2 holds. □

Remark 2 In the proof of Theorem 3.2, we further assume that $\gamma_t(G) \geqslant \frac{2n}{3}$. Then it follows from Theorem 3.1 that $l = \frac{n}{3}$ and $H \in \mathcal{F}_t^{\frac{n}{3}}$, and so H is the (P_3, v)-corona of a tree where v is an endvertex of P_3. By tedious arguments, this implies that G is either C_3 or C_6 or the (P_3, v)-corona of a connected graph. Thus we also get Theorem G as a corollary of Theorem 3.1.

Now we argue a sharpness of Theorem 3.2. Fix an integer $l \geqslant 0$. Let n be an integer with $n \geqslant \max\{3l, 4\}$ and $n + 4l \equiv 0 \pmod 7$, and let $\mathcal{A}$ be a (γ_t, l)-optimal set with $|\mathcal{A}| = \frac{n+4l}{7}$. Then every minimal-connected graph G with respect to $\mathcal{A}$ has n vertices and l leaves, and satisfies $\gamma_t(G) = \frac{4n+2l}{7}$. Thus Theorem 3.2 for the case where $0 \leqslant l \leqslant \frac{n}{3}$ is best possible.

Assume that $l \geqslant 2$, and let n be an integer with $2l \leqslant n \leqslant 3l - 1$. For each i $(1 \leqslant i \leqslant n - 2l)$, let L_i be a path of order 3, and for each i $(n - 2l + 1 \leqslant i \leqslant l)$, let L_i be a path of order 2. For each i $(1 \leqslant i \leqslant l)$, we define the attachment vertex of L_i as one of the endvertices of L_i. Then every minimal-connected graph G with respect to $\{L_i : 1 \leqslant i \leqslant l\}$ has n vertices and l leaves, and satisfies $\gamma_t(G) = n - l$. Thus Theorem 3.2 for the case where $\frac{n+1}{3} \leqslant l \leqslant \frac{n}{2}$ is best possible.

Let n be an integer with $l + 2 \leqslant n \leqslant 2l - 1$. Let L'_1 be a star having exactly $2l - n + 1$ leaves, and for each $2 \leqslant i \leqslant n - l$, let L'_i be a star of order 2. For each i $(1 \leqslant i \leqslant n - l)$, we define the attachment vertex of L'_i as one of the central vertices of L'_i. Then every minimal-connected graph G with respect to $\{L'_i : 1 \leqslant i \leqslant n - l\}$ has n vertices and l leaves, and satisfies $\gamma_t(G) = n - l$. Thus Theorem 3.2 for the case where $\frac{n+1}{2} \leqslant l \leqslant n - 2$ is best possible. Moreover, since the total domination number of a star is 2, Theorem 3.2 for the case where $l = n - 1$ is best possible.

4 Roman domination

In this section, we find a relationship between Roman domination and the number of leaves.

4.1 Definition and known results

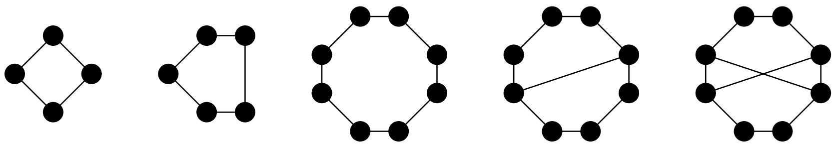
Let G be a graph. A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* of G if each vertex $y \in V(G)$ with $f(y) = 0$ is adjacent to a vertex $x \in V(G)$ with $f(x) = 2$. For a function $f : V(G) \to \{0, 1, 2\}$, the *weight* $w(f)$ of f is defined by $w(f) = \sum_{v \in V(G)} f(v)$. The minimum weight of a Roman dominating function of G is called the *Roman domination number* of G, and is denoted by $\gamma_R(G)$. The Roman domination number was introduced by Stewart [19], and was studied by Cockayne, Dreyer Jr., Hedetniemi and Hedetniemi, [4] in earnest. Recently, various properties on the Roman domination number has been explored in, for example, [8, 9, 14, 15].

Chambers, Kinnersley, Prince and West [2] gave a sharp upper bound of the Roman domination number for connected graphs with a characterization of the graphs attaining the equality.

Theorem J (Chambers et al. [2]) *Let G be a connected graph of order n ($\geqslant 3$). Then $\gamma_R(G) \leqslant \frac{4n}{5}$.*

Theorem K (Chambers et al. [2]) *Let G be a connected graph of order n. Then $\gamma_R(G) = \frac{4n}{5}$ if and only if G is either C_5 or the (P_5, v)-corona of a connected graph where v is the unique central vertex of P_5.*

They also proved the following theorem (here $\mathcal{B}_R$ is the set consisting of graphs depicted in Figure 10).

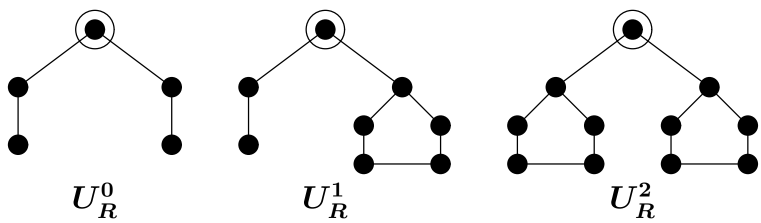


B_R^1 B_R^2 B_R^3 B_R^4 B_R^5

Figure 10: The graphs belonging to $\mathcal{B}_R$

Theorem L (Chambers et al. [2]) *Let G be a connected graph of order n with $\delta(G) \geqslant 2$. Then either $G \in \mathcal{B}_R$ or $\gamma_R(G) \leqslant \frac{8n}{11}$.*

Let U_R^0, U_R^1 and U_R^2 be the graphs depicted in Figure 11. We define the *attachment vertex* of U_R^i as the vertex of U_R^i enclosed with a circle.



U_R^0 U_R^1 U_R^2

Figure 11: The graphs U_R^0, U_R^1 and U_R^2

Let $l \geqslant 0$ be an integer. A set $\mathcal{A}$ of vertex-disjoint graphs is (γ_R, l)-*optimal* if

(O1″) each graph in $\mathcal{A}$ is a copy of one of U_R^0, U_R^1 and U_R^2, and

(O2″) $2|\{A \in \mathcal{A} : A \simeq U_R^0\}| + |\{A \in \mathcal{A} : A \simeq U_R^1\}| = l$.

Let $\mathfrak{F}_R^l$ be the set of minimal-connected graphs with respect to a (γ_R, l)-optimal set. Then we can easily check that every graph in $\mathfrak{F}_R^l$ is l-leaf minimally connected.

Chambers et al. [2] proved the following theorem, which gives Theorem L as a corollary.

Theorem M (Chambers et al. [2]) *Let G be a 0-leaf minimally connected graph of order n. Then $\gamma_R(G) \geqslant \frac{8n}{11}$ if and only if $G \in \{B_R^1, B_R^2, B_R^3\} \cup \mathfrak{F}_R^0$. In particular, either $G \in \{B_R^1, B_R^2, B_R^3\}$ or $\gamma_R(G) \leqslant \frac{8n}{11}$.*

4.2 Main result for Roman domination

The main result in this section is the following.

Theorem 4.1 *Let $l \geqslant 0$ be an integer, and let G be an l-leaf minimally connected graph of order n ($\geqslant 3$). Then $\gamma_R(G) \geqslant \frac{8n+2l}{11}$ if and only if either $l = 0$ and $G \in \{B_R^1, B_R^2, B_R^3\}$ or $G \in \mathfrak{F}_R^l$. In particular, either $l = 0$ and $G \in \{B_R^1, B_R^2, B_R^3\}$ or $\gamma_R(G) \leqslant \frac{8n+2l}{11}$.*

Proof. If $l = 0$, then by Theorem M, the desired result holds. Thus we may assume that $l \geqslant 1$. If $G \in \mathfrak{F}_R^l$, then by tedious arguments, we can check that $\gamma_R(G) = \frac{8n+2l}{11}$. Thus it suffices to show that if $\gamma_R(G) \geqslant \frac{8n+2l}{11}$, then $G \in \mathfrak{F}_R^l$. For $v \in L(G)$, let u_v be the unique neighbor of v. Let H be the graph obtained from G by adding for each $v \in L(G)$, a new path $Q_v = x_v y_v z_v$ and edges $vx_v, u_v z_v$ (see Figure 12). Then $\delta(H) \geqslant 2$ and

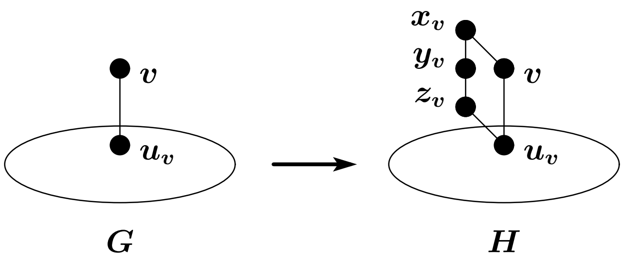


Figure 12: Construction of H

H has a cutvertex u_v where v is a leaf of G. In particular, $H \notin \{B_R^1, B_R^2, B_R^3\}$. Furthermore, by Lemma 1.2 and the construction of H, we see that H is 0-leaf minimally connected. This together with Theorem M leads to $\gamma_R(H) \leqslant \frac{8|V(H)|}{11}$.

Let $f : V(H) \to \{0, 1, 2\}$ be a Roman dominating function of H with $w(f) = \gamma_R(H)$. Then the following claim holds.

Claim 4.1 *Let $v \in L(G)$, and suppose that $\sum_{a \in \{v, u_v\} \cup V(Q_v)} f(a) \leqslant 3$. Then we have $\sum_{a \in \{v, u_v\} \cup V(Q_v)} f(a) = 3$, $f(u_v) = 0$, $f(v) \neq 2$, $f(z_v) \neq 2$ and also $(N_H(u_v) - \{v, z_v\}) \cap \{b \in V(H) : f(b) = 2\} \neq \varnothing$.*

Proof. Since each vertex $a \in \{v\} \cup V(Q_v)$ with $f(a) = 0$ is adjacent to a vertex $b \in \{v, u_v\} \cup V(Q_v)$ with $f(b) = 2$, we see that $\sum_{a\in\{v,u_v\}\cup V(Q_v)} f(a) = 3$. Let H' be the subgraph of H induced by $\{v, u_v\} \cup V(Q_v)$, and let f' be the restriction of f to $V(H')$. Since $\gamma_R(C_5) = 4$ and $H' \simeq C_5$, if either $f(u_v) \neq 0$ or $f(v) = 2$ or $f(z_v) = 2$, then f' is a Roman dominating function of H', and hence $\sum_{a\in V(H')} f(a) \geqslant 4$, which is a contradiction. Thus $f(u_v) = 0$, $f(v) \neq 2$ and $f(z_v) \neq 2$. This leads to the desired conclusion. ■

By Claim 4.1, $\sum_{a\in\{v,u_v\}\cup V(Q_v)} f(a) \geqslant 3$ for every $v \in L(G)$. Now we define the function $g : \{v, u_v : v \in L(G)\} \to \{0, 1, 2\}$ as follows: For $u \in \{u_v : v \in L(G)\}$, if there exists a vertex $v \in N_G(u) \cap L(G)$ such that $\sum_{a\in\{v,u_v\}\cup V(Q_v)} f(a) \geqslant 4$, let $g(u) = 2$; otherwise, let $g(u) = 0$. For $v \in L(G)$, if $g(u_v) = 2$, let $g(v) = 0$; if $g(u_v) = 0$, let $g(v) = 1$. Let f_0 be the function with $f_0 : V(G) \to \{0, 1, 2\}$ and

$$f_0(a) = \begin{cases} g(a) & (a \in \{v, u_v : v \in L(G)\}) \\ f(a) & (\text{Otherwise}). \end{cases}$$

It follows from Claim 4.1 and the definition of f_0, we have

$$f_0(a) \geqslant f(a) \text{ for all } a \in V(G) - L(G). \tag{4.1}$$

Claim 4.2 *The function f_0 is a Roman dominating function of G.*

Proof. Let $p \in V(G)$ be a vertex with $f_0(p) = 0$. It suffices to show that p is adjacent to a vertex of G assigned 2 by f_0. If $p \in L(G)$, then we have $f_0(u_p) = g(u_p) = 2$; if $p \in V(G) - \{v, u_v : v \in L(G)\}$, then it follows from (4.1) that there exists a vertex $q \in N_G(p)$ with $f_0(q) = f(q) = 2$. Thus we may assume that $p \in \{u_v : v \in L(G)\}$. For every $v \in N_G(p) \cap L(G)$, since $f_0(p) = 0$, we have $\sum_{a\in\{v,u_v\}\cup V(Q_v)} f(a) = 3$, and hence $f(p) = 0$, $f(v) \neq 2$ and $f(z_v) \neq 2$ by Claim 4.1. By the fact that f is a Roman dominating function of H and (4.1), there exists a vertex $q \in N_H(p) - \{v, z_v : v \in L(G)\}$ $(\subseteq N_G(p))$ with $f_0(q) = f(q) = 2$. ■

Claim 4.3 *We have $w(f_0) \leqslant w(f) - 2l$.*

Proof. Let $u \in \{u_v : v \in L(G)\}$, and set $X_u = N_G(u) \cap L(G)$. We first show that

$$\sum_{v\in X_u} f_0(v) + f_0(u) \leqslant \sum_{v\in X_u} \left(\sum_{a\in\{v\}\cup V(Q_v)} f(a) \right) + f(u) - 2|X_u|. \tag{4.2}$$

For the moment, we assume that $f_0(u) = 0$ (i.e., $g(u) = 0$). Then by the definition of $g(u)$ and Claim 4.1, $f(u) = 0$ and $\sum_{a\in\{v\}\cup V(Q_v)} f(a) = 3$ for every $v \in X_u$. Hence $\sum_{v\in X_u}(\sum_{a\in\{v\}\cup V(Q_v)} f(a)) + f(u) - 2|X_u| = |X_u|$. On the other hand, $f_0(v) = 1$ for every $v \in X_u$. Hence $\sum_{v\in X_u} f_0(v) + f_0(u) = |X_u|$. Consequently we get (4.2).

Thus we may assume that $f_0(u) = 2$ (i.e., $g(u) = 2$). Then there exists a vertex $v_0 \in X_u$ such that $\sum_{a\in\{v_0\}\cup V(Q_{v_0})} f(a) + f(u) \geqslant 4$. For a vertex $v \in X_u - \{v_0\}$, if $\sum_{a\in\{v,u\}\cup V(Q_v)} f(a) = 3$, then by Claim 4.1, $f(u) = 0$, and hence $\sum_{a\in\{v\}\cup V(Q_v)} f(a) = 3$; if $\sum_{a\in\{v,u\}\cup V(Q_v)} f(a) \geqslant 4$, then $\sum_{a\in\{v\}\cup V(Q_v)} f(a) \geqslant 2$ because $f(u) \leqslant 2$. In either case, we have $\sum_{a\in\{v\}\cup V(Q_v)} f(a) \geqslant 2$. Hence

$$\begin{aligned}
&\sum_{v\in X_u}\left(\sum_{a\in\{v\}\cup V(Q_v)} f(a)\right) + f(u) - 2|X_u| \\
&\qquad= \sum_{a\in\{v_0\}\cup V(Q_{v_0})} f(a) + f(u) + \sum_{v\in X_u-\{v_0\}}\left(\sum_{a\in\{v\}\cup V(Q_v)} f(a)\right) - 2|X_u| \\
&\qquad\geqslant 4 + 2(|X_u| - 1) - 2|X_u| \\
&\qquad= 2. \qquad (4.3)
\end{aligned}$$

On the other hand, $f_0(v) = 0$ for every $v \in X_u$. Hence $\sum_{v\in X_u} f_0(v) + f_0(u) = 2$. It follows from (4.3) that (4.2) holds.

Note that

$$\sum_{u\in\{u_v : v\in L(G)\}}\left(\sum_{v\in X_u} f_0(v) + f_0(u)\right) = \sum_{a\in\{v,u_v : v\in L(G)\}} f_0(a),$$

$$\sum_{u\in\{u_v : v\in L(G)\}}\left(\sum_{v\in X_u}\left(\sum_{a\in\{v\}\cup V(Q_v)} f(a)\right)\right) = \sum_{v\in L(G)}\left(\sum_{a\in\{v\}\cup V(Q_v)} f(a)\right)$$

and

$$\sum_{u\in\{u_v : v\in L(G)\}} |X_u| = l.$$

Since $u \in \{u_v : v \in L(G)\}$ is arbitrary, it follows from (4.2) that

$$\begin{aligned}
\sum_{a\in\{v,u_v : v\in L(G)\}} f_0(a) &\leqslant \sum_{v\in L(G)}\left(\sum_{a\in\{v\}\cup V(Q_v)} f(a)\right) + \sum_{v\in L(G)} f(u_v) - 2l \\
&= \sum_{v\in L(G)}\left(\sum_{a\in\{v,u_v\}\cup V(Q_v)} f(a)\right) - 2l.
\end{aligned}$$

This together with the fact that $f_0(a) = f(a)$ for every $a \in V(G) - \{v, u : v \in L(G)\}$ leads to the desired conclusion. ■

Since $|V(H)| = n + 3l$, it follows from Claims 4.2 and 4.3 that

$$\begin{aligned}\gamma_R(G) &\leqslant w(f_0) \\ &\leqslant w(f) - 2l \\ &\leqslant \frac{8|V(H)|}{11} - 2l \\ &= \frac{8(n+3l)}{11} - 2l \\ &= \frac{8n+2l}{11}. \end{aligned} \tag{4.4}$$

Since $\gamma_R(G) \geqslant \frac{8n+2l}{11}$, the equality of (4.4) holds. In particular, $\gamma_R(H) = \frac{8|V(H)|}{11}$. By the construction of H, there exist l disjoint cycles of H having 5 vertices such that G is obtained from H by deleting 3 consecutive vertices of each of those cycles. It follows that $G \in \mathcal{F}^l_R$. This completes the proof of Theorem 4.1. □

As a corollary of Theorem 4.1, we get the following result.

Theorem 4.2 *Let $l \geqslant 0$ be an integer, and let G be a connected graph of order n $(\geqslant 3)$ having exactly l leaves. Then either $l = 0$ and $G \in \mathcal{B}_R$ or*

$$\gamma_R(G) \leqslant \begin{cases} \frac{8n+2l}{11} & (0 \leqslant l \leqslant \frac{2n}{5}) \\ n - \frac{l}{2} & (\frac{2n+1}{5} \leqslant l \leqslant \frac{2n}{3}) \\ 2n - 2l & (\frac{2n+1}{3} \leqslant l \leqslant n-1). \end{cases}$$

We start with a lemma. A tree obtained from a star by subdividing some edges is called a *spider*. Note that any stars and any paths are spiders. We show the following lemma.

Lemma 4.3 *Let $l \geqslant 2$ be an integer, and let G be a connected graph having exactly l leaves. Then there exists a subgraph H of G such that each component of H is a spider and $L(H) = L(G)$.*

Proof. For $u, v \in L(G)$ with $u \neq v$, a path P of G joining u and v is a spider with $L(P) \subseteq L(G)$. Thus there exists a subgraph H of G such that each component of H is a spider and $L(H) \subseteq L(G)$. Choose H so that $|L(H)|$ is as large as possible. Suppose that $L(H) \neq L(G)$, and let $x \in L(G) - L(H)$. Since $d_G(x) = 1$, we have $x \in V(G) - V(H)$. Let Q be a shortest path of G joining x and $V(H)$. Write $V(H) \cap V(Q) = \{y\}$, and let T be the component of H containing y. Note that $d_T(y) \geqslant 2$. If $d_T(y) = \Delta(T)$, then $T' = T \cup Q$ is a spider with $L(T') = L(T) \cup \{x\}$, and hence $H' = H \cup Q$ is a subgraph of G such that each component of H' is a spider and $L(H') = L(H) \cup \{x\} \subseteq L(G)$, which contradicts the maximality of $|L(H)|$. Thus $d_T(y) = 2$ and $\Delta(T) \geqslant 3$. Let z be the vertex of T with $d_T(z) = \Delta(T)$, and let $w \in L(T)$ be the vertex such that the path P_z of T joining w and z contains y.

Since $d_T(z) \geqslant 3$, $T_1 = T - (V(P_z) - \{z\})$ is a spider with $L(T_1) = L(T) - \{w\}$. Let P_y be the path of P_z joining w and y. Then $T_2 = P_y \cup Q$ is a path of G with endvertices w and x. In particular, T_2 is a spider with $L(T_2) \subseteq L(G)$. Thus $H'' = (H - V(T)) \cup T_1 \cup T_2$ is a subgraph of G such that each component of H'' is a spider and $L(H'') = L(H) \cup \{x\} \subseteq L(G)$, which contradicts the maximality of $|L(H)|$. $\square$

Proof of Theorem 4.2. Let G be a connected graph of order n $(\geqslant 3)$ having exactly l leaves. It is known that $\gamma_R(G_1) \leqslant 2\gamma(G_1)$ for any graph G_1 (see [4]). This together with (2.2) in the proof of Theorem 1.1 leads to

$$\gamma_R(G) \leqslant 2n - 2l. \tag{4.5}$$

We show that $\gamma_R(G) \leqslant n - \frac{l}{2}$. If $l \in \{0,1\}$, then we can easily check that $\gamma_R(G) \leqslant n - 1$. Thus we may assume that $l \geqslant 2$. Then by Lemma 4.3, G has a subgraph G' such that each component of G' is a spider and $L(G') = L(G)$. For each component T of G', let x_T be a vertex of T with $d_T(x_T) = \Delta(T)$, and let $X = \{x_T : T \text{ is a component of } G'\}$. Note that $|X| \leqslant \frac{l}{2}$ and $|\bigcup_{x\in X}(N_G(x) - X)| \geqslant \sum_{x\in X} |N_{G'}(x)| = l$. Hence the function $f : V(G) \to \{0,1,2\}$ with

$$f(a) = \begin{cases} 2 & (a \in X) \\ 0 & (a \in \bigcup_{x\in X}(N_G(x) - X)) \\ 1 & (\text{Otherwise}) \end{cases}$$

is a Roman dominating function of G with $w(f) = n + |X| - |\bigcup_{x\in X}(N_G(x) - X)| \leqslant n - \frac{l}{2}$. Consequently we have

$$\gamma_R(G) \leqslant n - \frac{l}{2}. \tag{4.6}$$

By Fact 1.1, G has a spanning l-leaf minimally connected subgraph H. Since deleting an edge cannot decrease the Roman domination number, we have $\gamma_R(G) \leqslant \gamma_R(H)$. This together with Theorem 4.1 implies that either $l = 0$ and $G \in \{B_R^1, B_R^2, B_R^3\}$ or $\gamma_R(G) \leqslant \frac{8n+2l}{11}$. Since $\mathfrak{B}_R$ is the set of graphs B containing one of B_R^1, B_R^2 and B_R^3 as a spanning subgraph and satisfying $\gamma(B) > \frac{8|V(B)|}{11}$, this together with (4.5) and (4.6) implies that either $l = 0$ and $H \in \mathfrak{B}_R$ or $\gamma(G) \leqslant \min\{\frac{8n+2l}{11}, n - \frac{l}{2}, 2n - 2l\}$. Consequently Theorem 4.2 holds. $\square$

Remark 3 In the proof of Theorem 4.2, we further assume that $\gamma_R(G) \geqslant \frac{4n}{5}$. *Then it follows from Theorem 4.1 that* $l = \frac{2n}{5}$ *and* $H \in \mathfrak{F}_R^{\frac{2n}{5}}$, *and so* H *is the* (P_5, v)*-corona of a tree where* v *is the unique central vertex of* P_5. *By tedious arguments, this implies that* G *is either* C_5 *or the* (P_5, v)*-corona of a connected graph. Thus we also get Theorem K as a corollary of Theorem 4.1.*

Now we argue a sharpness of Theorem 4.2. Fix an integer $l \geqslant 0$. Let n be an integer with $n \geqslant \max\{\frac{5l}{2}, 5\}$ and $n + 3l \equiv 0 \pmod{11}$, and let $\mathcal{A}$ be a (γ_R, l)-optimal set with $|\mathcal{A}| = \frac{n+3l}{11}$. Then every minimal-connected graph G with respect to $\mathcal{A}$ has n vertices and l leaves, and satisfies $\gamma_R(G) = \frac{8n+2l}{11}$. Thus Theorem 4.2 for the case where $0 \leqslant l \leqslant \frac{2n}{5}$ is best possible.

Assume that l is even, and let n be an integer with $2l \leqslant n \leqslant \frac{5l}{2} - 1$. For each i $(1 \leqslant i \leqslant n - 2l)$, let L_i be a path of order 5, and for each i $(n - 2l + 1 \leqslant i \leqslant \frac{l}{2})$, let L_i be a path of order 4. For each i $(1 \leqslant i \leqslant \frac{l}{2})$, we define the attachment vertex of L_i as one of the central vertices of L_i. Then every minimal-connected graph G with respect to $\{L_i : 1 \leqslant i \leqslant \frac{l}{2}\}$ has n vertices and l leaves, and satisfies $\gamma_R(G) = n - \frac{l}{2}$. Thus Theorem 4.2 for the case where $\frac{2n+1}{5} \leqslant l \leqslant \frac{n}{2}$ is best possible.

Again assume that l is even, and let n be an integer with $\frac{3l}{2} \leqslant n \leqslant 2l - 1$. For each i $(1 \leqslant i \leqslant n - \frac{3l}{2})$, let L'_i be a path of order 4, and for each i $(n - \frac{3l}{2} + 1 \leqslant i \leqslant \frac{l}{2})$, let L'_i be a path of order 3. For each i $(1 \leqslant i \leqslant \frac{l}{2})$, we define the attachment vertex of L'_i as one of the central vertices of L'_i. Then every minimal-connected graph G with respect to $\{L'_i : 1 \leqslant i \leqslant \frac{l}{2}\}$ has n vertices and l leaves, and satisfies $\gamma_R(G) = n - \frac{l}{2}$. Thus Theorem 4.2 for the case where $\frac{n+1}{2} \leqslant l \leqslant \frac{2n}{3}$ is best possible.

Let n be an integer with $l + 1 \leqslant n \leqslant \frac{3l-1}{2}$. Let L''_1 be a star having exactly $3l - 2n + 2$ leaves, and for each $2 \leqslant i \leqslant n - l$, let L''_i be a star of order 3. For each i $(1 \leqslant i \leqslant n - l)$, we define the attachment vertex of L''_i as one of the central vertices of L''_i. Then every minimal-connected graph G with respect to $\{L''_i : 1 \leqslant i \leqslant n - l\}$ has n vertices and l leaves, and satisfies $\gamma_R(G) = 2n - 2l$. Thus Theorem 4.2 for the case where $\frac{2n+1}{3} \leqslant l \leqslant n - 1$ is best possible.

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