Do triangle-free planar graphs have exponentially many 3-colorings?*

Zdeněk Dvořák[†]
Computer Science Institute
Charles University
Prague, Czech Republic
rakdver@iuuk.mff.cuni.cz

Jean-Sébastien Sereni
Centre National de la Recherche Scientifique
LORIA
Vandœuvre-lès-Nancy, France
sereni@kam.mff.cuni.cz

Submitted: Jan 13, 2017; Accepted: Aug 23, 2017; Published: Sep 8, 2017
Mathematics Subject Classifications: 05C15, 05C10

Abstract

Thomassen conjectured that triangle-free planar graphs have an exponential number of 3-colorings. We show this conjecture to be equivalent to the following statement: there exists a positive real α such that whenever G is a planar graph and A is a subset of its edges whose deletion makes G triangle-free, there exists a subset A' of A of size at least $\alpha|A|$ such that $G-(A\setminus A')$ is 3-colorable. This equivalence allows us to study restricted situations, where we can prove the statement to be true.

Keywords: Many 3-colorings; Triangle-free planar graphs; Request graphs

1 Introduction

A now classical theorem of Grötzsch [5] asserts that every triangle-free planar graph is 3-colorable. This statement spurred a lot of interest and, over the years, many ingenious proofs have been found [3, 8, 11]. The new proofs are simpler than the original argument, and often target further developments — algorithmic aspects or extension to other surfaces. In particular, refining some of his arguments, Thomassen [12] established that every planar graph of girth at least five has exponentially many — in terms of the number of vertices — list colorings provided all lists have size at least three. This statement cannot be extended to planar graphs of girth at least four, that is, triangle-free planar graphs,

*This work was done within the scope of the International Associated Laboratory STRUCO.
[†]Supported by the Center of Excellence – Inst. for Theor. Comp. Sci., Prague, project P202/12/G061 of Czech Science Foundation and by the project LL1201 (Complex Structures: Regularities in Combinatorics and Discrete Mathematics) of the Ministry of Education of Czech Republic.

as Voigt [13] exhibited a triangle-free planar graph G along with an assignment L of lists of size three to the vertices of G such that G is not L-colorable. However, it could still be true that triangle-free planar graphs admit exponentially many 3-colorings. This was actually conjectured in 2007 by Thomassen [12, Conjecture 2.1(b)]. The formulation we give implicitly uses a theorem by Jensen and Thomassen [6, Theorem 10] that the 3-color matrix of a planar graph has full rank if and only if the graph has no triangle.

Conjecture 1. There exists a positive real number β such that every triangle-free planar graph G has at least $2^{\beta|V(G)|}$ different 3-colorings.

As reported earlier, Thomassen [12] proved the statement under the additional assumption that G has no 4-cycle. In addition, he proved that every triangle-free planar graph G admits at least $2^{|V(G)|^{1/12}/20\,000}$ different 3-colorings. This lower bound, which is sub-exponential, was later improved by Asadi, Dvořák, Postle and Thomas [1] to $2^{\sqrt{|V(G)|/212}}$. In addition, Dvořák and Lidický [4, Corollary 1.3] proved the existence of an integer D such that every triangle-free planar graph G with maximum degree at most Δ has at least $3^{|V(G)|/\Delta^D}$ different 3-colorings, thereby confirming the analogue of Conjecture 1 for all classes of triangle-free planar graphs with bounded maximum degree. Actually, this statement follows from another result of theirs [4, Corollary 1.2], which states the existence of an integer D such that if G is a triangle-free planar graph and $V' \subset V(G)$ is a subset of vertices of G such that every two distinct vertices in V' are at distance at least D in G, then any 3-precoloring of the vertices in V' extends to a 3-coloring of the whole graph G. As we will see later on, precoloring extension might be a useful tool to study the number of 3-colorings of triangle-free planar graphs.

Summing-up, we see that Conjecture 1 is still widely open. Our goal is to show the equivalence between Conjecture 1 and another statement dealing with a variation—a very natural one, in our opinion—of the usual notion of coloring, which we now introduce.

For a function $w\colon X \to \mathbf{Q}^+$ and a set $X' \subseteq X$, let $w(X') = \sum_{x\in X'} w(x)$. A *request graph* $(G, R_=, R_{\neq}, w)$ consists of a graph G, disjoint sets $R_=$ and $R_{\neq}$ of vertices of G of degree two such that $R_= \cup R_{\neq}$ is an independent set in G, and a function $w\colon R_= \cup R_{\neq} \to \mathbf{Q}^+$. The vertices in $R_= \cup R_{\neq}$ are referred to as the *requests* or *request vertices*. Let φ be a proper coloring of G. We say that a vertex $r \in R_=$ is *satisfied* if both its neighbors have the same color, and a vertex $r \in R_{\neq}$ is *satisfied* if its neighbors have different colors. For $\alpha > 0$, we say that a 3-coloring φ *satisfies α-fraction of the requests* if, letting R' be the set of satisfied vertices in $R_= \cup R_{\neq}$, we have $w(R') \geqslant \alpha w(R_= \cup R_{\neq})$. The following problem arises from the work of Asadi *et al.* [1].

Problem 2. Is there a positive real number α such that every planar triangle-free request graph admits a 3-coloring satisfying α-fraction of its requests?

As it turns out, Problem 2 admits a positive answer if and only if Conjecture 1 is true.

Theorem 3. *The following assertions are equivalent.*

(RGEN) *There exists a positive real number α such that every planar triangle-free request graph admits a 3-coloring that satisfies α-fraction of its requests.*

(EXP) *There exists a positive real number β such that every planar triangle-free graph G has at least $2^{\beta|V(G)|}$ 3-colorings.*

Before going any further, we pause to clarify the relation between (RGEN) and the statement given in the abstract of this article, namely:

(TRIA) *There is a positive real number α such that for every planar graph G and every subset X of edges such that $G-X$ is triangle-free, there exists a 3-coloring c of $G-X$ such that at least $\alpha|X|$ edges in X join vertices of different colors under c.*

It suffices to subdivide each edge in X by a vertex placed in $R_{\neq}$ to see that (TRIA) is implied by (RGEN). We thus realize that (TRIA) is equivalent to the special case of (RGEN) where $R_{=}$ is empty and w assigns each vertex in $R_{\neq}$ weight 1. As we see below in Theorem 6, this special case is in fact equivalent to (RGEN), establishing the equivalence between (RGEN) and (TRIA).

Theorem 3 is proved in Section 3. Request graphs allow for different ways to address Conjecture 1, making it possible to focus on finding just one coloring subject to given constraints rather than many. It is unclear whether this will turn out to be advantageous, as Problem 2 appears to be quite difficult. For example, in Section 5, we consider the special case under the additional assumption that there are only non-equality requests and all the requests are incident with the same vertex (that is, $R_{=} = \varnothing$ and all the vertices in $R_{\neq}$ have a common neighbor). We manage to establish the following.

Corollary 4. *Let $\alpha_0 = 1/5058$. Consider a request graph $(G, \varnothing, R_{\neq}, w)$, where G is planar triangle-free. If all vertices of $R_{\neq}$ have a common neighbor, then there exists a 3-coloring of G satisfying α_0-fraction of the requests.*

As strong as the hypothesis of Corollary 4 are, the argument turns out to be unexpectedly involved. Let v be a common neighbor to all requests in $R_{\neq}$, let T be the set of vertices other than v adjacent to the requests in $R_{\neq}$, and let S be the set of non-request neighbors of v. Without loss of generality, we can give v color 3, and thus we seek a coloring of the graph $G' = G - v - R_{\neq}$ in which all vertices in S and a constant fraction of the vertices in T only use colors from the list $\{1, 2\}$.

Since the vertices in $S \cup T$ are incident with the same face of G', this is reminiscent of a well-known result of Thomassen [9] (Theorem 16 below), which implies that such a coloring exists whenever G' has girth at least 5 and $S \cup T$ is an independent set. As it turns out, the graph G' actually can have 4-cycles, but these are relatively easy to deal with (we can eliminate separating 4-cycles *via* a precoloring extension argument, and 4-faces can be reduced in a standard way by collapsing). Nevertheless, while the set S is independent since G is triangle-free, the vertices in T can be adjacent to other vertices in $S \cup T$.

Suppose for a moment that the outer face of G' is bounded by an induced cycle C. By ignoring a constant fraction of the requests, we can assume that the distance in C between any two distinct vertices in T is at least three. Consequently, $G'[S \cup T]$ does not contain a path on four vertices; it still can, however, contain 3-vertex paths with endvertices in S and the middle vertex in T. It would be convenient to have a variation of Thomassen's result that allows such 3-vertex paths with lists of size 2; but no such variation is known or even likely to hold. Even a quite involved result of Dvořák and Kawarabayashi [2] for 3-list-coloring only allows 2-vertex paths with lists of size two (and even that only subject to the additional restriction that the distance between such paths is at least three). Overcoming these issues requires a combination of several partial coloring arguments together with elimination of a part of interfering constraints in T using a result of Naserasr [7] on odd distance coloring of planar graphs.

The paper is structured as follows. In Section 2, we perform some ground work on Problem 2 where we show that it actually suffices to restrict the attention to request graphs with only non-equality (or only equality) requests, and to unit weights; that is, it is sufficient to consider request graphs of the form $(G, R_=, \varnothing, \text{unit})$ or, equivalently, of the form $(G, \varnothing, R_{\neq}, \text{unit})$, where unit: $R_= \cup R_{\neq} \to \mathbf{Q}^+$ is the function constantly equal to 1. Section 3 is devoted to proving our main result, Theorem 3. After introducing and strengthening some auxiliary results on list colorings in Section 4, we prove Corollary 4 in Section 5.

2 Ground work on Problem 2

We start by proving the following equivalences.

Theorem 5. *Let α be a positive real number. The following assertions are equivalent.*

(RGEN) *Every planar triangle-free request graph has a 3-coloring that satisfies α-fraction of its requests.*

(RE) *Every planar triangle-free request graph $(G, R_=, \varnothing, w)$ has a 3-coloring that satisfies α-fraction of its requests.*

(REU) *Every planar triangle-free request graph $(G, R_=, \varnothing, \mathit{unit})$ admits a 3-coloring that satisfies α-fraction of its requests.*

Proof. The implications (RGEN) $\Rightarrow$ (RE) $\Rightarrow$ (REU) are trivial.

Suppose that (REU) holds, and let $(G, R_=, \varnothing, w)$ be a planar triangle-free request graph. Without loss of generality, we can multiply all the values of w by some integer, so that the values of w become integral. Let G' be the graph obtained from G by replacing each vertex $r \in R_=$ by $w(r)$ clones, and let $R'_=$ be the set of all such clones. By (REU) applied to $(G', R'_=, \varnothing, \text{unit})$, there exists a 3-coloring of G' satisfying α-fraction of its requests, and its restriction to G satisfies α-fraction of requests of $(G, R_=, \varnothing, w)$. Hence, (REU) implies (RE).

Suppose that (RE) holds, and let $(G, R_=, R_{\neq}, w)$ be a request graph. Let G' be the graph obtained from G by replacing each vertex of $R_{\neq}$ as depicted in Figure 1(a). Let $R'_=$ be the set of created vertices that are depicted in the figure by a square containing "=". Let w' be the function matching w on $R_=$ and giving each vertex of $R'_=$ the weight of the

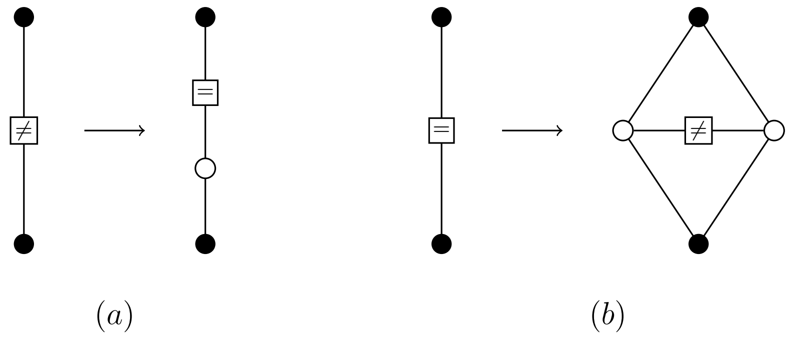


Figure 1: Gadgets showing equivalence of equality and inequality requests.

vertex of $R_{\neq}$ it replaces. Then $(G', R_= \cup R'_=, \varnothing, w')$ is a planar triangle-free request graph, and any 3-coloring of G' corresponds to a 3-coloring of G satisfying the same fraction of the requests. Hence, (RE) implies (RGEN). $\square$

Analogously (using the replacement from Figure 1(b)) we obtain the following.

Theorem 6. *Let α be a positive real number. The following assertions are equivalent.*

(RGEN) *Every planar triangle-free request graph admits a 3-coloring that satisfies α-fraction of its requests.*

(RN) *Every planar triangle-free request graph $(G, \varnothing, R_{\neq}, w)$ has a 3-coloring that satisfies α-fraction of its requests.*

(RNU) *Every planar triangle-free request graph $(G, \varnothing, R_{\neq}, unit)$ admits a 3-coloring that satisfies α-fraction of its requests.*

Let us note that (RNU) is just a reformulation of the statement from the abstract, discussed as (TRIA) earlier.

3 Satisfying requests is equivalent to having exponentially many 3-colorings

Theorem 5 implies that we can establish Theorem 3 by proving the following statement.

Theorem 7. *The following assertions are equivalent.*

(REU) *There exists a positive real number α such that every planar triangle-free request graph $(G, R_=, \varnothing, unit)$ has a 3-coloring satisfying α-fraction of its requests.*

(EXP) *There exists a positive real number β such that every planar triangle-free graph G has at least $2^{\beta|V(G)|}$ 3-colorings.*

Showing (EXP) $\Rightarrow$ (REU) is quite easy—we replace each request by a large number of vertices of degree two with the same neighbors, and observe that these vertices of degree two can only be colored in many ways if the neighbors are assigned the same color, i.e., the request is satisfied. Thus, if the graph after the replacement has exponentially many 3-colorings, then a constant fraction of the requests must be satisfied. The other implication (REU) $\Rightarrow$ (EXP) is more involved and it uses a number of auxiliary statements devised in order to prove the sub-exponential bounds of Thomassen [12, Theorem 5.8] and Asadi *et al.* [1, Theorem 1.3]. Essentially, the idea is to be able to place requests such that a 3-coloring satisfying a linear proportion of them will ensure properties that produce many different 3-colorings of the original graph. Mainly, we want the 3-coloring to produce 4-faces the vertices of which avoid one of the three colors. We shall thus pinpoint forced configurations of a minimal counter-example that allow us to put requests which, if satisfied, produce such faces. We also need to prove that there will be many such configurations, which is done using a decomposition of the graph based on its separating 5-cycles, as in the previous works on the topic.

We start by explaining why having many 4-faces as mentioned above helps us, through the following strengthening of a result of Thomassen [12]. For a 3-coloring of a plane graph, a face f is *bichromatic* if the set of colors assigned to the vertices incident to f has size two.

Lemma 8. *Let G be a connected plane triangle-free graph with $n \geqslant 3$ vertices, and for $i \geqslant 4$, let s_i be the number of faces of G of length exactly i. Let φ be a 3-coloring of G, and let q be the number of bichromatic 4-faces of G. Then G has at least $2^{(s^+ +8+q)/6}$ distinct 3-colorings, where $s^+ = s_5 + 2s_6 + \ldots = \sum_{i\geqslant 5}(i-4)s_i$.*

Proof. Let e be the number of edges of G and s the number of faces of G. By Euler's formula, $e + 2 = n + s$. Furthermore, $2e = 4s + s^+$, and thus $e = 2n - 4 - s^+/2$.

For $a, b \in \{1, 2, 3\}$ with $a < b$, we define V_{ab} to be the set of vertices of G colored by a or by b, and we let Q_{ab} be the set of 4-faces of G with all incident vertices in V_{ab}. Let X_{ab} be a minimal set of edges such that each face of Q_{ab} is incident with an edge of X_{ab}. By the minimality of X_{ab}, for every $e \in X_{ab}$ there exists a bichromatic 4-face f such that e is the only edge of X_{ab} incident with f, and thus $G[V_{ab}] - X_{ab}$ has the same components as $G[V_{ab}]$. Furthermore, e may only be incident with two 4-faces of Q_{ab}, and thus $|X_{ab}| \geqslant |Q_{ab}|/2$. Let c_{ab} be the number of components of $G[V_{ab}]$, set $n_{ab} = |V_{ab}|$ and $e_{ab} = |E(G[V_{ab}])|$. Then $e_{ab} - |X_{ab}| \geqslant n_{ab} - c_{ab}$, and thus $e_{ab} \geqslant n_{ab} - c_{ab} + |X_{ab}| \geqslant n_{ab} - c_{ab} + |Q_{ab}|/2$.

Summing these inequalities over all pairs of colors, we obtain

$$2n - 4 - s^+/2 = e = e_{12} + e_{23} + e_{13} \geqslant 2n - (c_{12} + c_{23} + c_{13}) + q/2,$$

and thus

$$c_{12} + c_{23} + c_{13} \geqslant s^+/2 + 4 + q/2.$$

By symmetry, we can assume that $c_{12} \geqslant c_{23} \geqslant c_{13}$, and thus

$$c_{12} \geqslant (s^+ + 8 + q)/6.$$

We can independently interchange the colors 1 and 2 on each component of $G[V_{12}]$, thereby obtaining $2^{c_{12}}$ different colorings of G. The statement of the lemma follows. $\square$

We also use the following result from Thomassen's paper.

Lemma 9 (Thomassen [12, Theorem 5.1])**.** *Let G be a plane triangle-free graph with outer face bounded by a cycle C of length at most 5, and let ψ be a 3-coloring of C. If $G \neq C$ and ψ does not extend to at least two 3-colorings of G, then there exists a vertex $v \in V(G) \setminus V(C)$ adjacent to two vertices of C of distinct colors.*

We need the following observation, which implicitly appears in the paper of Asadi *et al.* [1].

Lemma 10. *Let β be a positive real number and let n be an integer such that every planar triangle-free graph H with less than n vertices has at least $2^{\beta|V(H)|}$ distinct 3-colorings. Let $d_0 = \lfloor 1/\beta \rfloor$. Let G be a planar triangle-free graph with n vertices. If G has less than $2^{\beta n}$ distinct 3-colorings, then every vertex of G of degree at most d_0 is contained in a 5-cycle.*

Proof. We prove the contrapositive. Assume that the graph G contains a vertex v that has degree at most d_0 and is not contained in any 5-cycle. Let H be the graph obtained from $G - v$ by identifying all the neighbors of v to a single vertex. Note that H is planar and triangle-free, and every 3-coloring of H extends to two distinct 3-colorings of G, as we can freely choose two different colors for v. By assumptions, we know that H has at least $2^{\beta|V(H)|}$ distinct 3-colorings; hence G has at least $2^{\beta|V(H)|+1}$ distinct 3-colorings. Since $|V(H)| \geqslant n - d_0 \geqslant n - 1/\beta$, we deduce that $\beta\,|V(H)| + 1 \geqslant \beta n$, which concludes the proof. $\square$

A 5-*cycle decomposition* of a plane graph G is a pair (T, Λ), where T is a rooted tree and Λ is a function mapping each vertex of T to a subset of the plane, such that the following conditions hold.

- Let v be a vertex of T. If v is the root of T, then $\Lambda(v)$ is the whole plane, and otherwise $\Lambda(v)$ is the open disk bounded by a separating 5-cycle of G.
- Let u and v be distinct vertices of T. If u is a descendant of v, then $\Lambda(u) \subset \Lambda(v)$, that is, $\Lambda(u)$ is a proper subset of $\Lambda(v)$. If u is neither a descendant nor an ancestor of v, then $\Lambda(u) \cap \Lambda(v) = \varnothing$.

A vertex $x \in V(G)$ is *caught by the decomposition* if there exists $v \in V(T)$ such that x is contained in the boundary cycle of $\Lambda(v)$. The following is a consequence of the proof of a lemma by Asadi *et al.* [1, Lemma 2.1].

Lemma 11. *Every triangle-free plane graph G has a 5-cycle decomposition (T, Λ) such that every vertex of G that is incident with a 5-cycle is either incident with a 5-face of G or caught by (T, Λ).*

Combining these results, we obtain the following.

Corollary 12. *Let $\beta \in (0, 1/4)$ and let n be an integer such that every planar triangle-free graph H with less than n vertices has at least $2^{\beta|V(H)|}$ distinct 3-colorings. Set $d_0 = \lfloor 1/\beta \rfloor$ and $\gamma = \frac{d_0-3}{5(d_0-1)}$. Let G be a plane triangle-free graph with n vertices and s_5 faces of length 5. If G has less than $2^{\beta n}$ distinct 3-colorings, then G has a 5-cycle decomposition (T, Λ) satisfying $|V(T)| + s_5 \geqslant \gamma n$.*

Proof. By Lemma 10, every vertex of G of degree at most d_0 is contained in a 5-cycle, so in particular G has minimum degree at least 2. Let n_0 be the number of vertices of G of degree greater than d_0. Since G is planar and triangle-free, its average degree is less than 4, and thus $4n > (d_0 + 1)n_0 + 2(n - n_0) = 2n + (d_0 - 1)n_0$, and $n_0 < \frac{2}{d_0-1}n$. Hence, G contains more than $\frac{d_0-3}{d_0-1}n$ vertices of degree at most d_0, which are all contained in 5-cycles. Let (T, Λ) be a 5-cycle decomposition obtained by Lemma 11. Note that at most $5(|V(T)| + s_5)$ vertices are caught by (T, Λ) or incident with a 5-face of G, and thus the bound follows. □

Given a 5-cycle decomposition (T, Λ) of a graph G and a vertex $v \in V(T)$ with children $v_1, \ldots, v_k$ in T, we define G_v to be the subgraph of G drawn in the subset of the plane obtained from the closure of $\Lambda(v)$ by removing $\bigcup_{i=1}^{k} \Lambda(v_i)$. We say that the decomposition is *maximal* if for every $v \in V(T)$, the graph G_v contains no separating 5-cycle. A vertex v of $V(T)$ is *rich* if either v is the root of T or every precoloring of the outer face of G_v extends to at least two distinct 3-colorings of G_v; otherwise, v is *poor*. These notions are illustrated in Figure 2.

Lemma 13. *Let G be a plane triangle-free graph and let (T, Λ) be a maximal 5-cycle decomposition of G. If $v \in V(T)$ is poor, then G_v consists of the 5-cycle K_v bounding its outer face and another vertex adjacent to two vertices of K_v.*

Proof. Since v is poor, there exists a 3-coloring ψ of K_v that extends to a unique 3-coloring φ of G_v. Let $K_v = y_1y_2 \ldots y_5$. The definitions imply that $G_v \neq K_v$. Thus Lemma 9 yields that there exists a vertex $x \in V(G_v) \setminus V(K_v)$ adjacent to two vertices of K_v of distinct colors, which can be assumed to be y_1 and y_3. Since the decomposition is maximal, the 5-cycle $y_1xy_3y_4y_5$ bounds a face of G_v. If the 4-cycle $Q = y_1y_2y_3x$ also bounds a face, then the conclusion of the lemma holds. Hence assume that Q does not bound a face. Because v is poor, the precoloring of Q given by φ extends to exactly one 3-coloring of the subgraph of G_v drawn inside Q. So by Lemma 9, there exists a vertex $x' \in V(G_v) \setminus (V(K_v) \cup \{x\})$ adjacent to two vertices of Q with different colors. Since $\varphi(y_1) \neq \varphi(y_3)$, we have $\varphi(y_2) = \varphi(x)$ and thus x' is adjacent to y_1 and y_3. However, this implies that G_v contains a separating 5-cycle, namely $y_1x'y_3y_4y_5$, which contradicts the assumption that the decomposition (T, Λ) is maximal. □

Lemma 13 implies that in a maximal 5-cycle decomposition (T, Λ), each poor vertex of T has at most one son. For a poor vertex v, the *inner face* of G_v is its 5-face different from the outer face. A path $P = v_1v_2 \ldots v_k$ of poor vertices of T such that v_1 is the ancestor of all the vertices of the path is called a *k-suburb*. Let $G_P = G_{v_1} \cup \cdots \cup G_{v_k}$, and define the *inner face of G_P* to be the inner face of G_{v_k}. In the example shown in Figure 2,

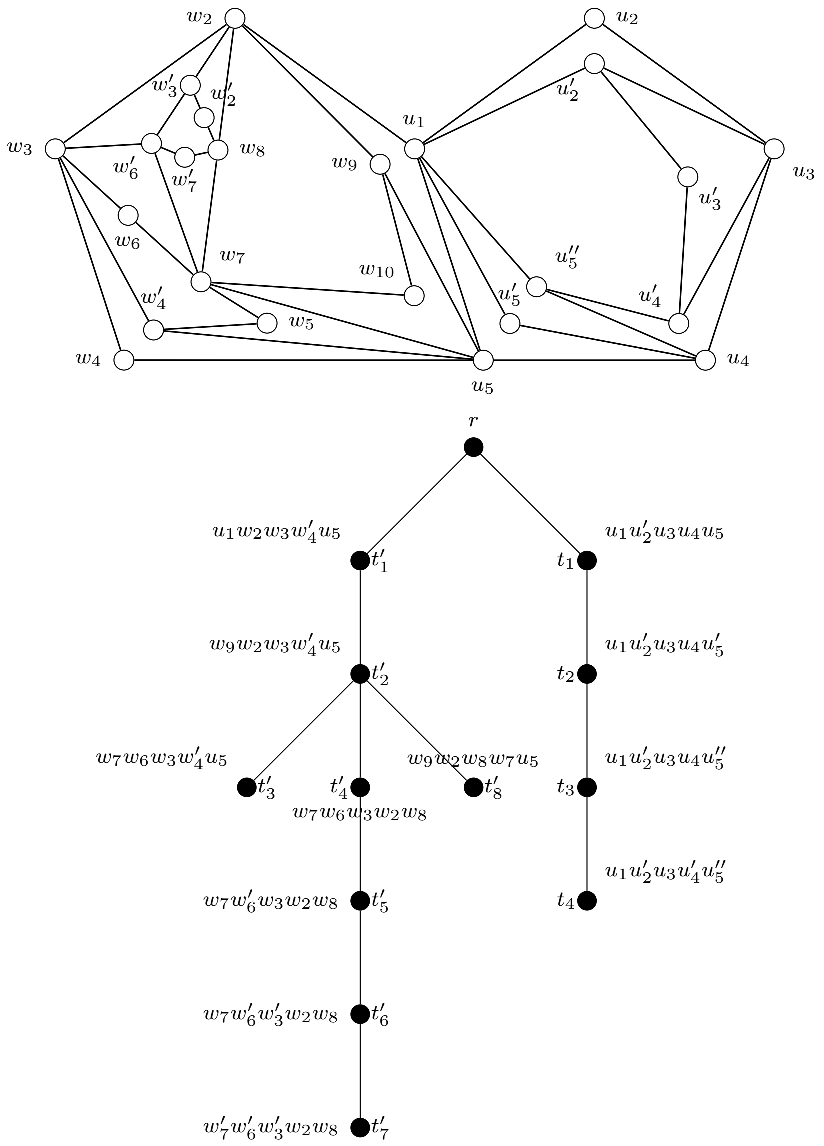


Figure 2: A graph G (top) along with its maximal 5-cycle decomposition (bottom): for each vertex of the tree, bar the root r, is shown the corresponding separating 5-cycle of G. The rich vertices are the root r and t_2'; all other vertices of the tree being poor.

the path $P = t_1 \dots t_4$ is a 4-suburb, and the graph G_P is the subgraph of G induced by $\{u_1, u_3, u_4, u_5, u_2', u_3', u_4', u_5', u_5''\}$. We say that the k-suburb P is *upwardly mobile* if every precoloring of the outer face of G_P extends to at least two distinct 3-colorings of G_P. In the example shown in Figure 2, the path $P' = t_4' \dots t_7'$ is a 4-suburb and it is updwardly mobile; the graph $G_{P'}$ being the subgraph of G induced by $\{w_2, w_3, w_6, w_7, w_8, w_2', w_3', w_6', w_7'\}$.

Let H be a plane graph with a plane subgraph F. A 3-coloring φ of H is *rearrangeable* with respect to F if there exists a 3-coloring φ' of H such that $\varphi'(v) = \varphi(v)$ for all

$v \in V(F)$ and some 4-face of H is bichromatic in φ'.

Lemma 14. *Let G be a plane triangle-free graph and let (T, Λ) be a maximal 5-cycle decomposition of G. Suppose that $P = v_1 v_2 \dots v_{11}$ is an 11-suburb in T and let F be the union of the boundary cycles of the outer and the inner face of G_P. If P is not upwardly mobile, then there exist distinct non-adjacent vertices x and y of G_P incident with a common 4-face, such that every 3-coloring φ of G_P that gives to x and y the same color is rearrangeable with respect to F.*

Proof. First, we argue that the conclusion of the lemma holds if G_P contains one of the following configurations.

- (i) A vertex $z \notin V(F)$ of degree two incident with a 4-face.
- (ii) Two adjacent vertices $z, z' \notin V(F)$ of degree three, such that z is only incident with 4-faces.
- (iii) A vertex $z \notin V(F)$ of degree four incident only with 4-faces, such that two neighbors $z_1, z_2 \notin V(F)$ of z that are not incident with the same 4-face at z have degree three, and z_1 is incident only with 4-faces.

In each of these cases, we find two non-adjacent vertices x and y incident to a 4-face f in G_P and next we let φ be an arbitrary 3-coloring of G_P that gives x and y the same color. In case (i) let $f = xzyu$ be a 4-face incident with z. We can recolor z with $\varphi(u)$ so that f is now bichromatic since $\varphi(x) = \varphi(y)$. In case (ii), let $f = zxuy$, $xzz'x'$, and $yzz'y'$ be the 4-faces incident with z. Since $\varphi(x) = \varphi(y)$, we can assume that $\varphi(x) = \varphi(y) = 1$ and $\varphi(u) = 2$. Consequently, $\varphi(x') \neq 1 \neq \varphi(y')$, and we can recolor z' by color 1 and z by color 2 to make f bichromatic. In case (iii), let zz_1xx', zz_1yy', $zz_2x''x'$, and $zz_2y''y'$ be the 4-faces incident with z, and let $f = xz_1yu$ be the further 4-face incident with z_1. Suppose that $\varphi(x) = \varphi(y) = 1$ and $\varphi(u) = 2$. If $\varphi(z) \neq 2$, then we can recolor z_1 by color 2 to make f bichromatic. If $\varphi(z) = 2$, then $\varphi(x') = \varphi(y') = 3$ and $\varphi(x'') \neq 3 \neq \varphi(y'')$. Therefore we can recolor z_2 by color 3, z by color 1, and z_1 by color 2 to make f bichromatic.

Note that Lemma 13 applies to each of $v_1, \dots, v_{11}$. For $i \in \{1, \dots, 11\}$, let the vertices of the outer face of G_{v_i} be labelled $u_1^{i-1} u_2^{i-1} \dots u_5^{i-1}$ and let the vertices of the inner face of $G_{v_{11}}$ be labelled $u_1^{11} u_2^{11} \dots u_5^{11}$, with the labels chosen so that for each $i \in \{1, \dots, 11\}$, there is a unique index $d_i \in \{1, \dots, 5\}$ such that $u_{d_i}^{i-1} \neq u_{d_i}^i$. Hence, $u_j^{i-1} = u_j^i$ for precisely four values of $j \in \{1, \dots, 5\}$.

Suppose that the suburb P is not upwardly mobile, and let ψ_0 be a precoloring of its outer face that extends to a unique 3-coloring ψ of G_P. Observe that for $i \in \{1, \dots, 11\}$ the neighbors of $u_{d_i}^i$ in the outer face of G_{v_i} must have different colors, and thus $\psi(u_{d_i}^i) = \psi(u_{d_i}^{i-1})$. We conclude that $\psi(u_j^i) = \psi(u_j^0)$ for each $i \in \{1, \dots, 11\}$ and each $j \in \{1, \dots, 5\}$.

By symmetry, we can assume that $\psi(u_1^0) = 1$, $\psi(u_2^0) = 2$, $\psi(u_3^0) = 3$, $\psi(u_4^0) = 1$, and $\psi(u_5^0) = 3$. It follows that $d_i \in \{1, 2, 3\}$ for $i \in \{1, \dots, 11\}$, hence $u_4^0 = \dots = u_4^{11}$ and $u_5^0 = \dots = u_5^{11}$. Consider the sequence $D = d_1, \dots, d_{11}$. If two consecutive elements of this sequence are equal, or if D contains a consecutive subsequence equal to $1, 3, 1$ or $3, 1, 3$, then G_P contains the configuration (i). If D contains a consecutive subsequence a, b, a, b for some distinct $a, b \in \{1, 2, 3\}$ with $|a - b| = 1$, then G_P contains the configuration (ii). In both cases, the conclusion of the lemma holds; hence, assume that no such consecutive subsequences appear in D. Furthermore, if D contains the consecutive subsequence $3, 1$, then the same graph G_P arises when this subsequence is replaced by $1, 3$. Hence we can assume that D does not contain the consecutive subsequence $3, 1$, and thus every appearance of 3 in D is followed by 2, except possibly for the one in the last position of D.

If D contains the consecutive subsequence $1, 3, 2, 1, 3$ not containing any of the last two elements of D, then by the previous paragraph D contains, as a consecutive subsequence, either $1, 3, 2, 1, 3, 2, 1$ or $1, 3, 2, 1, 3, 2, 3$. This implies that G_P contains the configuration (iii), and so the conclusion of the lemma holds. Hence we assume that D does not contain such a consecutive subsequence.

Suppose that D contains a consecutive subsequence $1, 3$, not containing the last five elements of D. The next element following 3 is necessarily 2. The next element cannot be 3, as it would be followed by 2 and D would contain a consecutive subsequence $3, 2, 3, 2$. Hence, the next element is 1 and by the previous paragraph the next one is 2, and so G_P contains the configuration (ii). It follows that we can assume that D does not contain a consecutive subsequence $1, 3$ disjoint from the last five elements of D. Hence, every appearance of 1 not contained in the last six elements of D is followed by 2.

It follows that D starts with one of the following sequences:

- $1, 2, 3, 2, 1, 2, 3, 2$;
- $2, 1, 2, 3, 2, 1, 2$; or
- $2, 1, 2, 3, 2, 1, 3$; or
- $2, 3, 2, 1, 2, 3, 2$; or
- $3, 2, 1, 2, 3, 2, 1, 2$; or
- $3, 2, 1, 2, 3, 2, 1, 3$.

In all the cases, G_P contains the configuration (ii) or (iii), and thus the conclusion of the lemma follows. □

We are now ready to demonstrate Theorem 7.

Proof of Theorem 7. We start by showing that (EXP) implies (REU), for any $\alpha \in (0, \beta)$. Fix a planar triangle-free request graph $(G, R_=, \varnothing, \text{unit})$ with $n + |R_=|$ vertices. Set $r = |R_=|$ and $N = \left\lceil \frac{n(\log_2 3 - \beta)}{\beta - \alpha} \right\rceil$. We can assume that $r \geqslant 1$. Every 3-coloring φ of $G - R_=$ greedily extends to a 3-coloring of G: let $s(\varphi)$ be the number of requests in $R_=$ satisfied by any such extension. Let G' be the graph obtained from G by replacing each vertex of $R_=$ by N clones, so $|V(G')| = n + Nr$. Observe that φ extends to exactly $2^{s(\varphi)N}$

3-colorings of G'. Let s_0 be the maximum of $s(\varphi)$ taken over all 3-colorings φ of $G - R_=$. As the number of 3-colorings of $G - R_=$ is at most 3^n, it follows that the number of 3-colorings of G' is at most $2^{s_0 N + n\log_2 3}$. On the other hand, (EXP) implies that the number of 3-colorings of G' is at least $2^{\beta(n+Nr)}$, and thus

$$\begin{aligned} s_0 N + n\log_2 3 &\geqslant \beta(n+Nr) \\ s_0 &\geqslant \beta r - \frac{(\log_2 3 - \beta)n}{N} \geqslant \alpha r. \end{aligned}$$

Hence, some 3-coloring φ of $G - R_=$ extends to a 3-coloring of G that satisfies at least $\alpha\,|R_=|$ of the requests, as required.

Next, we show that (REU) implies (EXP), for $\beta = \alpha/388$. Suppose for a contradiction that there exists a planar triangle-free graph G with less than $2^{\beta|V(G)|}$ 3-colorings. We choose such a graph G with the least possible number n of vertices. Let $d_0 = \lfloor 1/\beta \rfloor$ and $\gamma = \frac{d_0-3}{5(d_0-1)}$. Note that $d_0 \geqslant 388$, so $\gamma \geqslant \frac{77}{387}$. Let s_5 be the number of 5-faces of G. By Corollary 12, the graph G has a 5-cycle decomposition (T, Λ) satisfying $|V(T)| + s_5 \geqslant \gamma n$, and we can without loss of generality assume that the decomposition is maximal. Let r be the number of rich vertices of T and let ℓ be the number of poor leaves of T. Note that $s_5 \geqslant \ell$. Let S be a largest collection of pairwise disjoint 11-suburbs in (T, Λ). Note that at most $10(r+\ell)$ poor vertices of T belong to no member of S. Let m be the number of upwardly mobile suburbs in S, and let S_0 be the subset of S consisting of those suburbs that are not upwardly mobile.

For each rich vertex v and each upwardly mobile suburb P, every coloring of the outer face of G_v and of G_P extends to at least two 3-colorings. Hence, we conclude that G has at least 2^{r+m} 3-colorings, and thus $r + m < \beta n$. Hence

$$\begin{aligned} |S_0| &\geqslant \frac{|V(T)| - r - 10(r+\ell) - 11m}{11} \\ &= \frac{|V(T)| - 11(r+m) - 10\ell}{11} \\ &> \frac{77/387 - 11\beta}{11} n - s_5. \end{aligned}$$

Let $(G', R_=, \varnothing, \mathrm{unit})$ be the request graph obtained from G by adding, for every suburb in S_0, a vertex to $R_=$ adjacent to the two vertices x and y obtained from Lemma 14. By (REU), there exists a 3-coloring satisfying α-fraction of the requests, and by Lemma 14, we conclude that G has a 3-coloring with at least $\alpha\,|S_0|$ bichromatic faces. But then Lemma 8 implies that G has more than $2^{(s_5 + \alpha|S_0|)/6} \geqslant 2^{\frac{\alpha(77/387 - 11\beta)}{66} n} \geqslant 2^{\beta n}$ 3-colorings, which is a contradiction. □

4 Auxiliary results

In the rest of the paper, we will use a number of results on coloring and list coloring, which we present here. Let us formally state Grötzsch's theorem with one of its extensions.

Theorem 15 (Grötzsch [5], Thomassen [8]). *A planar triangle-free graph G is 3-colorable. Moreover, any precoloring of an $(\leqslant 5)$-cycle in G extends to a 3-coloring of G.*

Let us recall that Thomassen [9] proved the following generalization of 3-choosability of planar graphs of girth at least 5.

Theorem 16. *Let G be a plane graph of girth at least 5, let P be a subpath of G drawn in the boundary of the outer face of G with at most three vertices, and let L be an assignment of lists to the vertices of G, satisfying the following conditions. All vertices not incident with the outer face have lists of size three, vertices incident with the outer face not belonging to $V(P)$ have lists of size two or three, and vertices of P have lists of size one giving a proper coloring of P. If the vertices with list of size two form an independent set, then G is L-colorable.*

Theorem 16 can be strengthened as follows.

Theorem 17 (Dvořák and Kawarabayashi [2]). *Let G be a plane graph of girth at least 5, let $P = p_1 \dots p_k$ be a subpath of G drawn in the boundary of the outer face of G with $k \leqslant 3$, and let L be an assignment of lists to the vertices of G, satisfying the following conditions.*

(i) All vertices not incident with the outer face have lists of size three, vertices incident with the outer face not belonging to $V(P)$ have lists of size two or three, and vertices of P have lists of size one giving a proper coloring of P.

(ii) The graph G has no path $v_1v_2v_3$ with $|L(v_1)| = |L(v_2)| = |L(v_3)| = 2$.

(iii) The graph G has no path $v_1v_2v_3v_4v_5$ with $|L(v_1)| = |L(v_2)| = |L(v_4)| = |L(v_5)| = 2$ and $|L(v_3)| = 3$.

(iv) If $|V(P)| = 3$, then at least one endvertex p of P is contained in no path pv_2v_3 with $|L(v_2)| = |L(v_3)| = 2$ and no path $pv_2v_3v_4v_5$ with $|L(v_2)| = |L(v_4)| = |L(v_5)| = 2$ and $|L(v_3)| = 3$.

Then G is L-colorable.

We need the following variant of this result. If P is a path with $|V(P)| = 3$, we call the vertex of P of degree 2 the *middle vertex* of P. When $|V(P)| \leqslant 2$, we do not consider any vertex of P to be the middle one.

Lemma 18. *Let G be a plane graph of girth at least 5, let $P = p_1 \dots p_k$ be a subpath of G drawn in the boundary of the outer face of G with $k \leqslant 3$, and let L be an assignment of lists to the vertices of G, satisfying the following conditions.*

(i) All vertices not incident with the outer face have lists $\{1, 2, 3\}$, vertices incident with the outer face not belonging to $V(P)$ have lists $\{1, 2\}$ or $\{1, 2, 3\}$, and vertices of P have lists of size one giving a proper 3-coloring of P.

(ii) *The graph G has no path $v_1v_2v_3$ with $|L(v_1)| = |L(v_2)| = |L(v_3)| = 2$.*

(iii) *If $|V(P)| = 3$, then for one of the endvertices p of P, the graph G contains no path pv_1v_2 with $|L(v_1)| = |L(v_2)| = 2$.*

Then G is L-colorable.

Proof. We prove the statement by induction, assuming that it holds for all graphs with fewer than $|V(G)|$ vertices.

We can assume that G is 2-connected, the cycle K bounding its outer face has no chords except for those incident with the middle vertex of P, and there is no path xyz such that $x, z \in V(K)$, $y \notin V(K)$, x is not the middle vertex of P and $|L(z)| = 2$ — let us show the last assertion, the other ones follow similarly. If G contains such a path, then $G = G_1 \cup G_2$ for proper induced subgraphs G_1 and G_2 with $xyz = G_1 \cap G_2$ and $P \subseteq G_1$. We L-color G_1 by the induction hypothesis, modify the lists of x, y and z to single-element lists given by this coloring, and extend the coloring to G_2 by the induction hypothesis (G_2 satisfies (iii), since a path zv_1v_2 with $|L(v_1)| = |L(v_2)| = 2$ is forbidden by the assumption (ii) for G).

We exclude with a similar argument a chord incident with the middle vertex of P: let $P = p_1p_2p_3$, where G contains no path $p_3v_1v_2$ with $|L(v_1)| = |L(v_2)| = 2$. Write $G = G_1 \cup G_2$ for proper induced subgraphs G_1 and G_2 intersecting in a chord p_2v, such that $p_3 \in V(G_2)$. By the induction hypothesis, G_1 is L-colorable (since it contains only two vertices p_1 and p_2 with a list of size one). We modify the list of v to the singleton matching this L-coloring, and color G_2 by the induction hypothesis, thereby obtaining an L-coloring of G. Hence, we can assume that K is an induced cycle.

Next, suppose that G contains a path $v_1v_2v_3$ with $|L(v_1)| = |L(v_3)| = 2$ and $|L(v_2)| = 3$. By the previous arguments, $v_1v_2v_3$ is a subpath of K, each neighbor u_2 of v_2 distinct from v_1 and v_3 has a list of size three, and every neighbor of u_2 has a list of size different from two. Define N to be the set of neighbors of v_2 distinct from v_1 and v_3. Since G has girth greater than 3, N is in independent set. Let L' be obtained from L by setting the list of each vertex in N to $\{1, 2\}$. By the induction hypothesis, $G - v_2$ is L'-colorable, and we obtain an L-coloring of G by giving v_2 color 3.

Hence, we can assume that G does not contain any such path. It follows that G and L satisfy the assumptions of Theorem 17, so G is L-colorable. □

We also need the following result on extendability of 3-colorings in plane graphs of girth at least 5.

Theorem 19 (Thomassen [10])**.** *Let G be a plane graph of girth at least 5 with outer face bounded by a cycle K of length at most 9. Let L be an assignment of lists of size one to vertices of K yielding a proper coloring of K, and of lists of size three to all other vertices of G. If G is not L-colorable, then either $|K| \in \{8, 9\}$ and K has a chord, or $|K| = 9$ and a vertex of $V(G) \setminus V(K)$ has three neighbors in K.*

Let G be a plane graph, let P be a subpath of the boundary of the outer face of G, and let X be a set of edges contained in the boundary of the outer face of G forming a matching vertex-disjoint from P. Let Z be the set of vertices of G incident with P or an edge in X. Let G' be a plane graph such that G is an induced subgraph of G', $G' - V(G)$ is an induced cycle K of length $|Z|$ bounding the outer face of G', and the edges of G' between $V(K)$ and $V(G)$ form a perfect matching between $V(K)$ and Z. For each $z \in Z$, let k_z be the vertex of K matched to z. We say that G' is a *casing* for G, P and X if for all edges $xy \in X \cup E(P)$, the vertices k_x and k_y are adjacent in K and the 4-cycle k_xxyk_y bounds a face of G'. Let p be any vertex of P. For two vertices x and y incident with edges of X, we write $x \prec y$ if k_x precedes k_y in the clockwise ordering of vertices of K starting with k_p.

Let us remark that when G is 2-connected, its casing is uniquely determined and the ordering $\prec$ matches the ordering of the vertices around the outer face of G; casings are just a technical device to enable us to keep track of the order also when the boundary of the outer face of G is not a cycle.

We now give one more variation of Theorem 17 (note the change in (iii), which now permits some paths $v_1v_2v_3v_4v_5$ with $|L(v_1)| = |L(v_2)| = |L(v_4)| = |L(v_5)| = 2$, as well as the modifications to (i) and (iv)). In the situations of these theorems, we say that an edge $e = xy$ joining two vertices with lists of size two *blocks* a vertex p if there exists a path $puvxy$ with $|L(u)| = 2$ and $|L(v)| = 3$.

Lemma 20. *Let G be a plane graph of girth at least 5, let $P = p_1 \ldots p_k$ be a subpath of G drawn in the boundary of the outer face of G with $k \leqslant 3$, and let L be an assignment of lists to vertices of G, satisfying the following conditions.*

- *(i') All vertices not incident with the outer face have lists of size three, vertices incident with the outer face not belonging to $V(P)$ have lists of size two or three, and vertices of P have lists of size one giving a proper coloring of P. Furthermore, each edge of G that joins two vertices with list of size less than three is contained in the boundary of the outer face of G.*
- *(ii) The graph G has no path $v_1v_2v_3$ with $|L(v_1)| = |L(v_2)| = |L(v_3)| = 2$.*
- *(iii') Let X be the set of edges of G joining vertices with a list of size two. There exists a casing G' (with outer face K) for G, P and X, such that the following holds for the ordering $\prec$ defined by the casing. If v_1v_2 and v_4v_5 are distinct edges of X with $v_1 \prec v_2 \prec v_4 \prec v_5$, then v_2 and v_4 have no common neighbor, and v_1 and v_5 have no common neighbor.*
- *(iv') If $k = 3$, then G contains no path $p_1v_2v_3$ with $|L(v_2)| = |L(v_3)| = 2$. Furthermore, every edge $xy \in X$ of G that blocks p_1 such that $xp_3, yp_3 \notin E(G)$ also blocks p_3 and satisfies $L(p_2) \subseteq L(x) \cup L(y)$.*

Then G is L-colorable.

Proof. We prove the statement by induction on $|V(G)|$, assuming that it holds for all graphs with fewer than $|V(G)|$ vertices. Clearly, we can assume that G is connected. Also we can assume that $k \geqslant 2$, as otherwise we can add to P another vertex incident with the outer face of G.

Furthermore, we can assume that G is 2-connected and every chord of the cycle bounding the outer face of G is incident with the middle vertex of P: otherwise, suppose for instance that the outer face of G has a chord xy with neither x nor y being the middle vertex of P, and write $G = G_1 \cup G_2$ for induced subgraphs G_1 and G_2 intersecting in xy such that $P \subseteq G_1$. By the induction hypothesis, the graph G_1 has an L-coloring φ_1 (let us remark that a casing for G_1, P and $X_1 = X \cap E(G_1)$ postulated by the assumption (iii') can be obtained from G' by removing the vertices of $G_2 - \{x, y\}$, possibly removing edges between x or y and K if x or y is not incident with an edge in $E(P) \cup X_1$, and suppressing vertices of degree two in K). Let L_2 be the list assignment obtained from L by giving x and y singleton lists prescribed by φ_1, and find an L_2-coloring of G_2 by the induction hypothesis (letting X_2 be the set of edges of G_2 joining vertices with list of size two according to L_2, a casing for G_2, $P_2 = xy$ and X_2 can be constructed from G' by removing the vertices of $G_1 - \{x, y\}$ and the edges between $V(G_1)$ and $V(K)$ not incident with the edges o X_2, adding edges xk_{p_1} and yk_{p_2}, and suppressing vertices of degree two in K). This yields an L-coloring of G.

A similar argument shows that we can assume the following.

(4.1) There is no path $Q = q_1q_2q_3$ of length two with q_1 and q_3 incident with the outer face of G and not equal to the middle vertex of P, and q_2 not incident with the outer face, such that writing $G = G_1 \cup G_2$ for induced subgraphs G_1 and G_2 with intersection Q and $P \subseteq G_1$, no neighbor of q_1 in G_2 has a list of size two.

This implies that G and L satisfy the assumption (iii) of Theorem 17. Indeed, suppose that G contains a path $v_1v_2v_3v_4v_5$ with $|L(v_1)| = |L(v_2)| = |L(v_4)| = |L(v_5)| = 2$ and $|L(v_3)| = 3$. By the assumption (iii') and symmetry, we can assume that $v_1 \prec v_2 \prec v_5 \prec v_4$. Since all chords of the outer face are incident with the middle vertex of P, it follows that v_3 is not incident with the outer face. Let G_1 and G_2 be proper induced subgraphs of G such that $G_1 \cup G_2 = G$, $G_1 \cap G_2 = v_2v_3v_4$, and $P \subseteq G_1$. Note that $v_1 \in V(G_1) \setminus V(G_2)$, and by the assumption (ii) for G, we conclude that v_2 has no neighbor with a list of size two in G_2. Then the path $v_2v_3v_4$ contradicts (4.1) (with $q_i = v_{i+1}$ for $i \in \{1, 2, 3\}$).

If G and L satisfy the assumption (iv) of Theorem 17, it follows from that theorem that G is L-colorable. Hence, suppose this is not the case. Thus (iv') implies that $P = p_1p_2p_3$ and G contains an edge xy joining vertices with lists of size two that blocks p_1. Furthermore, (iv') also implies that either p_3 has a neighbor in $\{x, y\}$ or the edge xy blocks p_3. Let $p_1u_1v_1xy$ with $|L(u_1)| = 2$ and $|L(v_1)| = 3$ be a path showing that xy blocks p_1. Note that u_1 has no neighbor with a list of size two, since we showed in the previous paragraph that G satisfies the assumption (iii) of Theorem 17. By (4.1) and the absence of chords not incident with p_2, we conclude that $p_1u_1v_1xy$ is contained in the boundary of the outer face of G. By a symmetric argument at p_3, we conclude that the outer face of G is bounded by either a 7-cycle $p_1u_1v_1xyp_3p_2$ or a 9-cycle $p_1u_1v_1xyv_3u_3p_3p_2$ with $|L(u_3)| = 2$ and $|L(v_3)| = 3$. By Theorem 19, we conclude that G is L-colorable, unless its outer face is bounded by a 9-cycle and G contains a vertex z adjacent to p_2, v_1, and v_3. However, in that case G is L-colorable as well, since $L(p_2) \subseteq L(x) \cup L(y)$ by the assumption (iv'). $\square$

Finally, we consider distance colorability of planar triangle-free graphs. The *Clebsch graph* is the graph with vertex set equal to the elements of the finite field GF(16) and edges joining two elements if their difference is a perfect cube.

Theorem 21 (Naserasr [7])**.** *Every planar triangle-free graph has a homomorphism to the Clebsch graph.*

Since the Clebsch graph is triangle-free, Theorem 21 has the following consequence, also noted by Naserasr [7].

Corollary 22. *Every planar triangle-free graph has a proper coloring by* 16 *colors such that any two vertices joined by a path of length* 3 *have different colors.*

5 Requests at a vertex

In this section, we consider the case of a request graph with only non-equality requests and all requests adjacent to one vertex v. Let T be the set of vertices other than v adjacent to the requests and let S be the set of non-request neighbors of v. We can without loss of generality assign to v color 3, and thus we equivalently ask for all vertices of S as well as a constant fraction of the vertices of T to be colored from the list $\{1, 2\}$. After removing v and the request vertices, the vertices of $S \cup T$ will be incident with a single face of the graph, say the outer one. If the request graph had girth at least 5 and $S = \varnothing$, we could satisfy all requests in any independent subset of T using Theorem 16, and this would allow us to satisfy at least 1/3-fraction of all the requests. However, the graphs is only assumed to be triangle-free, and thus a more involved argument is needed.

Let us introduce a definition motivated by the situation described in the previous paragraph. Let G be a graph, let S and T be disjoint subsets of its vertices, let P be a path in G disjoint from $S \cup T$, and let $w\colon T \to \mathbf{Q}^+$ be an assignment of positive weights to the vertices in T. If S is an independent set in G, we say that $C = (G, P, S, T, w)$ is a *cog*, and the elements of T are its *demands*. A 3-*coloring* of the cog is a 3-coloring φ of G such that $\varphi(v) \in \{1, 2\}$ for all $v \in S$. For a real number α, we say that φ *satisfies* α*-fraction of demands* if $w(\varphi^{-1}(\{1, 2\}) \cap T) \geqslant \alpha w(T)$. We say that the cog is *plane* if G is a plane graph, P is a subpath of the boundary of the outer face of G, and S and T consist only of vertices incident with the outer face of G. The *girth* of the cog is defined as the length of the shortest cycle in G.

In all forthcoming figures, vertices of P are depicted by filled circles, vertices of S are depicted by squares, vertices of T are depicted by squares containing a question mark, and all other vertices are depicted by empty circles.

Let $C = (G, P, S, T, w)$ be a plane cog and let Q be an induced path in G such that the ends of Q are incident with the outer face and no other vertex or edge of Q is incident with the outer face. Then $G = G_1 \cup G_2$ for proper induced subgraphs G_1 and G_2 with

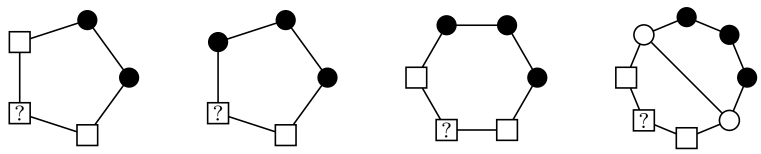


Figure 3: Obstructing cogs.

intersection Q. Suppose that $P \subseteq G_1$, and define $C_1 = (G_1, P, S \cap V(G_1), T \cap V(G_1), w \restriction (T \cap V(G_1)))$, and $C_2 = (G_2, Q, S \cap V(G_2) \setminus V(Q), T \cap V(G_2) \setminus V(Q), w \restriction (T \cap V(G_2) \setminus V(Q)))$. We say that C_1 and C_2 are the *Q-components of C*, and that C_2 is *cut off* by Q. If Q has length 2 and one of its ends belongs to $S \cup T$, we say that Q is a *weak 2-chord.* A cog $C' = (G', P', S', T', w')$ is a *subcog* of C if $G' \subseteq G$, $P' = P \cap G'$, $S' \subseteq S \cap V(G')$, $T' \subseteq T \cap V(G')$, and w' is the restriction of w to T'.

We observe that Theorem 16 implies that if $C = (G, P, S, T, w)$ is a plane cog of girth at least 5 with $|V(P)| \leqslant 3$, then every 3-coloring of P extends to a 3-coloring of the cog. In Lemma 24, we extend this to show that when $|V(P)| = 2$ (and with a few exceptions), such a 3-coloring can satisfy a constant fraction of the demands, even if the cog has girth 4. This directly implies the result for request graphs with only non-equality requests at a single vertex, Corollary 4.

A plane cog (G, P, S, T, w) is *polished* if T is an independent set and G does not contain a path $v_1v_2v_3$ with $v_1, v_3 \in T$ and $v_2 \in S$. Let us first deal with the special case of satisfying demands in polished cogs of girth at least five.

Lemma 23. *Let $\alpha_1 = 1/562$. Let $C = (G, P, S, T, w)$ be a polished plane cog of girth at least 5, where $|V(P)| \leqslant 3$. Let ψ be a 3-coloring of P. If C does not contain any of the subcogs depicted in Figure 3, then ψ extends to a 3-coloring of C satisfying α_1-fraction of the demands.*

Proof. Suppose on the contrary that C and ψ form a counterexample with $|V(G)|$ as small as possible. Clearly, G is connected and vertices not belonging to $S \cup T \cup V(P)$ have degree at least three.

Also, G is 2-connected: otherwise, let v be a cutvertex of G. If v is not the middle vertex of P, then let C_1 and C_2 be the v-components of C. Note that neither C_1 nor C_2 contains a subcog depicted in Figure 3. By the minimality of C, the precoloring ψ extends to a 3-coloring φ_1 of C_1 satisfying α_1-fraction of its demands. Furthermore, the 3-coloring of v by color $\varphi_1(v)$ extends to a 3-coloring φ_2 of C_2 satisfying α_1-fraction of its demands. The combination of φ_1 and φ_2 is a 3-coloring of C satisfying α_1-fraction of its demands, which contradicts the assumption that (C, ψ) is a counterexample. A similar argument excludes the case that v is the middle vertex of P and thus G contains no cutvertices. In particular, the outer face of G is bounded by a cycle K. Similarly, Theorem 19 implies the following.

(5.1) Every cycle in G of length at most 7 bounds a face, and the open disk bounded by any 8-cycle in G contains no vertices.

Suppose that K has a chord uv. Let us first consider the case that neither u nor v is the middle vertex of P. Let C_1 and C_2 be the uv-components of C, and let G_2 be the graph of C_2. Note that C_1 does not contain a subcog depicted in Figure 3, so the induction hypothesis ensures that ψ extends to a 3-coloring of C_1. Considering now C_2 with u and v precolored as prescribed by this extension, we deduce that that C_2 must contain the subcog depicted in Figure 3(a)—if C_2 did not contain such a subcog, we obtain a contradiction as in the previous paragraph, since C_2 has only two precolored vertices. Hence, G_2 contains a path $ux_1x_2x_3v$ with $x_1, x_3 \in S$ and $x_2 \in T$. Since C is polished, $u, v \notin S \cup T$. We obtain the following.

(5.2) The cycle K has no chord with an end in $S \cup T$, unless the other end of the chord is the middle vertex of P.

In particular, the edges ux_1, x_1x_2, x_2x_3, and x_3v are not chords, and since every 5-cycle in G bounds a face by (5.1), we conclude that G_2 is equal to the 5-cycle $vux_1x_2x_3$.

(5.3) If uv is a chord of the cycle K not incident with the middle vertex of P, then the uv-component of C cut off by uv is the cog depicted in Figure 3(a).

(5.2) implies that each vertex of T is incident with at most two vertices of S (consecutive to it in K). Since C is polished, each component of $G[S \cup T]$ is a path of length at most two contained in K, and if its length is two, then its middle vertex belongs to T. We next show the following.

(5.4) Suppose that $Q = uvz$ is a weak 2-chord of C, where $z \in S \cup T$ and u is not the middle vertex of P. Then the Q-component C' of C cut off by Q is equal to the cog depicted in Figure 3(b), and since C is polished, it follows that $u \notin S \cup T$ and $z \in S$.

Suppose for a contradiction that this is not the case, and let $Q = uvz$ be a weak 2-chord satisfying the assumptions that fails the conclusion of (5.4) with C' minimal. As before, we argue that C' contains a subcog C'' depicted in Figure 3. If C'' is the subcog from Figure 3(a), then since C is polished, C'' contains the edge uv (and not vz). Let $u' \in S$ be the neighbor of v in C'' distinct from u. However, then the cut-off $u'vz$-component of C contradicts the minimality of C' (it cannot be equal to the cog depicted in Figure 3(b) since C is polished and $u', z \in S \cup T$). Similarly, as C is polished, C'' is not the cog depicted in Figure 3(c). If C'' is the cog depicted in Figure 3(b), then (5.1) and (5.2) yield that $C' = C''$, which contradicts the definition of Q.

Finally, suppose that C'' is the cog depicted in Figure 3(d). As C is polished, the minimality of C' along with (5.1) and (5.3) imply that either $C' = C''$ or C' is the cog depicted in Figure 4. Let β be the weight of the unique demand of C''. Let $C_1 = (G_1, P, S_1, T_1, w_1)$ be the Q-component of C distinct from C'. If $z \in S$, then let $C_1' = C_1$; otherwise (when $z \in T$), let C_1' be obtained from C_1 by increasing the weight of z by β. By the minimality of C, any 3-coloring of P extends to a 3-coloring φ of C_1' satisfying

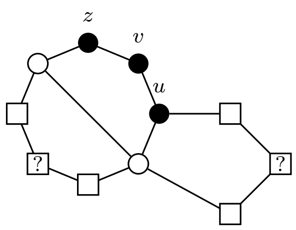
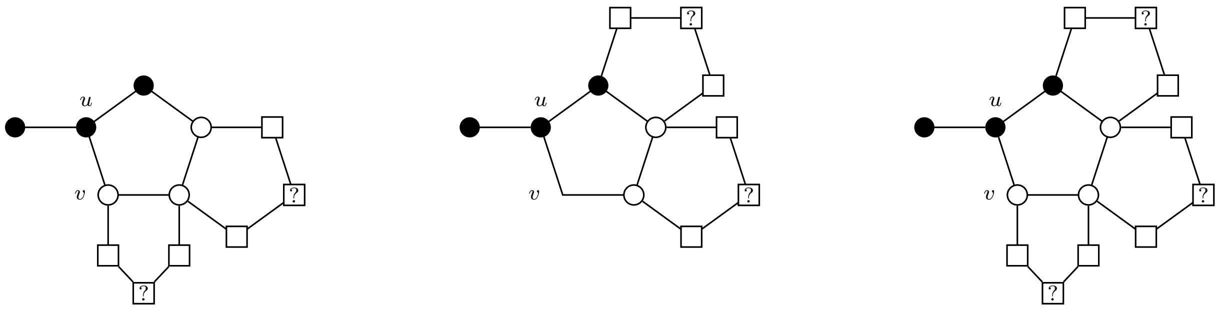


Figure 4: A cog split off by a weak 2-chord.



(a) (b) (c)

Figure 5: Compositions of the cog (d) with cogs (a) from Figure 3.

α_1-fraction of its demands. If $\varphi(u) \neq 3$, then we can color the neighbor of u in C' with a list of size three by color 3 and extend the coloring so that all demands in C' are satisfied, and the resulting 3-coloring satisfies α_1-fraction of demands of C. Hence, suppose that $\varphi(u) = 3$. If $z \in T$ and $\varphi(z) = 3$ (so that the demand of z is not satisfied), then we extend φ to C'' without satisfying its unique demand; otherwise $\varphi(z) \in \{1, 2\}$, and we observe that φ can be extended to a 3-coloring of C'' satisfying its demand. In either case, if $C' \neq C''$, then the coloring extends to a 3-coloring of C' satisfying the demand of C' not in C'', since $\varphi(u) = 3$. Observe that in all the cases, the resulting 3-coloring of C satisfies α_1-fraction of its demands. This is a contradiction, showing that (5.4) holds.

Suppose now that $|V(P)| = 3$ and K has a chord uv, where u is the middle vertex of P. Let G_1 and G_2 be proper induced subgraphs of G such that $G = G_1 \cup G_2$ and $uv = G_1 \cap G_2$. For $i \in \{1, 2\}$, let P_i be the path in G_i consisting of uv and an edge of P; let $C_i = (G_i, P_i, S \cap V(G_i) \setminus \{v\}, T \cap V(G_i) \setminus \{v\}, w \restriction (T \cap V(G_i) \setminus \{v\}))$. If C_j, for some $j \in \{1, 2\}$, does not contain any of the subcogs depicted in Figure 3, then let $C'_{3-j} = (G_{3-j}, P \cap G_{3-j}, S \cap V(G_{3-j}), T \cap V(G_{3-j}), w \restriction (T \cap V(G_{3-j})))$, extend ψ to a 3-coloring of C'_{3-j} satisfying α_1-fraction of its demands by the minimality of C, extend the resulting precoloring of P_j to a 3-coloring of C_j satisfying α_1-fraction of its demands by the minimality of C, and obtain a contradiction as before. Hence, we can assume that for each $i \in \{1, 2\}$, the cog C_i contains one of the subcogs depicted in Figure 3. If C_i contains one of the subcogs (b), (c), or (d) from that figure, it is actually equal to it by (5.1), (5.2) and (5.4), with the exception of the subcog (d), which can have copies of subcog (a) attached to two of its edges (see Figure 5). If C_i contains the subcog C_i' equal to (a) from the figure, then since C does not contain such a subcog, we conclude that C_i' contains the edge uv (and not the edge of P). But then G_i contains another chord incident with u, and we can repeat the same argument (at most once, since this chord is incident with a vertex in S and thus cannot be followed by another copy of the cog depicted in Figure 3(a)).

In conclusion, if $uv_1, \ldots, uv_m$ are all chords incident with u in cyclic order around u, then $m \leqslant 3$ and C consists of $P = p_1up_2$, these chords, a path $v_1x_1y_1v_2$ if $m = 2$ and $v_1x_1y_1v_2y_2x_2v_3$ if $m = 3$, with $y_1, y_2, v_1, v_3 \in S$ and $x_1, x_2 \in T$, and subcogs depicted in Figure 3 (b), (c), or (d) or Figure 5 attached to the paths p_1uv_1 and p_2uv_m. Note that if $m \geqslant 2$, then the demands $x_1, \ldots, x_{m-1}$ can be satisfied by giving the vertices $v_1, \ldots, v_m$ alternating colors different from $\psi(u)$, and if $m = 1$ and $v_1 \in T$, then we can always satisfy the demand of v_1 by giving it a color in $\{1, 2\} \setminus \{\psi(u)\}$. Similarly, at least a 2/3-fraction of the demands in each of the two subcogs at the ends can be satisfied with the proper choice of color of v_1 or v_m (if say $v_1 \in S$ so that its color may be forced by ψ, then since C is polished and does not contain the subcog (a), it follows that the subcog cut off by p_1uv_1 is either (d) or the one depicted in Figure 5(b); and for these, it suffices that v_1 will be colored by 1 or 2 to enable us to satisfy its demands). We conclude that every 3-coloring of P extends to a 3-coloring of C satisfying 1/4-fraction of its demands. This is a contradiction, showing the following.

(5.5) No chord of K is incident with the middle vertex of P.

Suppose that a vertex $p \in V(P)$ is incident with a chord Q, and let C' be the Q-component of C cut off by Q. By (5.3), C' is the graph depicted in Figure 3(a). If $\psi(p) = 3$, then observe that any 3-coloring of C can be modified by recoloring within C' so that the demand of C' is satisfied. Hence, the minimality of C implies the following.

(5.6) If a chord of K is incident with a vertex $p \in V(P)$, then $\psi(p) \in \{1, 2\}$.

Note that we can assume that $|V(P)| \geqslant 2$, as otherwise we can include another vertex in P without creating the subcog depicted in Figure 3(a). Next, we prove the following.

(5.7) Let $v_1v_2v_3$ be a path of G with $v_1, v_3 \in S \cup T$ and $v_2 \notin T$. Then $v_1v_2v_3$ is a subpath of K. Furthermore, if $v_1, v_3 \in S$, then v_2 is either incident with a chord or a weak 2-chord of K (together with (5.3), (5.4), and (5.5), this implies that v_1 or v_3 is an endvertex of a path of length two in $G[S \cup T]$).

Suppose for a contradiction that this is not the case. Note that $v_1v_2v_3$ is a subpath of K by (5.2), (5.4), and (5.5), and $v_2 \notin V(P)$ since $|V(P)| \geqslant 2$. Assume that v_1 and v_3 belong to S, and that v_2 is neither incident with a chord nor a weak 2-chord of K. Let N be the set of neighbors of v_2 distinct from v_1 and v_3. Since v_2 is not incident with a chord, no vertex of N belongs to $S \cup T \cup V(P)$. Since v_2 is not incident with a weak 2-chord, no vertex in N is adjacent to a vertex in $S \cup T$. Since G is triangle-free, N is an independent set. Hence, $C' = (G - v_2, P, S \cup N, T, w)$ is a polished cog. If C' does not contain any of the subcogs depicted in Figure 3, then it follows from the minimality of C that ψ extends to a 3-coloring of C' satisfying α_1-fraction of its demands, which can be extended to a 3-coloring of C by giving v_2 the color 3. This contradicts the assumption that C is a counterexample. Hence C' contains a subcog C'' depicted in Figure 3. Clearly, C'' contains a vertex $y \in N$. Furthermore, y has a neighbor z in C'' that belongs to T. It follows that either v_2y is a chord or v_2yz is a weak 2-chord of K, a contradiction which establishes (5.7).

Without loss of generality, we can assume that $G[S \cup T]$ contains no isolated vertices belonging to T, as these can be moved into S. Let T_1 and T_2 be the vertices of T belonging to paths of lengths 1 and 2 in $G[S \cup T]$, respectively.

Suppose that $w(T_1) \geqslant 2\alpha_1 w(T)$. We let $t_1, \ldots, t_n$ be the vertices of T_1 in order around K, where P is between t_n and t_1; without loss of generality, $w(t_1) \geqslant w(t_n)$. Let $T_1' = T_1$ if $n = 1$ and $T_1' = T_1 \setminus \{t_n\}$ otherwise; we have $w(T_1') \geqslant w(T_1)/2$. Let L be the list assignment for G such that

$$L(v) = \begin{cases} \{\psi(v)\} & \text{if } v \in V(P), \\ \{1, 2\} & \text{if } v \in S \cup T_1', \\ \{1, 2, 3\} & \text{otherwise.} \end{cases}$$

An L-coloring of G would yield a 3-coloring of C that satisfies all demands in T_1', with weight at least $w(T_1)/2 \geqslant \alpha_1 w(T)$. This would contradict the assumption that C is a counterexample. Therefore, G is not L-colorable, and thus it violates one of the assumptions of Lemma 18. The assumptions (i) and (ii) are clearly satisfied. Hence, the assumption (iii) is violated, so G contains a walk $v_1v_2p_1p_2p_3v_3v_4$ (where $P = p_1p_2p_3$) with $|L(v_1)| = |L(v_2)| = |L(v_3)| = |L(v_4)| = 2$; i.e., $v_1, \ldots, v_4 \in S \cup T_1'$. Consequently, (5.2) ensures that this walk is a subwalk of K, and thus it contains both t_1 and t_n. Hence, $t_1, t_n \in T_1'$, and thus $n = 1$ and $v_1 = v_3$ and $v_2 = v_4$. But then C contains the subcog depicted in Figure 3(b). This is a contradiction, showing that the following holds.

(5.8) We have $w(T_1) < 2\alpha_1 w(T)$.

We also note the following direct corollary of (5.7).

(5.9) Let $v_1v_2v_3$ be a path of G with $v_1, v_3 \in S \cup T_2$ and $v_2 \notin T_2$. Then $v_1v_2v_3$ is a subpath of K, $v_1, v_3 \in S$, and v_2 is either incident with a chord or a weak 2-chord of K.

A vertex $z \in T_2$ is *peripheral* if there exists either a chord or a weak 2-chord Q such that z is contained in the Q-component C_z of C cut off by Q, and at least one of the endvertices of Q is adjacent to a vertex in S not belonging to C_z. We choose one of the endvertices of Q with this property and call it the *connector* of z. Note that (5.3), (5.4) and (5.5) imply that the graph of C_z is a 5-cycle.

Let T_p be the set of peripheral vertices and suppose that $w(T_p) \geqslant 48\alpha_1 w(T)$. Let Y be the set of connectors of the peripheral vertices, and for $y \in Y$, let us define

$$\omega(y) = \sum_{\substack{z \in T_p \\ \text{with connector } y}} w(z).$$

Note that $\omega(Y) = w(T_p)$. By Corollary 22, there exists an independent set $Y' \subseteq Y$ such that no two vertices of Y' are joined by a path of length 3 in G and $\omega(Y') \geqslant w(T_p)/16$. Let $y_1, \ldots, y_n$ be the vertices of Y' in order around K, with P being contained between y_n and y_1. We consider the cycle $y_1 \ldots y_n$ built on Y' and we let Y'' be an independent set in this cycle such that $\omega(Y'') \geqslant \omega(Y')/3$.

Let G_0 be the subgraph of G obtained by removing the vertices in T_p with their neighbors of degree 2. Let N be the set of composed of all vertices of $G_0 - P$ that are adjacent to a vertex in Y'' by an edge that does not belong to K. Note that N is an independent set by the choice of Y'. Also (5.6) yields that each vertex in P adjacent to a vertex in Y'' has color 1 or 2. Consider the graph $G_0 - Y''$ with the list assignment L such that

$$L(v) = \begin{cases} \{\psi(v)\} & \text{if } v \in V(P), \\ \{1, 2\} & \text{if } v \in (S \cap V(G_0 - Y'')) \cup N, \\ \{1, 2, 3\} & \text{otherwise.} \end{cases}$$

Any L-coloring of $G_0 - Y''$ can be extended to a 3-coloring of C by first giving vertices in Y'' color 3 and next coloring C_z for each $z \in T_p$; if C_z contains a vertex of Y'', we can extend the coloring so that the demand of C_z is satisfied. It follows that in the resulting 3-coloring of C, the weight of satisfied demands is at least $\omega(Y'') \geqslant w(T_p)/48 \geqslant \alpha_1 w(T)$, which contradicts the assumption that C is a counterexample.

Therefore, $G_0 - Y''$ is not L-colorable, and thus it violates one of the assumptions of Lemma 18. The assumption (i) is clearly satisfied. If a vertex $v \in S$ is adjacent to a vertex $x \in N$ with a neighbor $y \in Y''$, then either yx is a chord of K or yxv is a weak 2-chord of K, and thus yxv is a subpath of the outer face of G_0. Suppose that the assumption (ii) is violated for a path $v_1v_2v_3$. Then $v_1, v_3 \in N$, $v_2 \in S$, and the outer face of G_0 contains a subpath $yv_1v_2v_3y'$ with $y, y' \in Y''$. However, this contradicts the choice of Y'', as y and y' would then be consecutive in the cycle $y_1 \ldots y_n$. Finally, suppose that the assumption (iii) is violated, and thus the outer face of G_0 contains a walk $yv_1v_2p_1p_2p_3v_3v_4y'$ (where $P = p_1p_2p_3$) with $y, y' \in Y''$, $v_1, v_4 \in N$ and $v_2, v_3 \in S$. This implies that $\{y, y'\} = \{y_1, y_n\}$, and so the choice of Y'' implies that $n = 1$ and $y = y'$. By (5.1), the interior of the 8-cycle $yv_1v_2p_1p_2p_3v_3v_4$ in G contains no vertices, and hence $V(G_0) = V(P) \cup \{y, v_1, v_2, v_3, v_4\}$. This implies that $G_0 - y$ is L-colorable, a contradiction. We thus conclude the following.

(5.10) We have $w(T_p) < 48\alpha_1 w(T)$.

Let $S_0 = S \cap V(G_0)$ and $T_0 = T_2 \setminus T_p$. From now on, we consider the cog $C_0 = (G_0, P, S_0, T_0, w \restriction T_0)$. Note that any 3-coloring of C_0 extends to a 3-coloring of C (without necessarily satisfying any additional demands). Also, the outer face of G_0 is bounded by a cycle K_0.

(5.11) The graph G_0 contains no path $v_1v_2v_3$ with $v_1, v_3 \in S_0 \cup T_0$ and $v_2 \notin T_0$.

Indeed, by (5.9) such a path would be a subpath of K and v_2 would be incident with a chord or a weak 2-chord, implying that v_1 or v_3 belongs to $V(G) \setminus V(G_0)$.

For $t \in T_0$, let B_t be the set consisting of t and its two neighbors in S_0. By (5.11), if t and t' are two distinct vertices in T_0, then no vertex of G_0 has neighbors both in B_t and $B_{t'}$. Let G_0' be the graph obtained from G_0 by, for each $t \in T_0$, contracting the edges between t and its neighbors in S_0, and by removing all edges among the neighbors of t in the resulting graph (since G_0 has girth at least 5, we know by (5.1) that there may be only one such edge, in case that t has degree two and is incident with a 5-face). Note that G_0' is plane and triangle-free, and by Corollary 22, there exists a set $T_0' \subseteq T_0$ such that $w(T_0') \geqslant w(T_0)/16$ and no two vertices of T_0' are joined by a path of length 3 in G_0'. Consequently, if $t, t' \in T_0'$ are distinct, then G_0 contains no path of length 3 with one end in B_t and the other end in $B_{t'}$.

Let $B = \bigcup_{t \in T_0'} B_t$ and let N be the set of vertices in $V(G_0) \setminus B$ that have a neighbor in B. By the previous paragraph, N induces a partial matching in G_0 (with each edge of $G_0[N]$ being contained in the neighborhood of B_t for some $t \in T_0'$ of degree two, called the *origin* of the edge). Furthermore, vertices of N have no neighbors in $S_0 \setminus B$ by (5.11), and thus $G_0[S_0 \cup N]$ is a partial matching with the same edges as $G_0[N]$. Observe also that, by (5.3), (5.4) and the construction of G_0, the endvertices of P are not adjacent to vertices incident with an edge of $G_0[N]$.

Let $p_1, \ldots, p_k, s_1, \ldots, s_{2|N|}$ be the vertices of P and of $B \cap S_0$ in order around the outer face of G_0. Let $p_1', \ldots, p_k'$ be new vertices, and let G_0'' be the graph obtained from G_0 by adding the cycle $K' = p_1' \ldots p_k' s_1 \ldots s_{2|N|}$ as its outer face as well as the edges $p_i p_i'$ for $i \in \{1, \ldots, k\}$. Let G_0' be the graph obtained from $G_0'' - (B \cap T_0)$ by removing all edges between $B \cap S_0$ and $V(G_0') \setminus V(K')$ not incident with the vertices in N. Note that G_0' forms a casing for $G_0 - B$, P, and $E(G_0[N])$; let $\prec$ be the corresponding ordering on the vertices incident with the edges of $G_0[N]$.

Let w_N be the sum of the weights of the origins of the edges of $G_0[N]$. Let H be the bipartite graph with one part consisting of the vertices in N incident with the edges of $G_0[N]$, and the other part of the vertices in $V(G_0) \setminus B$ that are adjacent to them in G_0, and the edge set consisting exactly of the edges of G_0 between these two parts. Let H' be the graph obtained from H by, for each edge xy of $G_0[N]$ with $x \prec y$, subdividing all edges of H incident with x once and then identifying x and y to a single vertex. Note that H' is plane and triangle-free, and thus by Corollary 22, there exists a subset X of the edges of $G_0[N]$ such that the corresponding vertices of H' are not joined by paths of length 3 and the set T_X of the origins of the edges in X satisfies $w(T_X) \geqslant w_N/16$.

Let T_0'' be the set consisting of the vertices in T_X and of the vertices of T_0' that are not origins of any edge of $G_0[N]$. Note that $w(T_0'') \geqslant w(T_0')/16 \geqslant w(T_0)/256$. Let $B'' = \bigcup_{t \in T_0''} B_t$ and let N'' be the set of vertices of $V(G_0) \setminus B''$ that have a neighbor in B''. By the construction of H' and the choice of X, the following holds.

(5.12) If x_1y_1 and x_2y_2 are distinct edges in $G_0[N'']$ with $x_1 \prec y_1$ and $x_2 \prec y_2$, then x_1 and y_2 have no common neighbors in G_0, and y_1 and x_2 have no common neighbors in G_0.

If $|V(P)| \leqslant 2$, then let $T_0''' = T_0''$. Otherwise, if $P = p_1p_2p_3$, we choose $T_0''' \subseteq T_0''$ as follows. For $i \in \{1, 3\}$, let O_i be the set of edges $xy \in E(G_0[N''])$ such that there exists a path p_iuvxy in $G_0 - B''$ with $u \in S_0 \cup N''$; and let R_i denote the set of origins of the edges in $O_i \setminus O_{4-i}$. By symmetry, we can assume that $w(R_1) \leqslant w(R_3)$. We let $T_0''' = T_0'' \setminus R_1$, and note that $w(T_0''') \geqslant w(T_0'')/2 \geqslant w(T_0)/512$. Let $B''' = \bigcup_{t \in T_0'''} B_t$.

Let c be a color in $\{1, 2\}$, different from $\psi(p_2)$ when $|V(P)| = 3$. Let L be the list assignment for $G_0 - B'''$ such that

$$L(v) = \begin{cases} \{\psi(v)\} & \text{if } v \in V(P), \\ \{1,2\} & \text{if } v \in S_0 \setminus B''', \\ \{1,2,3\} \setminus \{3-c\} & \text{if } v \text{ is adjacent to a vertex in } T_0''', \\ \{1,2,3\} \setminus \{c\} & \text{if } v \text{ is adjacent to a vertex in } S_0 \cap B''', \\ \{1,2,3\} & \text{otherwise.} \end{cases}$$

Note that $G_0 - B'''$ and the list assignment L satisfy the assumptions of Lemma 20 (the condition (i') is obviously satisfied, the condition (ii) holds by the choice of T_0', the condition (iii') holds by (5.12), and the condition (iv') holds by the choice of T_0''' and the color c). Hence, $G_0 - B'''$ is L-colorable, and we can extend this coloring to a 3-coloring of C_0 by giving vertices of T_0''' the color $3 - c$ and the vertices of $B''' \cap S_0$ the color c. This satisfies all demands in T_0''', whose total weight is at least $w(T_0)/512$. As this 3-coloring extends to C, we have a contradiction unless $w(T_0)/512 < \alpha_1 w(T)$.

However, if $w(T_0)/512 < \alpha_1 w(T)$ then (5.8) and (5.10) yield that

$$w(T) = w(T_1) + w(T_p) + w(T_0) < (2 + 48 + 512)\alpha_1 w(T) = w(T),$$

which is a contradiction. This concludes the proof. □

We now generalize Lemma 23 to triangle-free non-polished cogs (allowing now only a path with two vertices to be precolored).

Lemma 24. *Let $\alpha_0 = \alpha_1/9$, where α_1 is the constant from Lemma 23 (i.e., $\alpha_0 = 1/5058$). Let $C = (G, P, S, T, w)$ be a plane cog of girth at least 4, where $|V(P)| \leqslant 2$. If either $|V(P)| \leqslant 1$ or at least one vertex of P has no neighbor in S, then every 3-coloring of P extends to a 3-coloring of C satisfying α_0-fraction of the demands.*

Proof. Suppose for a contradiction that C is a counterexample with $|V(G)|$ as small as possible, and let ψ be a 3-coloring of P that does not extend to a 3-coloring of C satisfying α_0-fraction of the demands. Clearly, G is connected and all vertices not belonging to $S \cup T \cup V(P)$ have degree at least three.

Also, G is 2-connected: otherwise, let v be a cutvertex of G, and let C_1 and C_2 be the v-components of C. By the minimality of C, the precoloring ψ extends to a 3-coloring φ_1 of C_1 satisfying α_0-fraction of its demands. Furthermore, the 3-coloring of v by color $\varphi_1(v)$ extends to a 3-coloring φ_2 of C_2 satisfying α_0-fraction of its demands. The combination of φ_1 and φ_2 is a 3-coloring of C satisfying α_0-fraction of its demands, which contradicts the assumption that C is a counterexample.

Hence, the outer face of G is bounded by a cycle K. If $|V(P)| \leqslant 1$, then let $S' = S$, otherwise let S' consist of S and a vertex of P that has no neighbor in S. Suppose that

K has a chord uv, where $u \in S'$. Let C_1 and C_2 be the uv-components of C. Note that u has no neighbor in S, and thus C_2 satisfies the assumptions of Lemma 24. Hence, we obtain a contradiction as in the previous paragraph, and we conclude that K has no chords incident with vertices in S'.

By Theorem 15, it similarly follows that the open subset of the plane contained inside any $(\leqslant 5)$-cycle in G is a face of G. Suppose that G contains a 4-face $f = v_1v_2v_3v_4$. If f is the outer face, then we conclude that $V(G) = \{v_1, v_2, v_3, v_4\}$ and it is easy to verify that every 3-coloring of P extends to a 3-coloring of C satisfying α_0-fraction of its demands. Hence, f is not the outer face.

Since S' is an independent set, we can by symmetry assume that $v_1, v_3 \notin S'$. Furthermore, G contains no path v_1xyv_3 of length three: otherwise, the face f would be contained in the interior of one of the 5-cycles $v_1xyv_3v_2$ and $v_1xyv_3v_4$, thereby contradicting our previous conclusion that the interior of each 5-cycle of G is a face. Let C' be the cog obtained from C by identifying v_1 with v_3 to a new vertex v (if both v_1 and v_3 belong to T, then v has weight $w(v_1) + w(v_3)$ in C'). Note that C' satisfies all the assumptions of Lemma 24, and by the minimality of C, every 3-coloring of P extends to a 3-coloring of C' satisfying α_0-fraction of its demands. We can extend this 3-coloring to C by giving both v_1 and v_3 the color of v. Observe that the resulting 3-coloring satisfies α_0-fraction of the demands of C, unless say $v_1 \in V(P)$, $\psi(v_1) = 3$ and $v_3 \in T$. Since C is a counterexample, the latter must be the case.

If $v_2, v_4 \notin S'$, we can identify v_2 with v_4 instead and obtain a contradiction in the same way. Hence, we can assume that $v_2 \in S'$. Since K has no chords incident with vertices in S', we conclude that $v_1v_2v_3$ is a subpath of K and v_2 has degree two. By the minimality of C, there exists a 3-coloring φ of the subcog of C obtained by removing v_2, extending $\psi \restriction (V(P) \setminus \{v_2\})$ and satisfying α_0-fraction of the demands. If $v_2 \in S$, then we can give v_2 a color in $\{1, 2\} \setminus \{\varphi(v_3)\}$, since $\psi(v_1) = 3$. If $v_2 \in V(P)$, then we can assume that $\varphi(v_3) \neq \psi(v_2)$, since $\psi(v_1) = 3$, $\psi(v_2) \in \{1, 2\}$, and exchanging colors 1 and 2 in the coloring φ keeps the same weight of satisfied demands. In either case, we obtain a contradiction with the assumption that C is a counterexample. It follows that G has girth at least five.

By Theorem 15, there exists a 3-coloring ψ_1 of G. We write $K = v_1v_2 \dots v_k$, and note that there exists an assignment ψ_2 of colors in $\{1, 2, 3\}$ to the vertices of K so that no two vertices at distance (in K) exactly two from each other have the same color. Let T_1 be a subset of T of maximum weight that is monochromatic both in ψ_1 and in ψ_2; clearly, $w(T_1) \geqslant w(T)/9$. Since T_1 is monochromatic in ψ_1, it is an independent set in G. Since K has no chords incident with vertices of S, if $v_i \in S$ has a neighbor $v_j \in T$, then $j \in \{i-1, i+1\}$, with indices taken cyclically, and since T_1 is monochromatic in ψ_2, at most one such neighbor belongs to T_1. Hence, G contains no path $u_1u_2u_3$ with $u_2 \in S$ and $u_1, u_3 \in T_1$. Therefore, $C' = (G, P, S, T_1, w \restriction T_1)$ is a polished plane cog of girth at least 5, and by Lemma 23, every 3-coloring of P extends to a 3-coloring φ of C' that satisfies α_1-fraction of its demands. Note that φ is also a 3-coloring of C, and since $w(T_1) \geqslant w(T)/9$, it satisfies $(\alpha_1/9)$-fraction of the demands of C. This contradicts the assumption that C is a counterexample. □

The result on request graphs with only non-equality requests all at a single vertex now readily follows.

Proof of Corollary 4. Let v be a common neighbor of vertices of $R_{\neq}$, and let T be the set of neighbors of vertices of $R_{\neq}$ not equal to v. For $t \in T$, let us define $w'(t) = \sum_{r\in R_{\neq}, tr\in E(G)} w(r)$. Let S be the set of neighbors of v not belonging to $R_{\neq}$. Let $C = (G - (R_{\neq} \cup \{v\}), \varnothing, S, T, w')$, and note that C is a plane cog of girth at least 4. By Lemma 24, there exists a 3-coloring of C satisfying α_0-fraction of its demands. By giving v the color 3 and coloring vertices of $R_{\neq}$ by colors different from the colors of their neighbors, we obtain a 3-coloring of G that satisfies α_0-fraction of its requests, as required. □

References

[1] A. Asadi, Z. Dvořák, L. Postle, and R. Thomas, *Sub-exponentially many 3-colorings of triangle-free planar graphs*, J. Combin. Theory, Ser. B, 103(6):706–712 (2013).

[2] Z. Dvořák and K. Kawarabayashi, *Choosability of planar graphs of girth* 5, arXiv:1109.2976.

[3] Z. Dvořák, K. Kawarabayashi, and R. Thomas, *Three-coloring triangle-free planar graphs in linear time*, Trans. on Algorithms, 7 (2011), article no. 41.

[4] Z. Dvořák and B. Lidický, *Fine structure of* 4*-critical triangle-free graphs II. Planar triangle-free graphs with two precolored* 4*-cycles*, SIAM J. Discrete Math., 31(2):865–874 (2015).

[5] H. Grötzsch, *Ein Dreifarbensatz für Dreikreisfreie Netze auf der Kugel*, Math.-Natur. Reihe, 8:109–120 (1959).

[6] T. Jensen and C. Thomassen, *The color space of a graph*, J. Graph Theory, 34(3):234–245 (2000).

[7] R. Naserasr, *Homomorphisms and edge-colourings of planar graphs*, Journal of Combinatorial Theory, Series B, 97(3):394–400 (2007).

[8] C. Thomassen, *Grötzsch's 3-color theorem and its counterparts for the torus and the projective plane*, J. Combin. Theory, Ser. B, 62(2):268–279 (1994).

[9] ——, *3-list-coloring planar graphs of girth 5*, J. Combin. Theory, Ser. B, 64(1):101–107 (1995).

[10] ——, *The chromatic number of a graph of girth 5 on a fixed surface*, J. Combin. Theory, Ser. B, 87(1):38–71 (2003).

[11] ——, *A short list color proof of Grötzsch's theorem*, J. Combin. Theory, Ser. B, 88(1):189–192 (2003).

[12] ——, *Many* 3*-colorings of triangle-free planar graphs*, J. Combin. Theory, Ser. B, 97(3):334–349 (2007).

[13] M. Voigt, *A not 3-choosable planar graph without 3-cycles*, Discrete Math., 146(1-3):325–328 (1995).