Combinatorial Reductions for the Stanley Depth of I and S/I

Mitchel T. Keller[*]

Department of Mathematics
Washington & Lee University
Lexington, Virginia, U.S.A.

kellermt@wlu.edu

Stephen J. Young[†]

Pacific Northwest National Laboratory
Richland, Washington, U.S.A.

stephen.young@pnnl.gov

Abstract

We develop combinatorial tools to study the relationship between the Stanley depth of a monomial ideal I and the Stanley depth of its compliment, S/I. Using these results we are able to prove that if S is a polynomial ring with at most 5 indeterminates and I is a square-free monomial ideal, then the Stanley depth of S/I is strictly larger than the Stanley depth of I. Using a computer search, we are able to extend this strict inequality up to polynomial rings with at most 7 indeterminates. This partially answers questions asked by Propescu and Qureshi as well as Herzog.

Keywords: Stanley depth; monomial ideal; subset lattice; Boolean lattice

1 Introduction

In [24], Richard P. Stanley introduced what has come to be known to be the Stanley depth of a particular class of modules. Following Herzog's survey article [5], we give the following restricted definition that suffices for our purposes.

Definition 1. Let K be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring over K. Let M be a $\mathbb{Z}^n$-graded S-module, $m \in M$ homogenous, and $Z \subset \{x_1, \ldots, x_n\}$. We call the $K[Z]$-submodule $mK[Z]$ of M a *Stanley space* of M if $mK[Z]$ is a free $K[Z]$-submodule of M. In this case, we call $|Z|$ the *dimension* of $mK[Z]$.

A *Stanley decomposition* $\mathcal{D}$ of M is a decomposition of M as a direct sum of $\mathbb{Z}^n$-graded K-vector spaces

$$\mathcal{D}\colon M = \bigoplus_{j=1}^{r} m_j K[Z_j],$$

where each $m_j K[Z_j]$ is a Stanley space of M. Given a Stanley decomposition $\mathcal{D}$, we define its *Stanley depth* to be $\mathrm{sdepth}(\mathcal{D}) = \min\{|Z_j|\colon j = 1, \ldots, r\}$. The *Stanley depth* of M is then $\mathrm{sdepth}(M) = \max_{\mathcal{D}} \mathrm{sdepth}(\mathcal{D})$, where the maximum is taken over all Stanley decompositions of M.

In this work we focus on the relationship between $\mathrm{sdepth}\,I$ and $\mathrm{sdepth}\,S/I$ for an arbitrary monomial ideal I. In [19], Rauf showed that if I is a complete intersection monomial ideal, then $\mathrm{sdepth}(I) > \mathrm{sdepth}(S/I)$. Subsequently, Herzog's survey article [5] stated the weaker inequality $\mathrm{sdepth}(I) \geqslant \mathrm{sdepth}(S/I)$ (for arbitrary monomial ideals) as Conjecture 64, noting that the inequality is strict in all known cases. The strict version of the inequality was conjectured by Popescu and Qureshi in [17]. They gave their motivation in Proposition 5.2, which states that this inequality, when combined with a proof of a conjecture on cyclic S modules, would yield a proof of a conjecture of Stanley's for all monomial ideals. (A more general version of Conjecture 64 in [5] is presented there as Question 63, which involves the relationship between a module and its syzygy module. That question has been answered in the negative by Ichim et al. in [8].)

The conjecture of Stanley dates back to [24], where he conjectured that $\mathrm{sdepth}(M) \geqslant \mathrm{depth}(M)$ for all S-modules M. This conjecture has been resolved affirmatively in a number of cases, but recently Duval et al. provided a counterexample in [2]. In particular, they disproved an earlier conjecture given independently by Stanley [23] and Garsia [3]. That conjecture proposed that every Cohen-Macaulay simplicial complex is partitionable. By a result of Herzog, Jahan, and Yassemi [6], this conjecture is equivalent to Stanley's conjecture when M is S/I for the Stanley-Reisner ideal I of a Cohen-Macaulay simplicial complex. However, the construction of Duval et al. gives a quotient ideal as the counterexample to Stanley's conjecture, and thus it remains possible that the conjecture holds for all monomial ideals. In fact, a recent conjecture of Katthän [11] would imply Stanley's conjecture for monomial ideals as well in addition to implying that Stanley's conjecture was "almost" right for quotients of monomial ideals (in that we would have $\mathrm{sdepth}\,S/I \geqslant \mathrm{depth}\,S/I - 1$).

Our primary tool in investigating the relationship between $\mathrm{sdepth}\,I$ and $\mathrm{sdepth}\,S/I$ will be the recent work of Herzog, Vladoiu, and Zheng [7] which allows us to focus on partitions of partially ordered sets instead of algebraic decompositions. Specifically, they showed that if the module M is a monomial ideal I of S or a quotient I/J of monomial ideals of S, it is possible to determine the Stanley depth of M precisely by considering special partitions of a poset naturally associated to the monomial ideal (or quotient).

Let $J \subseteq I \subseteq S$ be monomial ideals. Suppose that I is generated by the monomials $x^{\mathbf{a}_1}, x^{\mathbf{a}_2}, \ldots, x^{\mathbf{a}_r}$ and J is generated by $x^{\mathbf{b}_1}, x^{\mathbf{b}_2}, \ldots, x^{\mathbf{b}_s}$. Here the $\mathbf{a}_i$ and $\mathbf{b}_j$ are ordered n-tuples of nonnegative integers and $x^{\mathbf{c}} = x_1^{\mathbf{c}(1)} x_2^{\mathbf{c}(2)} \cdots x_n^{\mathbf{c}(n)}$ with $\mathbf{c}(i)$ the ith entry of $\mathbf{c}$. Notice that the usual componentwise partial order on $\mathbb{N}^n$ is algebraically meaningful here,

as $\mathbf{a} \leqslant \mathbf{b}$ if and only if $x^{\mathbf{a}}$ divides $x^{\mathbf{b}}$. To define a finite subposet of $\mathbb{N}^n$ associated with I/J, we find a $\mathbf{g} \in \mathbb{N}^n$ such that $\mathbf{a}_i \leqslant \mathbf{g}$ for all i and $\mathbf{b}_j \leqslant \mathbf{g}$ for all j. (Such a $\mathbf{g}$ clearly exists, as the componentwise maximum of the $\mathbf{a}_i$ and $\mathbf{b}_j$ satisfies the inequalities.) Define a subposet $P_{I/J}^{\mathbf{g}}$ of $\mathbb{N}^n$ associated to I/J by taking as the ground set of $P_{I/J}$ all $\mathbf{c} \in \mathbb{N}^n$ such that

1. $\mathbf{c} \leqslant \mathbf{g}$,

2. $\mathbf{c} \geqslant \mathbf{a}_i$ for some i, and

3. $\mathbf{c} \not\geqslant \mathbf{b}_j$ for all j.

We generally will take $\mathbf{g}$ to be as described above and then write $P_{I/J}$ for $P_{I/J}^{\mathbf{g}}$.

The idea of Herzog et al. in [7] was to partition the poset $P_{I/J}$ into intervals $[\mathbf{a}, \mathbf{b}] = \{\mathbf{c} \in \mathbb{N}^n \colon \mathbf{a} \leqslant \mathbf{c} \leqslant \mathbf{b}\}$ and make the following definitions.

Definition 2. Let $\mathcal{P}$ be a partition of $P_{I/J}^{\mathbf{g}}$ into intervals. We call such a partition a *Stanley partition*. For $\mathbf{c} \in P_{I/J}^{\mathbf{g}}$, the number of coordinates in which $\mathbf{c}$ is equal to $\mathbf{g}$ is denoted by $\alpha(\mathbf{c}) = |\{i \in [n] \colon \mathbf{c}(i) = \mathbf{g}(i)\}|$. The *Stanley depth* of $\mathcal{P}$ is $\mathrm{sdepth}(\mathcal{P}) = \min_{[\mathbf{a}, \mathbf{b}] \in \mathcal{P}} \alpha(\mathbf{b})$, and the *Stanley depth* of $P_{I/J}^{\mathbf{g}}$ is $\mathrm{sdepth}(P_{I/J}^{\mathbf{g}}) = \max_{\mathcal{P}} \mathrm{sdepth}(\mathcal{P})$, where the maximum is taken over all partitions of $P_{S/I}^{\mathbf{g}}$ into intervals. If $\mathrm{sdepth}(\mathcal{P}) = \mathrm{sdepth}(P_{I/J}^{\mathbf{g}})$, we say that $\mathcal{P}$ is *optimal*.

Herzog et al. showed in [7] that $\mathrm{sdepth}(I/J) = \mathrm{sdepth}(P_{I/J}^{\mathbf{g}})$, providing a mechanism (albeit not terribly efficient) for computing the Stanley depth of a monomial ideal (or quotient of monomial ideals). This result generated a flurry of new activity, including the calculation of the Stanley depth of the maximal ideal of $K[x_1, \ldots, x_n]$ [1], results on the Stanley depth of squarefree Veronese ideals of $K[x_1, \ldots, x_n]$ for some degrees [4, 12], results on the Stanley depth of complete intersection monomial ideals [13, 15, 22], and CoCoA implementations of the algorithm [10, 21].

In this paper, we investigate the conjectured inequality using the poset partition approach. We prove the conjecture (with strict inequality) for $n = 3$. We then restrict our arguments to the case where I is squarefree, since in this context the posets P_I and $P_{S/I}$ can be viewed as complementary subposets of the subset lattice $\mathbf{2}^n$. In addition to being easier to read, arguments and results focusing on squarefree monomial ideals were given additional emphasis when Ichim et al. showed in [9] that

$$\mathrm{sdepth}\, I/J - \mathrm{depth}\, I/J = \mathrm{sdepth}\, I^p/J^p - \mathrm{depth}\, I^p/J^p,$$

where $J \subseteq I \subseteq S$ are ideals and I^p denotes polarization.

To facilitate our arguments for the squarefree cases, Sections 3, 4, and 5 establish a number of ancillary results focused mainly on the structure of a minimal counterexample. For $n = 4$ and $n = 5$, we prove the strict conjecture in the case where I is squarefree. We also describe computational methods that have verified that if I is squarefree, the strict conjecture also holds for $n = 6$ and $n = 7$. Therefore, the principal result of this paper is:

Theorem. *Let K be a field and $n \leqslant 7$ an integer. If I is a squarefree monomial ideal of $S = K[x_1, \ldots, x_n]$, then $\mathrm{sdepth}\, I > \mathrm{sdepth}\, S/I$.*

Notation and Terminology

If P is a poset, we call the set

$$D(A) = \{x \in P \mid x < a \text{ for some } a \in A\}$$

the *down set* of A in P. The *closed down set* of A is $D[A] = D(A) \cup A$. Dually, the *up set* of A in P is $U(A) = \{x \in P \mid x > a \text{ for some } a \in A\}$ and $U[A] = U(A) \cup A$ is the *closed up set* of A in P. If S is a set, we denote by $\mathbf{2}^S$ the lattice of all subsets of S. In the case of $[n] = \{1, 2, \ldots, n\}$, we will often write $\mathbf{2}^n$ for $\mathbf{2}^{[n]}$.

If A is an antichain in $\mathbf{2}^n$, then $D[A]$ can be viewed a simplicial complex Δ with A as its set of facets. If I_Δ is the Stanley-Reisner ideal of a simplicial complex Δ, then P_{S/I_Δ} is the same as Δ. Similarly, for a squarefree monomial ideal I, $P_{S/I}$ is the Stanley-Reisner complex of I, and the antichain of maximal elements of $P_{S/I}$ is the set of facets of the simplicial complex. If all the facets of a simplicial complex have the same dimension, then the complex is said to be *pure*. The dimension of a simplicial complex is one less than the maximum size of a facet. Because the approach taken here focuses on the poset perspective, we will prefer that terminology and viewpoint for much of this paper, although we will freely pass between the different terminologies.

2   The $n = 3$ case

Before developing our collection of results that provide additional power in the squarefree case, we address the general ideal case for $n = 3$.

Theorem 3. *If $n = 3$ and $I \subseteq S$ is a monomial ideal, then* $\operatorname{sdepth} I > \operatorname{sdepth} S/I$.

Proof. Let $I = (x^{\mathbf{a}_1}, \ldots, x^{\mathbf{a}_t})$ and fix $\mathbf{g} \in \mathbb{N}^n$ such that $\mathbf{g} \geqslant \mathbf{a}_i$ for all i. Then P_I and $P_{S/I}$ are disjoint and their union is the subposet of $\mathbb{N}^n$ containing all elements less than or equal to $\mathbf{g}$. Let M be the set of maximal elements of $P_{S/I}$. For $\mathbf{c} \in P_I \cup P_{S/I} = D[\mathbf{g}]$, let

$$\alpha(\mathbf{c}) = |\{i \in [n] \mid \mathbf{c}(i) = \mathbf{g}(i)\}|.$$

If $C \subseteq P_I \cup P_{S/I}$, let $\alpha(C) = \min_{\mathbf{c} \in C} \alpha(\mathbf{c})$.

Since $\operatorname{sdepth}(P_{S/I}) \leqslant \alpha(M)$, it suffices to show that $\operatorname{sdepth}(P_I) > \alpha(M)$. Our proof will be by induction on $m = |M|$. The base case is $m = 1$. We take the single element of M to be an ordered triple $\mathbf{b}$ and consider cases determined by $\alpha(\mathbf{b})$.

0. Suppose $\alpha(\mathbf{b}) = 0$. Partition P_I into a collection of one-dimensional intervals $[(i, j, k), (i, j, \mathbf{g}(3))]$ for $0 \leqslant i \leqslant \mathbf{g}(1)$, $0 \leqslant j \leqslant \mathbf{g}(2)$, and k as small as possible so that $(i, j, k) \in P_I$. Each interval has at least one coordinate of its upper bound in agreement with $\mathbf{g}$, so $\operatorname{sdepth} P_I \geqslant 1$.

1. Suppose $\alpha(\mathbf{b}) = 1$. Without loss of generality, assume that $\mathbf{b}(1) = \mathbf{g}(1)$ and $\mathbf{b}(2) \geqslant \mathbf{b}(3)$. We partition P_I into two intervals. The first is

$$I_1 = [(0, \mathbf{b}(2) + 1, 0), (\mathbf{g}(1), \mathbf{g}(2), \mathbf{b}(3))],$$

which lies completely inside P_I because every element has second coordinate too large to belong to $P_{S/I}$. The second is $I_2 = [(0, 0, \mathbf{b}(3) + 1), \mathbf{g}]$. Again, I_2 lies inside P_I as the third coordinate of every point is too large to belong to $P_{S/I}$. Furthermore, $I_1 \cap I_2 = \{\}$ because $\mathbf{c}_2(3) > \mathbf{c}_1(3)$ for every $\mathbf{c}_1 \in I_1, \mathbf{c}_2 \in I_2$. To show that $I_1 \cup I_2 = P_I$, note that we must ensure that any $\mathbf{c}$ with $\mathbf{c}(i) > \mathbf{b}(i)$ for some i is in one of the intervals. If $\mathbf{c}(3) > \mathbf{b}(3)$, then $\mathbf{c} \in I_2$. If $\mathbf{c}(2) > \mathbf{b}(2)$ and $\mathbf{c}(3) \leqslant \mathbf{b}(3)$, then $\mathbf{c} \in I_1$. Since $\mathbf{b}(1) = \mathbf{g}(1)$, there are no elements of P_i with first coordinate greater than $\mathbf{b}(1)$. Thus sdepth $P_I \geqslant 2$.

2. If $\alpha(\mathbf{b}) = 2$, then we may simply partition P_I into a single interval of the form $[(0, 0, \mathbf{b}(3) + 1), \mathbf{g}]$, assuming without loss of generality that $\mathbf{b}(1) = \mathbf{g}(1)$ and $\mathbf{b}(2) = \mathbf{g}(2)$.

3. The case $\alpha(\mathbf{b}) = 3$ is absurd, as it would imply that P_I is empty.

Before proceeding to the inductive step, we must eliminate the case where $\alpha(M) = 2$. Note that in this case, $|M| \leqslant 3$ and for each $i = 1, 2, 3$, there is at most one element $\mathbf{z}_i$ of M that disagrees with $\mathbf{g}$ in coordinate i. We can partition P_I into the single interval $[\mathbf{c}, \mathbf{g}]$ with (for $i = 1, 2, 3$) $\mathbf{c}(i) = \mathbf{z}_i(i) + 1$ if $\mathbf{z}_i$ exists and $\mathbf{c}(i) = 0$ otherwise.

Now suppose that for some $m \geqslant 1$, if $|M| \leqslant m$, then sdepth $P_I > \alpha(M)$. We consider the situation where M has $m + 1$ elements and $\alpha(M) < 2$. Suppose there is $\mathbf{b} \in M$ with $\alpha(\mathbf{b}) = 2$. Without loss of generality, $\mathbf{b} = (\mathbf{g}(1), \mathbf{g}(2), \mathbf{b}(3))$. The interval $[(0, 0, 0), \mathbf{b}]$ may be used in the partition of $P_{S/I}$, so delete it from $P_{S/I}$ and reduce the third coordinate of every remaining point by $\mathbf{b}(3) + 1$ to leave a poset Q. Here we may consider the set M' consisting of the elements of $M - \mathbf{b}$ shifted by reducing their third coordinate by $\mathbf{b}(3) + 1$. Now we apply induction to determine that sdepth $P_I > \alpha(M') = \alpha(M)$ as required.

If $\alpha(M) = 0$, partition P_I using intervals $[(i, j, k), (i, j, g(3))]$ for all i, j, choosing k as small as possible so that $(i, j, k) \in P_I$. This partition is guaranteed to work since P_I is an up set. Hence sdepth $P_I \geqslant 1 > \alpha(M)$.

We may now assume that $\alpha(M) = 1$. We have already taken care of the case where there is $\mathbf{b} \in M$ with $\alpha(\mathbf{b}) = 2$, so $\alpha(\mathbf{b}) = 1$ for all $\mathbf{b} \in M$. For $\mathbf{b} \in M$, define $\gamma(\mathbf{b}) = \max_{i \colon \mathbf{b}(i) \neq \mathbf{g}(i)} \mathbf{b}(i)$. Now choose $\mathbf{b}_0$ so that $\gamma(\mathbf{b}_0)$ is maximum. Without loss of generality, $\mathbf{b}_0(1) = \mathbf{g}(1)$ and $\mathbf{b}_0(2) = \gamma(\mathbf{b}_0)$.

We wish to find an interval that can be used in a partition of P_I that will allow us to decrease the size of M. To do this, consider the interval $J = [\mathbf{s}, \mathbf{t}]$ where $\mathbf{s} = (c_1, c_2, 0)$ and $\mathbf{t} = (\mathbf{g}(1), \mathbf{g}(2), c_3)$ with

$$c_1 = 1 + \max_{\substack{\mathbf{b} \in M \colon \\ \mathbf{b}(1) \neq \mathbf{g}(1) \\ \mathbf{b}(2) = \mathbf{g}(2)}} \mathbf{b}(1), \quad c_2 = \mathbf{b}_0(2) + 1, \quad \text{and} \quad c_3 = \min_{\mathbf{b} \in M} \mathbf{b}(3).$$

If there is no $\mathbf{b} \in M$ with $\mathbf{b}(1) \neq \mathbf{g}(1)$ and $\mathbf{b}(2) = \mathbf{g}(2)$, then let $c_1 = 0$. We must argue that $\mathbf{s} \notin P_{S/I} = D[M]$. Our choice of $\mathbf{s}(2)$ ensures that $\mathbf{s}(2)$ is not less than any $\mathbf{m} \in M$ unless $\mathbf{m}(2) = \mathbf{g}(2)$. However, for such an $\mathbf{m}$, $\mathbf{s}(1)$ is too large to lie in $D[M]$ unless there is $\mathbf{m} \in M$ with $\alpha(\mathbf{m}) = 2$, a case we have already eliminated.
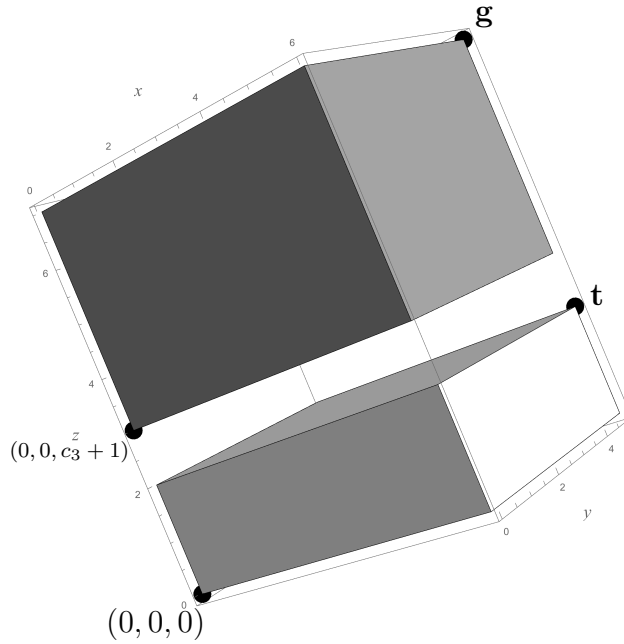
Figure 1: $D[\mathbf{g}]$ and $D[\mathbf{t}]$

In order to be able to apply the induction hypothesis, we must show that $D[\mathbf{t}] - J \subseteq D[M]$. To do this, take $\mathbf{x} \in D[\mathbf{t}] - J$. Since $\mathbf{x} \notin U[\mathbf{s}]$, we must have $\mathbf{x}(1) < c_1$ or $\mathbf{x}(2) < c_2$. If $\mathbf{x}(2) < c_2$, then $\mathbf{x} \leqslant \mathbf{b}_0$ since $\mathbf{b}_0(1) = \mathbf{g}(1) \geqslant \mathbf{x}(1)$ and $\mathbf{x}(3) \leqslant c_3 \leqslant \mathbf{b}_0(3)$. Thus $\mathbf{x} \in D[M]$. Suppose that $\mathbf{x}(2) \geqslant c_2$. Now we must have $\mathbf{x}(1) < c_1$. Since $\mathbf{x}(1) \geqslant 0$, this requires that $c_1 \neq 0$, and thus there is $\mathbf{b} \in M$ with $\mathbf{b}(1) \neq \mathbf{g}(1)$ and $\mathbf{b}(2) = \mathbf{g}(2)$. Take such a $\mathbf{b}$ with $\mathbf{b}(1) = c_1 - 1$. This $\mathbf{b}$ is greater than or equal to $\mathbf{x}$, implying $\mathbf{x} \in D[M]$.

For an illustration of the situation at this point, see Figure 1. Notice now that any $\mathbf{b} \in M$ with $\mathbf{b}(3) = c_3$ (and there is at least one such $\mathbf{b}$) must satisfy $\mathbf{b} \in D[\mathbf{t}]$. Thus, $D[\mathbf{g}] - D[\mathbf{t}] = [(0, 0, c_3 + 1), \mathbf{g}]$ contains at most m elements of M. Call this set of maximal elements $M' \subseteq M$. We may therefore, by induction, find an interval partition $\mathcal{P}'_I$ of $P_I - D[\mathbf{t}]$ with $\operatorname{sdepth}(P_I - D[\mathbf{t}]) > 1 = \alpha(M')$. Adding J to $\mathcal{P}'_I$ yields a partition $\mathcal{P}_I$ of P_I witnessing $\operatorname{sdepth} P_I = 2 > 1 = \alpha(M)$ since $\alpha(\mathbf{t}) = 2$. (Note that if $|M'| = 0$, then $D[\mathbf{g}] - D[\mathbf{t}]$ becomes the only interval in the partition $\mathcal{P}'_I$.) $\qquad\square$

3 Properties of a minimal counterexample

For the remainder of the paper, we will focus on the case where I is a squarefree monomial ideal of S. Combinatorially, this provides an advantage in that a squarefree monomial $x^{\mathbf{a}}$ has every entry in $\mathbf{a}$ equal to either 0 or 1. Thus, we may identify $\mathbf{a}$ with $A = \{i \in [n] : \mathbf{a}(i) = 1\}$. In doing so, P_I becomes an up set in the subset lattice $\mathbf{2}^n$, and $P_{S/I}$ is the complementary down set of P_I in $\mathbf{2}^n$. (From an algebraic perspective, $P_{S/I}$ can be viewed as the Stanley-Reisner simplicial complex of I.) We are then able to use more straightforward language and ideas connected to the subset lattice than the coordinate-based argument of the previous section. We begin with an elementary property of the subset lattice.

Proposition 4. *There is an interval partition of $2^n - \{\}$ in which the minimal element of every interval is a singleton.*

Proof. Our proof is by induction on n. The case $n = 1$ is trivial. To partition $2^{n+1} - \{\}$, start with the interval consisting of all subsets of $[n + 1]$ containing $n + 1$. This is an interval with a singleton as its minimal element. Removing it from $2^{n+1} - \{\}$ leaves $2^n - \{\}$, which we can partition into intervals with singletons as minimal elements by induction. $\qquad\square$

Theorem 5. *Let K be a field and let $S = K[x_1, \ldots, x_n]$. If there exists a squarefree monomial ideal $I \subseteq S$ such that $\operatorname{sdepth} I \leqslant \operatorname{sdepth} S/I$, then there exists an integer $n' \leqslant n$, a subset of indeterminates $\{y_1, \ldots, y_{n'}\} \subseteq \{x_1, \ldots, x_n\}$, and a squarefree monomial ideal $I' \subseteq K[y_1, \ldots, y_{n'}]$ having Stanley-Reisner simplicial complex Δ satisfying*

1. *Δ is pure with $\dim(\Delta) + 1 = \operatorname{sdepth} S/I' \geqslant \operatorname{sdepth} I'$,*

2. *the union of the facets of Δ is $[n']$, and*

3. *the intersection of the facets of Δ is empty.*

Proof. Let $I \subseteq S$ be a squarefree monomial ideal such that $k = \operatorname{sdepth}(S/I) \geqslant \operatorname{sdepth} I$. We also assume that amongst all such ideals, we have chosen S so that n is minimal.

We begin by proving the first statement by showing how to construct such a simplicial complex starting with a squarefree monomial ideal for which $\operatorname{sdepth} I \leqslant \operatorname{sdepth} S/I$. We assume that $P_{S/I}$ contains a set X of size at least $k + 1$, taking $|X|$ to be as large as possible. Let $\mathcal{P}_I$ be an optimal Stanley partition of P_I and $\mathcal{P}_{S/I}$ an optimal Stanley partition of $P_{S/I}$. Consider the interval I_X of $\mathcal{P}_{S/I}$ containing X. By the maximality of $|X|$, X is the upper bound of I_X. If I_X is trivial, simply move X from $P_{S/I}$ to P_I to form $P_{S/J}$ and P_J for the appropriate squarefree monomial J. Otherwise, notice that $I_X - X$ is isomorphic to the dual of $2^{|X|-1} - \{\}$, so Proposition 4 implies that there is a partition of $I_X - X$ into intervals in which each interval's upper bound has size $|X| - 1$. We form P_J from P_I by adding X to P_I (and to $\mathcal{P}_I$ as a trivial interval in $\mathcal{P}_J$) and remove X from $P_{S/I}$ to form $P_{S/J}$.

Replacing I_X in $\mathcal{P}_{S/I}$ by the interval partition provided by Proposition 4 gives a partition $\mathcal{P}_{S/J}$ that still has Stanley depth at least k since the maximal element of each interval added has size at least k. We thus must show that $\operatorname{sdepth} J \leqslant k$ to retain the desired inequality. Suppose that $\operatorname{sdepth} J > k$ and let $\mathcal{Q}_J$ be a witnessing partition. Because X was maximal in $P_{S/I}$, we know that X is minimal in P_J. In particular there is an interval $[X, X'] \in \mathcal{Q}_J$ with $X \neq X'$ because otherwise $\mathcal{Q}_J - [X, X]$ would be a partition of I having greater Stanley depth than $\mathcal{P}_I$. Let $\mathcal{P}_X$ be a the partition of $[X, X'] - X$ resulting from applying Proposition 4 to $[X, X'] - X$. Let $\mathcal{Q}_I = (\mathcal{P}_J \cup \mathcal{P}_X) - \{[X, X']\}$. Notice that $\mathcal{Q}_I$ is a partition of P_I in which the maximal element of every interval has size at least $\min\{\operatorname{sdepth} J, |X| + 1\} > k$. Therefore, $\operatorname{sdepth} I > k$, which contradicts our assumption that $\operatorname{sdepth} I \leqslant \operatorname{sdepth} S/I = k$.

If $P_{S/J}$ still contains a set of size greater than k, this process can be repeated until $P_{S/J}$ is a pure simplicial complex, and the inequality on Stanley depth will be preserved throughout.

We now prove the second property. Let $\mathcal{A}_I$ be the minimal elements of P_I and let $\mathcal{A}_{S/I}$ be the maximal elements of $P_{S/I}$ (the facets of the Stanley-Reisner complex of I). Assume without loss of generality that there is no $A \in \mathcal{A}_{S/I}$ such that $n \in A$. Let $\mathcal{A}'_I = \mathcal{A} \cap \mathbf{2}^{[n-1]}$. Then $D[\mathcal{A}_{S/I}] \cup U[\mathcal{A}'_I] = \mathbf{2}^{[n-1]}$, so by the minimality of n, sdepth $U[\mathcal{A}'_I] >$ sdepth $D[\mathcal{A}_{S/I}]$ is witnessed by partitions $\mathcal{P}'_I$ and $\mathcal{P}_{S/I}$. But now $\mathcal{P}_I = \mathcal{P}'_I \cup \{[\{n\},[n]]\}$ is a partition of $U[\mathcal{A}_I]$ with Stanley depth equal to sdepth $\mathcal{P}'_I$. Therefore, sdepth $I >$ sdepth S/I, contradicting that I and S/I are a counterexample to the conjecture.

We now proceed to prove the third property holds as well. Since we have already proved the first property, we may assume that Δ_I is pure with all facets of dimension $k - 1$ where $k = $ sdepth S/I. Further assume that some element of $[n]$, without loss of generality we will say it is n, appears in every facet of Δ_I. Notice that since n is in every facet of $\Delta_I = P_{S/I}$ and this poset is a down set in $\mathbf{2}^n$, $P_{S/I}$ can be viewed as the disjoint union of two isomorphic subposets of $P_{S/I}$—namely, all the subsets containing n and all those not containing n. Let $P'_{S/I}$ be the subposet of $P_{S/I}$ consisting only of subsets of $[n-1]$. Since $P'_{S/I}$ is a down set of $\mathbf{2}^{n-1}$, its complement P'_I is an up set corresponding to some monomial ideal J and $P'_{S/I} = P_{S'/J}$ where $S' = K[x_1,\ldots,x_{n-1}]$. By minimality, we know that sdepth $J >$ sdepth S'/J is witnessed by optimal Stanley partitions $\mathcal{P}_J$ and $\mathcal{P}_{S'/J}$. Now notice that adding n to the upper bound of each interval of $\mathcal{P}_{S'/J}$ gives a Stanley partition of $P_{S/I}$ witnessing sdepth $S/I = k$. Similarly, adding n to the upper bounds of the intervals of $\mathcal{P}_J$ gives a Stanley partition of P_I witnessing sdepth $I \geqslant 1 + $ sdepth $J \geqslant k + 1$. This contradicts that I and S/I were a counterexample. $\square$

Since satisfying Theorem 5 when Δ_I is a pure simplicial complex of dimension $n - 2$ requires that $P_I = \{[n]\}$, we have the following corollary.

Corollary 6. *Let $I \subseteq S = K[x_1,\ldots,x_n]$ be a squarefree monomial ideal. If the facets of Δ_I all have size $n - 1$, then sdepth $I >$ sdepth S/I.*

We conclude this section with a lemma that looks at what happens when P_I consists only of sets of size at least k.

Lemma 7. *Let $I \subseteq S = K[x_1,\ldots,x_n]$ be a squarefree monomial ideal. If $P_{S/I}$ contains all $(k-1)$-sets and $k \leqslant (n-1)/2$, then sdepth$(I) > k$.*

Proof. To show that sdepth $I > k$, it suffices to find a complete matching from the k-sets in P_I to the $(k+1)$-sets. Looking only at the k- and $(k+1)$-sets in P_I as a bipartite graph, we see that the partite set consisting of the k-sets is $(n - k)$-regular. The vertices in the partite set consisting of the $(k+1)$-sets all have degree at most $k+1$. Since $k \leqslant (n-1)/2$, $k + 1 \leqslant n - k$. Thus, by a corollary of Hall's Theorem, there is a complete matching from the k-sets to the $(k + 1)$-sets. Use the edges of the matching to define intervals in a Stanley partition of P_I with trivial intervals used to cover any set not in one of the matching intervals. $\square$

4 Combinatorial criteria

In this section, we continue to assume that $S = K[x_1, \ldots, x_n]$ and I is a squarefree monomial ideal of S. We consider the posets P_I and $P_{S/I}$ as complementary subposets of 2^n. If we are focusing on proving that $\operatorname{sdepth} I > \operatorname{sdepth} S/I$, Theorem 5 allows us to restrict our attention to the case where every maximal element of $P_{S/I}$ is a k-set where $k = \operatorname{sdepth} S/I$. In this section, we establish some necessary conditions on $P_{S/I}$ for this to happen. Although this conjecture is our motivation, the conditions discussed here apply to a broader class of posets and will be established in the fullest context.

The first criterion we establish is what we call the *combinatorial criterion*. Assuming $J \subseteq I$ are squarefree monomial ideals of S, there exist antichains $\mathcal{A}, \mathcal{B} \subseteq 2^n$ such that the poset $P_{I/J}$ is the intersection of $U[\mathcal{A}]$ and $D[\mathcal{B}]$. In particular, $\mathcal{A}$ is the antichain of minimal elements of $P_{I/J}$ and $\mathcal{B}$ is the antichain of maximal elements. If we wish to show that $\operatorname{sdepth} I/J \geqslant k$, we must find a partition $\mathcal{P}_{I/J}$ of $P_{I/J}$ into intervals so that each interval's upper bound has size at least k. In fact, it suffices by a straightforward generalization of property (1) of Theorem 5 to put every set of size at most k into an interval with upper bound of size k, leaving the remaining sets in trivial intervals.

For ease of explanation, let us temporarily focus on partitioning $P_{S/I}$ where the maximal elements are all of size k. This naturally requires an interval from the empty set to a k-set be included in $\mathcal{P}_{S/I}$. Therefore, $P_{S/I}$ must include at least k singletons. Proceeding now to consider the number of 2-sets in $P_{S/I}$, we notice that the first interval requires at least $\binom{k}{2}$ of them and that each singleton not covered by the first interval will be the minimal element of an interval in $\mathcal{P}_{S/I}$ covering $k - 1 = \binom{k-1}{1}$ 2-sets. Thus, we have a lower bound on the number of 2-sets in $P_{S/I}$. If $P_{S/I}$ does not have this number of 2-sets, we say that $P_{S/I}$ *fails to satisfy the combinatorial criterion* (and we must necessarily have $\operatorname{sdepth} S/I < k$).

Instead of introducing cumbersome notation to give equations that must have integer solutions if the combinatorial criterion is satisfied, we instead describe the process of verifying the combinatorial criterion for a poset $P_{I/J}$. We begin with a vector $\mathbf{a} = (a_0, a_1, \ldots, a_k)$ where a_i is the number of i-sets in $P_{I/J}$ for $i = 0, \ldots, k$. (If $P_{I/J}$ is considered as a (relative) simplicial complex, then this is its f-vector, truncated after counting the $(k-1)$-dimensional simplices.) As we proceed, $\mathbf{a}$ will track the number of i-sets not already in an interval of an interval partition in which all maximal elements have size i. (Note that we do not actually construct the partition; we only consider what structure it *must* have if the upper bound of each interval has size k.) Supposing that all sets of size less than i have been successfully covered, we know that there are a_i remaining sets of size i to cover. If we are to successfully complete the partition, each of these is the minimal element of an interval with maximal element of size k, so each interval consumes $\binom{k-i}{j-i}$ sets of size $j = i, \ldots, k$. The vector $\mathbf{a}$ is then updated by decreasing a_j by $a_i \binom{k-i}{j-i}$ for $j = i, \ldots, k$. If coordinate a_j of $\mathbf{a}$ is now negative for some j, we know that there were insufficiently many j-sets for a partition witnessing Stanley depth k to exist. In this case, we say that the *combinatorial criterion is violated*. If we can repeat this process up to $i = k$ without ever obtaining a negative entry in $\mathbf{a}$, then the *combinatorial criterion is*

satisfied and we know that there are enough sets of each size in $P_{I/J}$ to support a partition of the type sought. We note that satisfying the combinatorial criterion is equivalent to the h-vector of $P_{S/I}$ (when it can be viewed as a pure simplicial complex) being nonnegative by Proposition III.2.3 of Stanley's monograph [25].

Unfortunately, while the combinatorial criterion is necessary for $P_{S/I}$ to have Stanley depth k, it is not sufficient. To see why, suppose that $P_{S/I}$ is the down set of the antichain $\{123, 124, 125, 134, 345, 234\}$ in $\mathbf{2}^5$. The interval with the empty set as its minimal element covers three singletons and three 2-sets. This leaves two singletons, seven 2-sets, and five 3-sets uncovered. Intervals beginning at the remaining two singletons cover four more 2-sets (two each) and two more 3-sets. Now $\mathbf{a} = (0, 0, 3, 3)$. Covering the remaining 2-sets requires three 3-sets, reducing $\mathbf{a}$ to the zero vector. Thus, this poset does satisfy the combinatorial criterion. However, we will now see that sdepth $P_{S/I} < 3$. This is because the strict up set of $\{5\}$ in $P_{S/I}$ is $U = \{15, 25, 35, 45, 125, 345\}$. Whatever interval covers $\{5\}$ will consume two 2-sets from U and one 3-set, leaving behind two 2-sets and one 3-set. There is no way for a single interval not containing $\{5\}$ to cover these three remaining sets, so a Stanley partition of $P_{S/I}$ must contain an interval with maximal element of size less than 3.

The way in which the preceding example failed to have Stanley depth 3 is instructive in providing a stronger version of the combinatorial criterion. Since it is not sufficient to have enough sets of each size in $P_{I/J}$, we add the additional requirement that for each set A in $P_{I/J}$, the subposet $U[A] \cap P_{I/J}$ must satisfy the combinatorial criterion. If $P_{I/J}$ satisfies this more stringent requirement, we say that it satisfies the *strong combinatorial criterion*. The example given above violates the strong combinatorial criterion because the combinatorial criterion is not satisfied for the up set of $\{5\}$. To see that the strong combinatorial criterion is necessary for $P_{I/J}$ to have Stanley depth k, we consider the restriction of an optimal Stanley partition $\mathcal{P}_{I/J}$ of $P_{I/J}$ to the closed up set U (inside $P_{I/J}$) of some set $A \in P_{I/J}$. (Form this restriction by intersecting each interval of $\mathcal{P}_{I/J}$ with U and discarding empty intervals that result.) Notice that U has a single minimal element A and each maximal element of U is a superset of A. Thus, U is isomorphic to $P_{S'/I'}$ for some ring $S' = K[T]$ with $T \subset \{x_1, \ldots, x_n\}$ and squarefree monomial ideal $I' \subseteq S'$. Since the restriction of $\mathcal{P}_{I/J}$ to U gives an interval partition of U in which the upper bound of each interval has size k, we know from above that U must satisfy the combinatorial criterion. From the simplicial complex perspective, satisfying the strong combinatorial criterion is equivalent to the h-vector of every link being nonnegative.

Because the strong combinatorial criterion can be checked by computer via simple counting methods, it is much more efficient than verifying the existence of a particular Stanley partition by brute force. However, we also note that a very large fraction of squarefree monomial ideals for which the maximal elements of $P_{S/I}$ are all of the same size satisfy the strong combinatorial criterion for $k \leqslant n/2$ and for $k > n/2$, the majority violate the strong combinatorial criterion. See Tables 1 and 2 in Section 7 for relevant enumerations.

At one point, computer investigations emboldened us to believe that if Δ is a pure simplicial complex of dimension $k - 1$, then $\operatorname{sdepth}(S/I_\Delta) = k$ if and only if Δ satisfies

the strong combinatorial criterion. However, the counterexample C_3 of Duval et al. in [2, Theorem 3.5] has $n = 16$, is Cohen-Macaulay (hence pure) of dimension 3, and satisfies the strong combinatorial criterion. However, the Stanley depth of S/I_{C_3} is only 3 for that example. It would be interesting to determine if every pure k-dimensional simplicial complex Δ with $\mathrm{sdepth}(S/I_\Delta) < k+1$ contains an antichain that provides a combinatorial witness to the fact that the Stanley depth is not $k + 1$.

5   Splitting

Our final reduction result focuses on the scenario where $P_{S/I}$ can be split into two pieces, one consisting of all sets containing an element $x \in [n]$ and the other consisting of all sets not containing x, in a particular way that allows for a Stanley partition to be constructed. This technique will be useful in computational results that come later. In what follows, we will refer to an antichain in the subset lattice satisfying the strong combinatorial criterion, by which we mean that its closed downset satisfies the strong combinatorial criterion.

Definition 8. Let $\mathcal{A}$ be an antichain in $\mathbf{2}^{[n]}$ and $x \in [n]$. Let $\mathcal{A}_x = \{A \in \mathcal{A} \mid x \in A\}$ and $\mathcal{A}'_x = \{A - x \mid A \in \mathcal{A}_x\}$. We say that $\mathcal{A}$ *splits over* x provided that

   (i) for all $S \in \mathcal{A}'_x$, there exists $T \in \mathcal{A} - \mathcal{A}_x$ such that $S \subseteq T$ and

   (ii) $\mathrm{sdepth}(D[\mathcal{A} - \mathcal{A}_x]) \geqslant \mathrm{sdepth}(D[\mathcal{A}])$.

The antichain $\mathcal{A}$ *splits* provided that there exists $x \in [n]$ such that $\mathcal{A}$ splits over x.

To illustrate this definition, we consider the following example.

Example 9. Let $\mathcal{A} = \{\{1, 2, 5\}, \{2, 4, 6\}, \{1, 2, 3\}, \{2, 4, 5\}, \{3, 4, 5\}, \{2, 3, 5\}\}$. We see immediately that $\mathcal{A}$ splits over 6, since $\mathcal{A}_6 = \{\{2, 4, 6\}\}$ and $\mathcal{A}'_6 = \{2, 4\} \subset \{2, 4, 5\}$. (Verification of the combinatorial criterion requirements is straightforward but tedious.) Similarly, $\mathcal{A}$ splits over 1 and 3. However, $\mathcal{A}$ does not split over 2 since $\mathcal{A} - \mathcal{A}_2 = \{\{3, 4, 5\}\}$ and $\{1, 5\} \in \mathcal{A}'_2$ but $\{1, 5\} \not\subseteq \{3, 4, 5\}$.

We now show that the search for a counterexample to the conjecture can be restricted to antichains that do not split.

Lemma 10. *Let $I \subseteq S = K[x_1, \ldots, x_n]$ be a squarefree monomial ideal. If $\mathrm{sdepth}\, I \leqslant \mathrm{sdepth}\, S/I$ and the antichain of maximal elements of $P_{S/I}$ splits, then there exists a squarefree monomial ideal $J \subseteq S' = K[y_1, \ldots, y_{n'}]$ with $n' \leqslant n$ and $\{y_1, \ldots, y_{n'}\} \subseteq \{x_1, \ldots, x_n\}$ such that $\mathrm{sdepth}\, J \leqslant \mathrm{sdepth}\, S'/J$ and the antichain of maximal elements of $P_{S'/J}$ does not split.*

Proof. Suppose that I and S/I are minimal in terms of n such that $\mathrm{sdepth}\, I \leqslant \mathrm{sdepth}\, S/I$ and the antichain $\mathcal{A}$ of maximal elements of $P_{S/I}$ splits. Without loss of generality, we assume that $\mathcal{A}$ splits over n. By Theorem 5, we may assume that every element of $\mathcal{A}$ is a k-set and $\mathrm{sdepth}(P_{S/I}) = k$. (Showing that the reductions of Theorem 5 preserve

splitting is a straightforward exercise.) Let $\mathcal{A}_n = \{A \in \mathcal{A} \mid n \in A\}$ and $\mathcal{B} = \mathcal{A} - \mathcal{A}_n$. Now $\mathcal{B}$ is an antichain of k-sets in $\mathbf{2}^{[n-1]}$, so by minimality there is a partition $\mathcal{P}_\mathcal{B}$ witnessing $\mathrm{sdepth}(\mathbf{2}^{[n-1]} - D[\mathcal{B}]) > \mathrm{sdepth}(D[\mathcal{B}]) \geqslant k$. Furthermore, the first condition in the definition of what it means for $\mathcal{A}$ to split over n ensures that no set covered by $\mathcal{P}_\mathcal{B}$ belongs to $D[\mathcal{A}]$. Similarly, $\mathcal{A}'_n = \{A - n \mid A \in \mathcal{A}_n\}$ is an antichain of $(k-1)$-sets in $\mathbf{2}^{[n-1]}$, and there exists a partition $\mathcal{P}_{\mathcal{A}'_n}$ witnessing $\mathrm{sdepth}(\mathbf{2}^{[n-1]} - D[\mathcal{A}'_n]) \geqslant k$. Form a new partition

$$\mathcal{P}_n = \left\{ [S \cup \{n\}, T \cup \{n\}] \mid [S, T] \in \mathcal{P}_{\mathcal{A}'_n} \right\}.$$

Let $\mathcal{D} = \{S \cup \{n\} \mid S \in D[\mathcal{A}'_n]\}$. Then $\mathcal{D} \cup \mathcal{P}_n = [\{n\}, [n]]$ and $\mathcal{P}_\mathcal{B} \cup D[\mathcal{B}] = [\{\}, [n-1]]$. Thus $\mathcal{P}_\mathcal{B} \cup \mathcal{P}_n$ is a partition of $\mathbf{2}^{[n]} - D[\mathcal{A}]$ witnessing $\mathrm{sdepth}(I) \geqslant k + 1$. But now we have $\mathrm{sdepth}(S/I) \leqslant k$, contradicting that $\mathrm{sdepth}\, I \leqslant \mathrm{sdepth}\, S/I$. $\qquad\square$

6 The $n = 4$ and $n = 5$ squarefree cases

In this section, we address the conjecture that $\mathrm{sdepth}\, I > \mathrm{sdepth}\, S/I$ for the cases where S is the polynomial ring over a field K in four or five indeterminates. The first proof is immediate from our reduction results, but we identify which result takes care of each of the various cases.

Theorem 11. *Let $S = K[x_1, \ldots, x_4]$. If I is a squarefree monomial ideal of S, then $\mathrm{sdepth}\, I > \mathrm{sdepth}\, S/I$.*

Proof. We consider the posets P_I and $P_{S/I}$ as complementary subposets of $\mathbf{2}^4$. Let $\mathcal{A}_I$ be the minimal elements of P_I and let $\mathcal{A}_{S/I}$ be the maximal elements of $P_{S/I}$. By Theorem 5, we may assume that all elements of $\mathcal{A}_{S/I}$ have size k. The cases where $k = 0$ and $k = 4$ are trivial.

If $k = 1$, then by Theorem 5, a minimal counterexample to the theorem would require that $\mathcal{A}_{S/I}$ consists precisely of the four singleton subsets of $[4]$. But then $\mathcal{A}_I$ contains only elements of size 2, and the inequality of the theorem holds trivially.

If $k = 2$, it suffices to notice that $P_{S/I}$ must contain the four singleton subsets of $[4]$ by Theorem 5. But then all $(k-1)$-sets are contained in $P_{S/I}$ and $k \leqslant n/2$, so the inequality holds by Lemma 7.

The case when $k = 3$ follows from Corollary 6. $\qquad\square$

We now proceed to the $n = 5$ case, which takes a more intricate argument. We know from our earlier results that the facets of Δ_I (equivalently, maximal elements of $P_{S/I}$) form a uniform hypergraph on $[n]$. Thus, for ease of reading, most of the discussion is phrased in the language of hypergraphs instead of monomial ideals, simplicial complexes, or posets. Recall that a hypergraph is *k-uniform* if every hyperedge has size k. The *degree* of a vertex in a (hyper)graph is the number of edges containing that vertex. A (hyper)graph is called *d-regular* if every vertex has degree d. The *degree sequence* of a (hyper)graph is the nonincreasing sequence of its vertex degrees.

Theorem 12. *Let $S = K[x_1, \ldots, x_5]$. If I is a squarefree monomial ideal of S, then $\mathrm{sdepth}\, I > \mathrm{sdepth}\, S/I$.*

Proof. Similarly to the proof of Theorem 11, we consider the posets P_I and $P_{S/I}$ as complementary subposets of $\mathbf{2}^5$ with $\mathcal{A}_I$ the minimal elements of P_I and $\mathcal{A}_{S/I}$ the maximal elements of $P_{S/I}$. Theorem 5 allows us to assume that all elements of $\mathcal{A}_{S/I}$ have size k for some k with $0 \leqslant k \leqslant 5$. The desired inequality is trivially true in the cases $k = 0$ and $k = 5$. In the case with $k = 1$, the inequality also follows immediately from Theorem 5 since either some element of $[5]$ does not appear in any set in $\mathcal{A}_{S/I}$ or all elements of $\mathcal{A}_I$ have size at least 2.

If $k = 2$, we note that if every element of $[5]$ appears in some set in $\mathcal{A}_{S/I}$, then all singletons are in $P_{S/I}$. Since $k - 1 = 1$ here, Lemma 7 yields the inequality. On the other hand, if some element of $[5]$ does not appear in any element of $\mathcal{A}_{S/I}$, then Theorem 5 implies the desired inequality. For $k = 4$, the result follows from Corollary 6.

To address the remaining case where $k = 3$, we will consider $\mathcal{A}_{S/I}$ to be a hypergraph $\mathcal{H} = ([5], \mathcal{E})$. We first note that $\mathcal{H}$ must have at least four hyperedges. Three hyperedges are necessary to satisfy conditions (2) and (3) of Theorem 5. The degree sequence of a three-edge 3-uniform hypergraph consistent with Theorem 5 must be $(2, 2, 2, 2, 1)$. Up to isomorphism, there is a unique 3-uniform hypergraph with this degree sequence; its edge set is $\{123, 124, 345\}$, which does not satisfy the combinatorial criterion. Therefore, $\mathcal{H}$ has at least four hyperedges.

We now show that there is a set $\mathcal{S} \subseteq \mathcal{E}$ of four hyperedges giving a subhypergraph $\mathcal{H}' = ([5], \mathcal{S})$ with degree sequence $(3, 3, 3, 2, 1)$. To do so, we consider the possible degree sequences for a four-edge 3-uniform subhypergraph $\mathcal{H}'$ of $\mathcal{H}$.

1. If $(4, 2, 2, 2, 2)$ is the degree sequence of $\mathcal{H}'$, then one vertex v is in all four hyperedges in $\mathcal{S}$. We know by Theorem 5 that there is an edge $X \in \mathcal{E}$ such that $v \notin X$. Deleting any hyperedge of $\mathcal{H}'$ yields a subhypergraph with degree sequence $(3, 2, 2, 1, 1)$. Adding X to this subhypergraph then yields a subhypergraph that must have degree sequence $(3, 3, 3, 2, 1)$ or $(3, 3, 2, 2, 2)$. In the former case, we're done. The latter case will be addressed last.

2. If $(3, 3, 3, 3, 0)$ is the degree sequence of $\mathcal{H}'$, then some vertex v is not in any hyperedge in $\mathcal{S}$. However, by Theorem 5, some hyperedge $Y \in \mathcal{E}$ contains v. Replacing any hyperedge of $\mathcal{H}'$ with Y yields progress, as the new subhypergraph must have degree sequence $(3, 3, 3, 2, 1)$, in which case we're done, or $(4, 3, 2, 2, 1)$, and we will consider this case shortly.

3. The sequence $(4, 3, 3, 1, 1)$ is not the degree sequence of a four-edge hypergraph, since removing the vertex of degree four would leave a graph with degree sequence $(3, 3, 1, 1)$, but there is no graph with this degree sequence. Therefore, no such hypergraph can exist.

4. If $(4, 3, 2, 2, 1)$ is the degree sequence of $\mathcal{H}'$, then a vertex v is in all four edges of $\mathcal{H}'$. Removing v leaves a graph with degree sequence $(3, 2, 2, 1)$. The only graph with this degree sequence is a triangle with a pendant edge attached to one of its vertices. Thus, $\mathcal{H}'$ must contain a hyperedge consisting of the vertex of degree 4, the vertex of degree 3, and a vertex of degree 2. Replacing this hyperedge with a

hyperedge $X \in \mathcal{E}$ such that $v \notin X$, which must exist since v has degree 4 in $\mathcal{H}'$ but cannot be in all hyperedges of $\mathcal{H}$, yields a new subhypergraph that has degree sequence $(3, 3, 3, 2, 1)$ or $(3, 3, 2, 2, 2)$.

5. It remains only to consider when $\mathcal{H}'$ has degree sequence $(3, 3, 2, 2, 2)$. In this case, there are two possibilities. The easier to resolve occurs when the three vertices of degree 2 all appear together in a hyperedge of $\mathcal{H}'$ and the other three hyperedges contain the two vertices of degree 3 and a vertex of degree 2. Let v be a vertex of degree 2 in $\mathcal{H}'$. Since the two hyperedges of $\mathcal{H}'$ containing v intersect only in v, we know that $\mathcal{E}$ has another hyperedge $X = \{v, u, w\}$ not in $\mathcal{H}'$ because the strong combinatorial criterion must be satisfied (particularly, the combinatorial criterion for $U[v]$). Notice that without loss of generality, u has degree 3 in $\mathcal{H}'$ and w has degree 2 in $\mathcal{H}'$. Letting x be the other vertex with degree 2 in $\mathcal{H}'$ and y the other vertex of degree 3 in $\mathcal{H}'$, we replace the hyperedge $\{u, x, y\}$ by X to form a new subhypergraph of $\mathcal{H}$ with degree sequence $(3, 3, 3, 2, 1)$ as desired.

The other possible way for $\mathcal{H}'$ to have a degree sequence of $(3, 3, 2, 2, 2)$ is depicted in Figure 2. We notice $\mathcal{H}$ must have more hyperedges than $\mathcal{H}'$, as the strong combinatorial criterion is violated by the up set of x if these are all the hyperedges of $\mathcal{H}$. There are four possibilities for the hyperedge of $\mathcal{H}$ that does not appear in
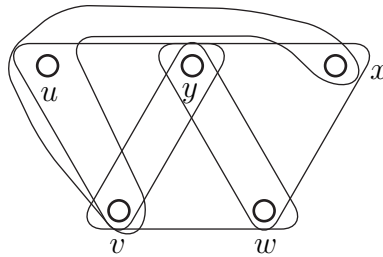

Figure 2: The second subhypergraph with degree sequence $(3, 3, 2, 2, 2)$

$\mathcal{H}'$ but must exist to ensure the strong combinatorial criterion is satisfied. They are $\{x, v, w\}$, $\{x, y, v\}$, $\{x, y, u\}$, and $\{x, w, u\}$. In the first three cases, it is possible to add the hyperedge to $\mathcal{H}'$ and remove another edge to obtain a new subhypergraph with degree sequence $(3, 3, 3, 2, 1)$. If $\{x, v, w\}$ is added, then $\{x, u, v\}$ should be removed. For both $\{x, y, v\}$ and $\{x, y, u\}$, the hyperedge to remove is $\{y, v, w\}$. The case where $\{x, w, u\}$ is the hyperedge available to add requires a bit more work. Adding $\{x, w, u\}$ to $\mathcal{H}'$ yields the (unique) 3-uniform, 3-regular hypergraph on 5 vertices. However, this hypergraph cannot be $\mathcal{H}$, as the combinatorial criterion would be violated if it were, since the f-vector would be $(1, 5, 10, 5)$. Without loss of generality, we may assume that $\mathcal{H}$ also contains the hyperedge $\{x, v, w\}$, so we remove $\{x, u, v\}$ and add $\{x, v, w\}$ to obtain a subhypergraph of $\mathcal{H}$ with degree sequence $(3, 3, 3, 2, 1)$ as desired.

We now proceed to consider what the four hyperedges in $\mathcal{S}$, the set of hyperedges of $\mathcal{H}'$, can be. Our claim is that without loss of generality, $\mathcal{S} = \{123, 124, 145, 234\}$. We begin by

assuming that 3 is the vertex contained in precisely two hyperedges $X, Y \in \mathcal{S}$. If $X \cap Y = \{3\}$, notice that the only way for the hyperedges in $\mathcal{S}$ to form a subhypergraph with degree sequence $(3, 3, 3, 2, 1)$ would be to have the three vertices of degree three appear together in a hyperedge and to use two copies of that hyperedge. Since this is impossible, $X \cap Y = \{2, 3\}$ without loss of generality. In fact, we may assume $\{123, 234\} \subset \mathcal{S}$. Notice now that 5 can only appear in one element of $\mathcal{S}$, since if it appeared in three (the only other option), then each element of $\{1, 2, 4\}$ would appear in more than one hyperedge, contradicting the existence of a degree 1 vertex in $\mathcal{H}'$. Furthermore, notice that 2 and 5 cannot appear in the same hyperedge, as if they did, then vertices 2, 3, and 5 would all have the required degree, preventing the addition of a fourth hyperedge. Therefore, $145 \in \mathcal{S}$. At this point, vertices 3 and 5 have the desired degrees, but vertices 1, 2, and 4 all have degree 2. Therefore, $124 \in \mathcal{S}$.

If $\mathcal{A}_{S/I} = \mathcal{S}$, then the only 2-sets of $\mathbf{2}^5$ left uncovered by $P_{S/I}$ are 25 and 35. In this case, we may use the intervals $[25, 1245]$ and $[35, 1345]$ in the Stanley partition of P_I (and use trivial intervals for the remainder). If $\mathcal{S} \subsetneq \mathcal{A}_{S/I}$, then these two intervals may be divided as needed to obtain a Stanley partition of P_I witnessing sdepth $I \geqslant 4$. $\qquad\square$

7 Computational work for larger values of n

Through the results of Section 3, 4, and 5, we have been able to reduce the work of proving that $\mathrm{sdepth}(I) > \mathrm{sdepth}(S/I)$ for $I \subset S$ a squarefree monomial ideal of $S = K[x_1, \ldots, x_n]$ to a size readily manageable by computer when $n = 6$ and $n = 7$. Taking, as usual, k to be the uniform size of the maximum elements of $P_{S/I}$, the $k = 0$ and $k = n$ cases are trivial. As with $n = 5$, the $k = 1$ and $k = 2$ cases are immediate consequences of Theorem 5 and Lemma 7. The $k = n - 1$ case follows from Corollary 6.

We relied upon computer search for the $k = 3$ and $k = 4$ cases with $n = 6$ and $n = 7$ and the $k = 5$ case with $n = 7$. We used McKay's `nauty` package [14] to generate the necessary nonisomorphic hypergraphs on 6 and 7 vertices.

We developed a collection of Python routines to test the hypergraphs for the various properties discussed in Sections 3 through 5. Because of the repercussions for the degrees of vertices in the hypergraph, a hypergraph failing to satisfy properties (2) and (3) of Theorem 5 was labelled as having "bad degree". Hypergraphs for which the corresponding down set failed the strong combinatorial criterion were not considered further. In light of Lemma 10, only antichains that did not split were actually tested to find the Stanley depth of P_I. In every case where this examination was conducted (by a nearly brute force search of possible partitions), we confirmed that $\mathrm{sdepth}(P_I) \geqslant k + 1$. The counts of hypergraphs falling into each category are shown in Tables 1 and 2. Computations were done via distributed computing at Washington and Lee University and the University of Louisville. All source code as well as the partitions of hypergraphs into the categories below are available for download from https://github.com/mitchkeller/stanley-depth

As a consequence of these computations and earlier results in this paper, we have the following result

k	Total	Bad Degree	Fail SCC	Splits	$\mathrm{sdepth}(P_I) \geqslant k+1$
3	2136	57	527	1496	56
4	156	35	55	66	0

Table 1: Computational results for $n = 6$

k	Total	Bad Degree	Fail SCC	Splits	$\mathrm{sdepth}(P_I) \geqslant k+1$
3	7013319	2257	888308	5987476	135278
4	7013319	2257	4439735	2383294	188033
5	1043	156	589	298	0

Table 2: Computational results for $n = 7$

Theorem 13. *If I is a squarefree monomial ideal of $S = K[x_1, \ldots, x_n]$ with $n \leqslant 7$, then* $\mathrm{sdepth}\, I > \mathrm{sdepth}\, S/I$.

We also used Python code to explicitly calculate the Stanley depth of I and of S/I to look for trends in the difference between these values for different values of k. No clear trends emerged, but the results of the computational work is reflected in Table 3 and Figure 3. It is worth noting that, given the counts for bad degree from Table 2, there are situations where there is not a natural reduction of the problem to a smaller value of n but the gap between $\mathrm{sdepth}(I)$ and $\mathrm{sdepth}(S/I)$ is still greater than 1. Furthermore, the instances where the gap between the Stanley depth of S/I and I is precisely one include instances that satisfy the strong combinatorial criterion as well as instances that violate it.

	4	5
1	13	
2	1026	4

(a) $k = 2$

	4	5
2	886423	2424
3	4878319	1246153

(b) $k = 3$

	4	5	6
3	279	4440053	
4		2572970	17

(c) $k = 4$

	5	6
4	282	369
5		392

(d) $k = 5$

Table 3: Number of pure simplicial complexes Δ of dimension $k-1$ (for $S = K[x_1, \ldots, x_7]$) with $\mathrm{sdepth}(S/I_\Delta)$ as given by row headings and $\mathrm{sdepth}(I_\Delta)$ as given by column headings.

Attempts to extend the computational work to increase the range of n for which Theorem 13 is valid appears infeasible at this time. Resolving the case with $n = 8$ and $k = 3$ (which would allow for concurrently resolving $k = 5$) would require generating over 894 billion hypergraphs, based on the enumeration of k-uniform hypergraphs on n vertices done by Qian [18]. (This is in contrast to approximately 3.5 million hypergraphs generated for the entire $n = 7$ case.)
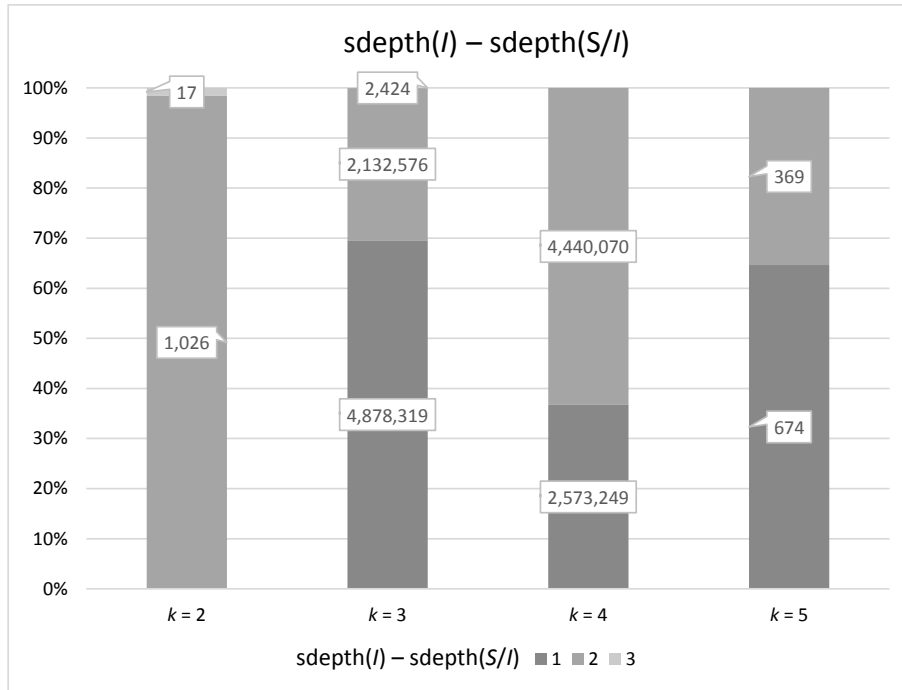
Figure 3: Quantifying the gap between $\mathrm{sdepth}(S/I)$ and $\mathrm{sdepth}(I)$

In addition to the computations necessary to prove Theorem 13, we also investigated the question raised by Duval et al. in [2] as to whether or not their counterexample to the partitionability conjecture was the smallest possible. We used our Python code to compute the Stanley depth, and whenever that result was less than the size of the sets in the antichain of maximal elements of $P_{S/I}$, we used SageMath [20] to determine if $P_{S/I}$ (viewed as a simplicial complex) was Cohen-Macaulay. As a consequence, we have Theorem 14.

Theorem 14. *Let K be a field, $S = K[x_1, \ldots, x_n]$, and $I \subseteq S$ a squarefree monomial ideal. Let Δ_I be the Stanley-Reisner complex of I. If $n \leqslant 7$ and Δ_I is Cohen-Macaulay, then $\mathrm{sdepth}\, S/I = \dim \Delta_I + 1$. Equivalently, Δ_I is partitionable.*

As an extension of Theorem 14, we have the following:

Corollary 15. *Let K be a field, $S = K[x_1, \ldots, x_n]$, and $I \subseteq S$ a squarefree monomial ideal. If $n \leqslant 7$, then $\mathrm{depth}\, S/I \leqslant \mathrm{sdepth}\, S/I$.*

This generalizes of a result of Popescu in [16], where it was proved for $n \leqslant 5$. Corollary 15 follows from Theorem 14 by Corollary 37 of [5], which reduces to the Cohen-Macaulay case by taking skeleta in a way that does not increase the number of vertices.

Acknowledgements

to the anonymous referee whose suggestions added additional connections to the existing literature.

References

[1] Biró, C., Howard, D. M., Keller, M. T., Trotter, W. T., and Young, S. J. Interval partitions and Stanley depth. *J. Combin. Theory Ser. A. 117*, 4 (May 2010), 475–482.

[2] Duval, A. M., Goeckner, B., Klivans, C. J., and Martin, J. L. A non-partitionable Cohen–Macaulay simplicial complex. *Advances in Mathematics 299* (Aug. 2016), 381–395.

[3] Garsia, A. M. Combinatorial methods in the theory of Cohen-Macaulay rings. *Advances in Mathematics 38*, 3 (1980), 229–266.

[4] Ge, M., Lin, J., and Shen, Y.-H. On a conjecture of Stanley depth of squarefree Veronese ideals. *Comm. Algebra 40*, 8 (2012), 2720–2731.

[5] Herzog, J. A survey on Stanley depth. In *Monomial Ideals, Computations and Applications*, vol. 2083 *Lecture Notes in Math.* Springer, Heidelberg, 2013, pp. 3–45.

[6] Herzog, J., Jahan, A. S., and Yassemi, S. Stanley decompositions and partitionable simplicial complexes. *Journal of Algebraic Combinatorics. 27*, 1 (2008), 113–125.

[7] Herzog, J., Vladoiu, M., and Zheng, X. How to compute the Stanley depth of a monomial ideal. *J. Algebra 322*, 9 (2009), 3151–3169.

[8] Ichim, B., Katthän, L., and José Moyano-Fernández, J. How to compute the Stanley depth of a module. *Mathematics of Computation 86*, 303 (2017), 455–472.

[9] Ichim, B., Katthän, L., and Moyano-Fernández, J. J. The behavior of Stanley depth under polarization. *Journal of Combinatorial Theory, Series A 135* (Oct. 2015), 332–347.

[10] Ichim, B., and Zarojanu, A. An algorithm for computing the multigraded Hilbert depth of a module. *Experimental Mathematics 23*, 3 (2014), 322–331.

[11] Katthän, L. Betti Posets and the Stanley Depth. *Arnold Mathematical Journal 2*, 2 (Feb. 2016), 267–276.

[12] Keller, M. T., Shen, Y.-H., Streib, N., and Young, S. J. On the Stanley depth of squarefree Veronese ideals. *J. Alg. Combin. 33*, 2 (2011), 313–324.

[13] Keller, M. T., and Young, S. J. Stanley depth of squarefree monomial ideals. *J. Algebra 322*, 10 (2009), 3789–3792.

[14] McKay, B. D., and Piperno, A. Practical graph isomorphism, II. *Journal of Symbolic Computation 60* (2014), 94–112.

[15] Okazaki, R. A lower bound of Stanley depth of monomial ideals. *Journal of Commutative Algebra 3*, 1 (2011), 83–88.

[16] POPESCU, D. An inequality between depth and Stanley depth. *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie. Nouvelle Série 52(100)*, 3 (2009), 377–382.

[17] POPESCU, D., AND QURESHI, M. I. Computing the Stanley depth. *Journal of Algebra 323*, 10 (2010), 2943–2959.

[18] QIAN, J. Enumeration of unlabeled uniform hypergraphs. *Discrete Mathematics 326* (2014), 66–74.

[19] RAUF, A. Depth and Stanley depth of multigraded modules. *Communications in Algebra 38*, 2 (2010), 773–784.

[20] THE SAGE DEVELOPERS. SageMath, the Sage Mathematics Software System (Version 7.2), 2016.

[21] RINALDO, G. An algorithm to compute the Stanley depth of monomial ideals. *Le Matematiche 63*, 2 (2008), 243–256 (2009).

[22] SHEN, Y. H. Stanley depth of complete intersection monomial ideals and upper-discrete partitions. *J. Algebra 321*, 4 (2009), 1285–1292.

[23] STANLEY, R. P. Balanced Cohen-Macaulay complexes. *Transactions of the American Mathematical Society 249*, 1 (1979), 139–157.

[24] STANLEY, R. P. Linear Diophantine equations and local cohomology. *Inventiones Mathematicae 68*, 2 (1982), 175–193.

[25] STANLEY, R. P. *Combinatorics and Commutative Algebra*, second ed., vol. 41 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 1996.