A note on the rainbow connection of random regular graphs

Michael Molloy*

Department of Computer Science
University of Toronto
Toronto, Canada

molloy@cs.toronto.edu.

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Abstract

We prove that a random 3-regular graph has rainbow connection number $O(\log n)$. This completes the remaining open case from *Rainbow connection of random regular graphs,* by Dudek, Frieze and Tsourakakis.

1 Introduction

A *rainbow path* in an edge-coloured graph is a path in which every edge has a different colour. The *rainbow connection number* of a graph G, denoted $rc(G)$, is the minimum number of colours required to colour the edges of G such that every pair of vertices is connected by a rainbow path. (The colouring is not required to be proper, although the edge-colourings in this paper will be.) This was introduced by Chartrand et al. in [2]; see [5] for a survey and motivations.

$G_{n,r}$ is the random r-regular graph on n vertices, where every such graph is selected with equal probability. (We assume that rn is even.) The diameter of $G_{n,r}$ is approximately $\log_{r-1} n$ with high probability (w.h.p.)[1][1]; and clearly this is a lower bound on $rc(G_{n,r})$. Dudek, Frieze and Tsourakakis[3] proved that $rc(G_{n,r}) = O(\log n)$ for $r \geqslant 4$. Kamcev̌, Krivelevich and Sudakov[4] provided a short elegant proof for $r \geqslant 5$ and extended the result to expander graphs and to vertex colourings (for $r \geqslant 28$). Here we briefly note that a small modification to the arguments in [3] proves their result for $r = 3$.

Let T_1, T_2 be two copies of a binary tree of height ℓ, where the roots x_1, x_2 have degree three. We allow an adversary to colour the edges of T_1, T_2 so that both colourings are

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[1]A property A holds with high probability if $\lim_{n\to\infty} \Pr(G_{n,r} \text{ has } A) = 1$.

rainbow; i.e. each colour appears at most once in each tree. L_i is the set of leaves of T_i. We say a pair of leaves $u_1 \in L_1, u_2 \in L_2$ is *compatible* if no colour appears both in the path from x_1 to u_1 and in the path from x_2 to u_2. We define M to be the number of compatible pairs.

Lemma 1. *For any two rainbow edge colourings of T_1, T_2 we have $M \geqslant 3 \times 2^{\ell-1}(2^{\ell-1}+1)$.*

Since $|L_1| = |L_2| = 3 \times 2^{\ell-1}$, more than $\frac{1}{3}$ of the pairs of leaves are compatible. Lemma 2 of [3] proves that the same holds for d-ary trees for all $d \geqslant 3$ (with a root of degree d), albeit with a different constant multiple.

Corollary 2. *W.h.p. $rc(G_{n,3}) = O(\log n)$.*

The proof goes exactly as in [3]. There they first obtain an edge-colouring such that for every vertex v, the set of edges within distance $\ell = K \log\log n$ of v is rainbow for a particular constant K. (That set of edges forms a tree for almost every v.) So consider any two vertices x_1, x_2, and let T_1, T_2 be the trees formed by their distance ℓ neighbourhoods. Applying their Lemma 2 (the analogue of our Lemma 1) they prove that at least one of the compatible pairs of leaves is joined by a path which contains none of the colours joining either leaf to its root. For the sake of brevity, we refer the reader to [3] for all the details.

Proof of Lemma 1. For each colour c we let $\rho(c)$ be the number of pairs $u_1 \in L_1, u_2 \in L_2$ such that c appears in both paths from x_i to u_i. Clearly $M \geqslant |L_1||L_2| - \sum_c \rho(c)$ where the sum is taken over all colours c appearing in both trees. For each such c, let $\lambda_i(c)$ denote the level of the edge in T_i coloured c, where the edges adjacent to the leaves are at level 0 and those adjacent to the root are at level $\ell - 1$. So $\rho(c) = 2^{\lambda_1(c)+\lambda_2(c)}$.

Now $\sum_c \rho(c)$ is clearly maximized when no colour appears in exactly one tree; so assume that each tree contains the colours $c_1, c_2, \ldots, c_{3\times 2^{\ell-1}}$. Because the trees are isomorphic, the sequences $(2^{\lambda_1(c_1)}, 2^{\lambda_1(c_2)}, \ldots)$ and $(2^{\lambda_2(c_1)}, 2^{\lambda_2(c_2)}, \ldots)$ are both permutations of the same multiset. $\sum_c \rho(c)$ is the sum of the products of the corresponding elements in those permutations, which is maximized when the permutations are identical. So noting that each tree has $3 \times 2^{\ell-\lambda-1}$ edges at level λ, we obtain

$$\sum_c \rho(c) \leqslant \sum_{\lambda=0}^{\ell-1} 3 \times 2^{\ell-\lambda-1} \times 2^{2\lambda} = 3 \times 2^{\ell-1} \times \sum_{\lambda=0}^{\ell-1} 2^{\lambda} = 3 \times 2^{\ell-1} \times (2^{\ell} - 1).$$

This yields the lemma as $|L_1||L_2| = (3 \times 2^{\ell-1})^2$. □

Lemma 1 is tight. We will demonstrate this with a recursive edge-colouring of two trees of height ℓ. In our construction, $s(u)$ denotes the sibling of u, i.e. the other vertex with the same parent as u (which is unique if u is distance at least 2 from the root). For each $u \in T_1$, the corresponding vertex in T_2 (i.e. same level and same place in left-to-right order) is labelled u'.

For $\ell = 1$, we colour the edges, left-to-right, 1,2,3 for T_1 and $3, 1, 2$ for T_2. To extend from height ℓ to $\ell+1$, we use a new set of colours on the new edges of T_1, and then use the same colours on T_2, this time exchanging the colours of the edges of each pair of siblings; i.e. $u, s(u')$ are joined to their parents by edges of the same colour and so are $s(u), u'$.

It is not hard to note first that (u, u') is always compatible, and then that each leaf $u \in L_1$ lies in $2^{\ell-1}+1$ non-compatible pairs: $2 \times (2^{\ell-2}+1)$ arising from children of nodes who were not compatible with its parent in the previous colouring, plus $s(u')$.

Remark: Lemma 2 of [3] is stated for d-ary trees, $d \geqslant 3$, where the root has degree d. Their lemma does not extend to binary trees, as they show with a counterexample (due to Alon) in their section 3. So the fact that the root has degree 3 in our Lemma 1 is crucial. Proving Lemma 1 when the root has degree 3 is sufficient for the remainder of the argument from [3] to apply. Indeed, [3] uses their Lemma 2 to derive Corollary 4 which applies to d-ary trees with a root of degree $d+1$, and then use Corollary 4 for the remainder of the paper. There is one exception – they use Lemma 3 (which is stated only for trees where the root has degree d) in section 2.6.3, but Corollary 4 applies there just as well.

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