

Multicolor Ramsey numbers and restricted Turán numbers for the loose 3-uniform path of length three

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Abstract

Let P denote a 3-uniform hypergraph consisting of 7 vertices a, b, c, d, e, f, g and 3 edges $\{a, b, c\}$, $\{c, d, e\}$, and $\{e, f, g\}$. It is known that the r -colored Ramsey number for P is $R(P; r) = r + 6$ for $r = 2, 3$, and that $R(P; r) \leq 3r$ for all $r \geq 3$. The latter result follows by a standard application of the Turán number $\text{ex}_3(n; P)$, which was determined to be $\binom{n-1}{2}$ in our previous work. We have also shown that the full star is the only extremal 3-graph for P . In this paper, we perform a subtle analysis of the Turán numbers for P under some additional restrictions. Most importantly, we determine the largest number of edges in an n -vertex P -free 3-graph which is not a star. These Turán-type results, in turn, allow us to confirm the formula $R(P; r) = r + 6$ for $r \in \{4, 5, 6, 7\}$.

1 Introduction

In this paper we prove results about both Ramsey numbers and Turán numbers for the *loose 3-uniform path of length 3* defined as the hypergraph $P := P_3^3$ consisting of 7 vertices, say, a, b, c, d, e, f, g , and 3 edges $\{a, b, c\}$, $\{c, d, e\}$, and $\{e, f, g\}$. This is a very special case of a more general notion of the k -uniform *loose path* P_m^k of length m , where $k, m \geq 2$, defined as a k -uniform hypergraph (or k -graph, for short) with m edges which can be linearly ordered in such a way that every two consecutive edges intersect in exactly one vertex while all other pairs of edges are disjoint. Note that some authors, e.g., in [5, 13] call such paths *linear*, while by loose they mean paths in which consecutive edges may intersect on more vertices.

The *complete k -graph* K_n^k is a k -graph on n vertices in which every k -element subset of the vertex set forms an edge. For a given k -graph F and an integer $r \geq 2$, the *Ramsey*

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number $R(F; r)$ is the least integer n such that every r -coloring of the edges of K_n^k results in a monochromatic copy of F .

In the classical case of two colors ($r = 2$), it was proved already in [6] that for graphs ($k = 2$) we have $R(P_m^2; 2) = \lfloor (3m + 1)/2 \rfloor$, while for 3-graphs ([14]) $R(P_m^3; 2) = \lfloor (5m + 1)/2 \rfloor$, both formulae holding for all $m \geq 2$. For higher dimensions ($k \geq 4$), only the numbers $R(P_m^k; 2)$, $m = 2, 3, 4$, have been determined exactly (see [7]), while in [8] an asymptotic formula $R(P_m^k; 2) \sim (k - 1/2)m$, k fixed, $m \rightarrow \infty$, was established. For more than two colors, the only existing results are $R(P; 3) = 9$ and $r + 6 \leq R(P; r) \leq 3r$ for $r \geq 3$ [10]. We include below a simple proof of the upper bound to recall the standard technique of using Turán numbers for bounding Ramsey numbers, as this is the starting point of the research presented in this paper.

For a given k -graph F and an integer $n \geq 1$, the Turán number $\text{ex}_k(n; F)$ is the largest number of edges in an n -vertex F -free k -graph (for a more general definition, see Section 2.) Every n -vertex F -free k -graph with $\text{ex}_k(n; F)$ edges is called *extremal*.

Clearly, if $\binom{n}{k} > r \cdot \text{ex}_k(n; F)$, then $R(F; r) \leq n$. This trivial observation can sometimes be sharpened, owing to a specific structure of the extremal k -graphs. A *star* is a hypergraph with a vertex, called *the center*, contained in all the edges. An n -vertex k -uniform star is called *full* and denoted by S_n^k if it has $\binom{n-1}{k-1}$ edges.

It has been proved in [11] that for $n \geq 8$, $\text{ex}_3(n; P) = \binom{n-1}{2}$ and that S_n^3 is the only extremal 3-graph. Thus, the above inequality is equivalent to $n > 3r$ and yields only that $R(P; r) \leq 3r + 1$. If $n = 3r$, then $\binom{n}{3} = r \cdot \text{ex}_3(n; P)$, meaning that for every r -coloring of K_n^3 either there is a monochromatic copy of P or every color forms a full star which, however, is impossible. This was good enough to claim that $R(P; 3) = 9$ in [10], but for $r = 4$ it only yielded the bound $R(P; 4) \leq 12$. To make further progress in pin-pointing the Ramsey numbers $R(P; r)$ one has to refine the analysis of the Turán numbers and extremal 3-graphs for P which, in our opinion, might be of independent interest.

Let us illustrate our approach by sticking to the case $r = 4$ for a while. The lower bound on $R(P; 4)$ is $r + 6 = 10$ and $\frac{1}{4} \binom{10}{3} = 30 < \binom{9}{2}$. This only tells us that in every 4-coloring of K_{10}^3 a color must have been applied to at least 30 edges. If we only knew that the edges of that color formed a star (not necessarily full), then we could remove the center of that star reducing the picture to a 3-coloring of K_9^3 about which we already know that it does contain a monochromatic copy of P .

In this paper we prove that this is, indeed, the case. In fact, we prove a much stronger result (Theorem 6 in Section 2.1) by determining precisely the largest number of edges in an n -vertex P -free 3-graph which is not a subset of a star. We call this number *the Turán number of the second order* and denote it by $\text{ex}^{(2)}(n; P)$. To determine the Ramsey numbers $R(P; r)$, the numbers $\text{ex}^{(2)}(n; P)$ work fine for $r = 5$ and $r = 7$, but, quite surprisingly, fail for $r = 6$. In this case, we need to define *the Turán number of the third order*, $\text{ex}^{(3)}(n; P)$ and compute it for $n = 12$ (see Definition 2 and Theorem 7 in Section 2.1).

Our contribution to the Ramsey theory of hypergraphs is summarized in the following result.

Theorem 1. *For all $r \leq 7$, $R(P; r) = r + 6$.*

Theorem 1 follows from Theorems 4, 6, and 7. The first one is taken from [11]. To prove Theorems 6 and 7 we invoke a new notion of *conditional Turán numbers* $\text{ex}_k(n; F|G)$ which are defined like the ordinary ones but with an additional restriction to k -graphs containing a copy of a prescribed k -graph G (see Section 2.2 for a formal definition).

Already in [11], when proving Theorem 4, we determined $\text{ex}_3(n; P|C)$ as a quadratic function of n , where C is the 3-uniform triangle defined below. Here, our main technical tool, Lemma 1, asserts that under a seemingly mild additional assumption of connectedness, this number drops down to a linear function of n . We also determine two conditional Turán numbers, one for P and one for the pair $\{P, C\}$, where the condition imposes the presence of a pair of disjoint edges, or, in other words, that the 3-graph is non-intersecting (Theorems 9 and 10).

In the next section we introduce higher order Turán numbers and conditional Turán numbers, and state Theorems 6, 7, 9, 10, as well as Lemma 1 and another technical result, Lemma 2. Section 3 contains a relatively straightforward proof of Theorem 1. Section 4 is devoted to deducing Theorems 6, 7, and 9 from Lemma 1 and Theorem 10. In Section 5 we prove Lemmas 1 and 2. The most challenging Theorem 10 is then proved in Section 6. Finally, in a short Section 7 we mention a couple of open problems.

2 Turán numbers

In this section, after providing some background, we define Turán numbers of the s -th order as well as conditional Turán numbers, and formulate our results concerning such numbers for P , the loose 3-uniform path of length 3. We begin by recalling the definition of the ordinary Turán number. Given a family of k -graphs \mathcal{F} , we call a k -graph H \mathcal{F} -free if for all $F \in \mathcal{F}$ we have $F \not\subseteq H$.

Definition 1. For a family of k -graphs \mathcal{F} and an integer $n \geq 1$, the *Turán number (of the 1st order)* is defined as

$$\text{ex}_k^{(1)}(n; \mathcal{F}) := \text{ex}_k(n; \mathcal{F}) = \max\{|E(H)| : |V(H)| = n \text{ and } H \text{ is } \mathcal{F}\text{-free}\}.$$

Every n -vertex \mathcal{F} -free k -graph with $\text{ex}_k(n; \mathcal{F})$ edges is called *extremal (1-extremal)* for \mathcal{F} . We denote by $\text{Ex}_k(n; \mathcal{F}) = \text{Ex}_k^{(1)}(n; \mathcal{F})$ the family of all n -vertex k -graphs which are extremal for \mathcal{F} .

In the case when $\mathcal{F} = \{F\}$, we will often write $\text{ex}_k(n; F)$ for $\text{ex}_k(n; \{F\})$ and $\text{Ex}_k(n; F)$ for $\text{Ex}_k(n; \{F\})$. If $k = 3$, for brevity of notation we will be omitting the subscript 3.

The Turán numbers for graphs have been harder to grasp in the case of bipartite F than when $\chi(F) \geq 3$. For k -graphs, $k \geq 3$, on the other hand, the k -partite case seems to be easier. Indeed, the numbers $\text{ex}_k(n; F)$ have been already computed for F being a pair of disjoint edges, a loose path and a loose cycle, while, e.g., $\text{ex}(n; K_4^3)$ is still not known, even asymptotically. Interestingly, the three k -partite cases of F mentioned above exhibit a whole lot of similarity.

A family F of sets is called *intersecting* if $e \cap e' \neq \emptyset$ for all $e, e' \in F$. Obviously, a star is intersecting. Restricting to n -vertex k -graphs, a celebrated result of Erdős, Ko, and Rado asserts that for $n \geq 2k + 1$, the full star S_n^k is, indeed, the unique largest intersecting family. Below, we formulate this result in terms of the Turán numbers. Let M_2^k be a k -graph consisting of two disjoint edges.

Theorem 2 ([2]). *For $n \geq 2k$, $\text{ex}_k(n; M_2^k) = \binom{n-1}{k-1}$. Moreover, for $n \geq 2k + 1$, $\text{Ex}_k(n; M_2^k) = \{S_n^k\}$.*

A *loose cycle* C_m^k is defined in the same way as a loose path P_m^k , except that this time also the first and the last edge share one vertex. When $k = m = 3$ it is sometimes called *a triangle*. For convenience we abbreviate our notation for triangles to $C := C_3^3$. The Turán number $\text{ex}(n; C)$ has been determined in [4] for $n \geq 75$ and later for all n in [1].

Theorem 3 ([1]). *For $n \geq 6$, $\text{ex}(n; C) = \binom{n-1}{2}$. Moreover, for $n \geq 8$, $\text{Ex}(n; C) = \{S_n^3\}$.*

Finally, we return to loose paths. For large n , the Turán number for P_m^k has been determined for $k \geq 4$ in [5] and for $m \geq 4$ in [13]. In [5] the authors mentioned that their method does not quite work for $k = 3$, while the authors of [13] credited [5] with that case. In [11] we closed this gap. Given two k -graphs F_1 and F_2 , by $F_1 \cup F_2$ we denote a vertex-disjoint union of F_1 and F_2 . Also, note that K_1^3 is just an isolated vertex.

Theorem 4 ([11]).

$$\text{ex}(n; P) = \begin{cases} \binom{n}{3} & \text{and} & \text{Ex}(n; P) = \{K_n^3\} & \text{for } n \leq 6, \\ 20 & \text{and} & \text{Ex}(n; P) = \{K_6^3 \cup K_1^3\} & \text{for } n = 7, \\ \binom{n-1}{2} & \text{and} & \text{Ex}(n; P) = \{S_n^3\} & \text{for } n \geq 8. \end{cases}$$

It was proved in [3] for large n and in [12] for all n that for $k \geq 4$ the Turán number for P_2^k , or the maximum number of edges in a k -graph with no singleton intersection, is $\text{ex}_k(n; P_2^k) = \binom{n-2}{k-2}$. In a couple of proofs we will need an easy analog of this result for $k = 3$, first observed in [12].

Fact 1. *For $n \geq 1$, we have $\text{ex}(n; P_2^3) \leq n$.*

2.1 A hierarchy of Turán numbers

Turán numbers of the first order are just the ordinary Turán numbers defined above. Here we introduce a hierarchy of Turán numbers, where in each generation we consider only k -graphs which are not sub- k -graphs of extremal k -graphs from all previous generations. The next definition is iterative.

Definition 2. For a family of k -graphs \mathcal{F} and integers $s, n \geq 1$, the *Turán number of the $(s + 1)$ -st order* is defined as

$$\text{ex}_k^{(s+1)}(n; \mathcal{F}) = \max\{|E(H)| : |V(H)| = n, H \text{ is } \mathcal{F}\text{-free, and} \\ \forall H' \in \text{Ex}_k^{(1)}(n; \mathcal{F}) \cup \dots \cup \text{Ex}_k^{(s)}(n; \mathcal{F}), H \not\subseteq H'\},$$

if such a k -graph H exists. An n -vertex \mathcal{F} -free k -graph H is called $(s+1)$ -extremal for \mathcal{F} if $|E(H)| = \text{ex}_k^{(s+1)}(n; \mathcal{F})$ and $\forall H' \in \text{Ex}_k^{(1)}(n; \mathcal{F}) \cup \dots \cup \text{Ex}_k^{(s)}(n; \mathcal{F}), H \not\subseteq H'$; we denote by $\text{Ex}_k^{(s+1)}(n; \mathcal{F})$ the family of n -vertex k -graphs which are $(s+1)$ -extremal for \mathcal{F} .

If $k = 3$, for brevity of notation we will be omitting the subscript $_3$. A historically first example of a Turán number of the 2nd order is due to Hilton and Milner [9] who determined the maximum size of a *nontrivial* intersecting family of k -sets, that is, one which is not a star. We state it here for $k = 3$ only and suppress the family $\text{Ex}^{(2)}(n; M_2^3)$ which was also found in [9]. Set $M := M_2^3$ for convenience.

Theorem 5 ([9]). *For $n \geq 7$, we have $\text{ex}^{(2)}(n; M) = 3n - 8$.*

In this paper we prove the following two results which we then use to compute some Ramsey numbers for P . First, we completely determine $\text{ex}^{(2)}(n; P)$, together with the corresponding 2-extremal 3-graphs. A *comet* $\text{Co}(n)$ is a 3-graph with n vertices consisting of a copy of K_4^3 to which a star S_{n-3}^3 is attached, the unique common vertex being the center of the star (see Figure 1). This vertex is called *the center* of the comet, while the set of the remaining three vertices of the 4-clique is called *the head*.

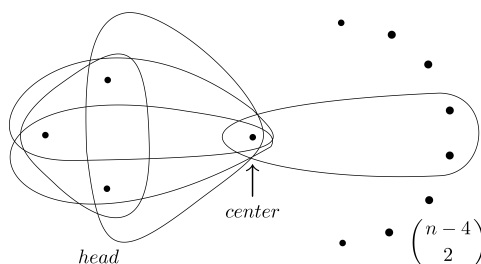


Figure 1: The comet $\text{Co}(n)$

Theorem 6.

$$\text{ex}^{(2)}(n; P) = \begin{cases} 15 & \text{and } \text{Ex}^{(2)}(n; P) = \{S_7^3\} & \text{for } n = 7, \\ 20 + \binom{n-6}{3} & \text{and } \text{Ex}^{(2)}(n; P) = \{K_6^3 \cup K_{n-6}^3\} & \text{for } 8 \leq n \leq 12, \\ 40 & \text{and } \text{Ex}^{(2)}(n; P) = \{K_6^3 \cup K_6^3 \cup K_1^3, \text{Co}(13)\} & \text{for } n = 13, \\ 4 + \binom{n-4}{2} & \text{and } \text{Ex}^{(2)}(n; P) = \{\text{Co}(n)\} & \text{for } n \geq 14. \end{cases}$$

Note that for $n \leq 6$ this number is not defined, since each 3-graph is a sub-3-graph of K_n^3 . Then, we calculate the 3rd Turán number for P , but only for $n = 12$ which is, however, just enough for our application.

Theorem 7.

$$\text{ex}^{(3)}(12; P) = 32 \quad \text{and} \quad \text{Ex}^{(3)}(12; P) = \{\text{Co}(12)\}.$$

2.2 Conditional Turán numbers

To determine the Turán numbers of higher order, it is sometimes useful to rely on Theorem 5 and divide all 3-graphs into those which contain M and those which do not. This leads us quickly to another variation on Turán numbers.

Definition 3. For a family of k -graphs \mathcal{F} , a family of \mathcal{F} -free k -graphs \mathcal{G} , and an integer $n \geq \min\{|V(G)| : G \in \mathcal{G}\}$, the *conditional Turán number* is defined as

$$\text{ex}_k(n; \mathcal{F}|\mathcal{G}) = \max\{|E(H)| : |V(H)| = n, H \text{ is } \mathcal{F}\text{-free, and } \exists G \in \mathcal{G} : H \supseteq G\}$$

Every n -vertex \mathcal{F} -free k -graph with $\text{ex}_k(n; \mathcal{F}|\mathcal{G})$ edges and such that $H \supseteq G$ for some $G \in \mathcal{G}$ is called \mathcal{G} -*extremal* for \mathcal{F} . We denote by $\text{Ex}_k(n; \mathcal{F}|\mathcal{G})$ the family of all n -vertex k -graphs which are \mathcal{G} -extremal for \mathcal{F} . (If $\mathcal{F} = \{F\}$ or $\mathcal{G} = \{G\}$, we will simply write $\text{ex}_k(n; F|\mathcal{G})$, $\text{ex}_k(n; \mathcal{F}|G)$, $\text{ex}_k(n; F|G)$, $\text{Ex}_k(n; F|\mathcal{G})$, $\text{Ex}_k(n; \mathcal{F}|G)$, or $\text{Ex}_k(n; F|G)$, respectively. If $k = 3$, to abbreviate the notation we will omit the subscript 3.)

In [11] we determined $\text{ex}(n; P|C)$ in terms of the ordinary Turán numbers $\text{ex}(n; P)$.

Theorem 8 ([11]). For $n \geq 6$,

$$\text{ex}(n; P|C) = 20 + \text{ex}(n - 6; P).$$

Moreover, $\text{Ex}(n; P|C) = \{K_6^3 \cup H_{n-6}\}$, where $\text{Ex}(n - 6; P) = \{H_{n-6}\}$, that is, H_{n-6} is the unique extremal P -free 3-graph on $n - 6$ vertices (cf. Theorem 4).

Theorem 8, combined with Theorem 4, yields immediately explicit values of $\text{ex}(n; P|C)$ along with the extremal sets $\text{Ex}(n; P|C)$.

Corollary 1.

$$\text{ex}(n; P|C) = \begin{cases} 20 + \binom{n-6}{3} & \text{and } \text{Ex}(n; P|C) = \{K_6^3 \cup K_{n-6}^3\} \text{ for } 6 \leq n \leq 12, \\ 40 & \text{and } \text{Ex}(n; P|C) = \{K_6^3 \cup K_6^3 \cup K_1^3\} \text{ for } n = 13, \\ 20 + \binom{n-7}{2} & \text{and } \text{Ex}(n; P|C) = \{K_6^3 \cup S_{n-6}^3\} \text{ for } n \geq 14. \end{cases}$$

Our next result reveals that the conditional Turán number $\text{ex}(n; P|C)$ drops significantly if we restrict ourselves to connected 3-graphs only. A 3-graph $H = (V(H), E(H))$ is *connected* if for every bipartition of the set of vertices $V(H) = V_1 \cup V_2$, $V_1 \neq \emptyset$, $V_2 \neq \emptyset$, there exists an edge $h \in E(H)$ such that $h \cap V_1 \neq \emptyset$ and $h \cap V_2 \neq \emptyset$.

Lemma 1. If H is a connected P -free 3-graph with $n \geq 7$ vertices and $H \supset C$, then

$$|E(H)| \leq 3n - 8.$$

It is not a coincidence that in Lemma 1 and Theorem 5 we see the same extremal number $3n - 8$. In fact, we prove Lemma 1 (see Section 5) by showing that the extremal 3-graph forms a nontrivial intersecting family.

Already in [11] we observed that, as a consequence of Theorem 5,

$$\text{ex}^{(2)}(n; P) = \text{ex}(n; P|M) \quad \text{and} \quad \text{ex}^{(2)}(n; C) = \text{ex}(n; C|M),$$

except for some very small values of n . We also found constructions yielding lower bounds and conjectured that these bounds are, indeed, the true values (see also Section 7). In this paper we confirm one of these conjectures.

Theorem 9.

$$\text{ex}(n; P|M) = \begin{cases} 20 + \binom{n-6}{3} & \text{and} \quad \text{Ex}(n; P|M) = \{K_6^3 \cup K_{n-6}^3\} & \text{for } 6 \leq n \leq 12, \\ 40 & \text{and} \quad \text{Ex}(n; P|M) = \{K_6^3 \cup K_6^3 \cup K_1^3, \text{Co}(13)\} & \text{for } n = 13, \\ 4 + \binom{n-4}{2} & \text{and} \quad \text{Ex}(n; P|M) = \{\text{Co}(n)\} & \text{for } n \geq 14. \end{cases}$$

Note that the Turán numbers $\text{ex}(n; P|M)$ and $\text{ex}^{(2)}(n; P)$ coincide for $n \geq 8$.

We also find it useful to determine the Turán number for the pair $\{P, C\}$ conditioning on 3-graphs H being non-intersecting.

Theorem 10.

$$\text{ex}(n; \{P, C\}|M) = \begin{cases} 2n - 4 & \text{for } 6 \leq n \leq 9, \\ 20 & \text{for } n = 10, \\ 4 + \binom{n-4}{2} & \text{and} \quad \text{Ex}(n; \{P, C\}|M) = \{\text{Co}(n)\} & \text{for } n \geq 11. \end{cases}$$

Note that the Turán numbers $\text{ex}(n; \{P, C\}|M)$, $\text{ex}(n; P|M)$, and $\text{ex}^{(2)}(n; P)$ coincide for $n \geq 13$.

To prove Theorem 10 we will need a lemma which states that if one, in addition to $\{P, C\}$, forbids also $P_2^3 \cup K_3^3$, then the formula, valid for $\text{ex}(n; \{P, C\}|M)$ only for $6 \leq n \leq 9$, takes over for all values of n .

Lemma 2. For $n \geq 6$

$$\text{ex}(n; \{P, C, P_2^3 \cup K_3^3\}|M) = 2n - 4.$$

3 Proof of Theorem 1

As mentioned in the Introduction, the inequality $R(P; r) \geq r + 6$, $r \geq 1$, has been already proved in [10]. We are going to show that $R(P; r) \leq r + 6$ for each $r = 4, 5, 6, 7$.

Case $r = 4$. Let us consider an arbitrary 4-coloring of the $\binom{10}{3} = 120$ edges of the complete 3-graph K_{10}^3 . There exists a color with at least $\frac{1}{4} \cdot 120 = 30$ edges. Denote the set of these edges by H . Since, by Theorem 4, $\text{Ex}^{(1)}(10; P) = \{S_{10}^3\}$, and, by Theorem 6, $\text{ex}^{(2)}(10; P) = 24 < 30$, either $P \subseteq H$ or $H \subseteq S_{10}^3$. In the latter case we delete the center of the star containing H , together with the incident edges, obtaining a 3-coloring of K_9^3 . Since $R(P; 3) = 9$, there is a monochromatic copy of P .

Case $r = 5$. The proof follows the lines of the previous one. We consider a 5-coloring of the complete 3-graph K_{11}^3 . There exists a color with at least $\binom{11}{3}/5 = 33$ edges. Denote the set of these edges by H . Again, by Theorems 4 and 6, either $P \subseteq H$ or $H \subseteq S_{11}^3$. In the latter case we delete the center of the star containing H , together with its incident edges, obtaining a 4-coloring of K_{10}^3 . Since, as we have just proved, $R(P; 4) = 10$, there is a monochromatic copy of P .

Case $r = 6$. This is the most difficult case in which we have to appeal to the 3rd Turán number. We begin, as before, by considering an arbitrary 6-coloring of the complete 3-graph K_{12}^3 on the set of vertices V and assuming that it does not yield a monochromatic copy of the path P . Then none of the color classes can be contained in a star S_{12}^3 , since otherwise we would delete this star, obtaining a 5-coloring of K_{11}^3 , which surely contains a monochromatic P . By Theorems 4 and 6, S_{12}^3 and $K_6^3 \cup K_6^3$ are, respectively, the unique 1-extremal and 2-extremal 3-graph for P . Consequently, by Theorem 7, every color class with more than 32 edges must be a sub-3-graph of $K_6^3 \cup K_6^3$.

There exists a color class with at least $\lceil \binom{12}{3}/6 \rceil = 37$ edges which, as explained above, is contained in a copy K of $K_6^3 \cup K_6^3$. After deleting all the edges of K from K_{12}^3 , we obtain a complete bipartite 3-graph B with bipartition $V = U \cup W$, $|U| = |W| = 6$, and with $|E(B)| = 220 - 40 = 180$ edges, colored by 5 colors. Note that any copy of $K_6^3 \cup K_6^3$ may share with B at most 36 edges. Consequently, since $180/5 = 36$, every color class has precisely 36 edges and, thus, is contained in $K_6^3 \cup K_6^3$.

Let $G_i, i = 1, 2, 3, 4, 5$, be the 5 color classes. Then, for each i , G_i is fully characterized by two partitions, $U = U'_i \cup U''_i$ and $W = W'_i \cup W''_i$. (G_i is then a disjoint union of two copies of K_6^3 , one on the vertex set $U'_i \cup W'_i$, the other one on $U''_i \cup W''_i$, with U'_i, U''_i, W'_i, W''_i being the 4 missing edges (see Figure 2).)

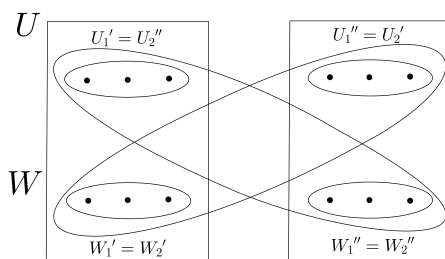


Figure 2: Illustration to the proof of Theorem 1, case $r = 6$

We now show that only 2 of the 5 color classes can be disjoint which is a contradiction (with a big cushion). For G_1 and G_2 to be disjoint, we need that $\{U'_1, U''_1\} = \{U'_2, U''_2\}$ and $\{W'_1, W''_1\} = \{W'_2, W''_2\}$, which simply means that one of the partitions, of U or of W , must be swapped. But this implies that G_1, G_2 , and G_3 cannot be pairwise disjoint.

Case $r = 7$. As $\lceil \binom{13}{3}/7 \rceil = 41 > 40 = \text{ex}^{(2)}(13; P)$, the proof in this case follows the lines of the proofs for $r = 4$ and $r = 5$, and therefore is omitted.

4 Proofs of Theorems 6, 7, and 9

In this section we first deduce Theorems 6 and 7 from Lemma 1 and Theorems 9 and 10, with a little help of some already known results (Theorems 3-5). Then we deduce Theorem 9 from Corollary 1 and Theorem 10. The proofs of Lemmas 1 and 2 will be presented in the next section, while the proof of the crucial Theorem 10, based on Lemma 2, is deferred to Section 6.

Throughout all the proofs, for convenience, we will be often identifying the edge set of a 3-graph with the 3-graph itself, writing, e.g., $|H|$ instead of $|E(H)|$.

Proof of Theorem 6. We consider the case $n = 7$ separately.

($n = 7$). By Theorem 4,

$$\text{ex}^{(1)}(7; P) = 20 \quad \text{and} \quad \text{Ex}^{(1)}(7; P) = \{K_6^3 \cup K_1^3\}.$$

Therefore, to determine $\text{ex}^{(2)}(7; P)$ we need to find the largest number of edges in a 7-vertex P -free 3-graph H which is not a sub-3-graph of $K_6^3 \cup K_1^3$. Note that $P \not\subseteq S_7^3 \not\subseteq K_6^3 \cup K_1^3$, and thus,

$$\text{ex}^{(2)}(7; P) \geq |S_7^3| = \binom{7-1}{2} = 15.$$

If H is a 7-vertex 3-graph with $|H| > 15$, then, by Theorem 3, $H \supset C$. But then, since $H \not\subseteq K_6^3 \cup K_1^3$, H must be connected. Consequently, since H is P -free, by Lemma 1, $|H| \leq 3 \times 7 - 8 = 13$, a contradiction. Checking that S_7^3 is the unique 2-extremal 3-graph for P and $n = 7$ is left to the reader.

($n \geq 8$). By Theorem 4 we have

$$\text{ex}^{(1)}(n; P) = \binom{n-1}{2} \quad \text{and} \quad \text{Ex}^{(1)}(n; P) = \{S_n^3\}.$$

Therefore, to determine $\text{ex}^{(2)}(n; P)$ for $n \geq 8$ we need to find the largest number of edges in an n -vertex P -free 3-graph H which is not a subgraph of the star S_n^3 . If H is an intersecting family, then, by Theorems 2 and 5, $|H| \leq \text{ex}^{(2)}(n; \{M, P\}) = \text{ex}^{(2)}(n; M) = 3n - 8$. Otherwise, $H \supset M$ and, therefore, $|H| \leq \text{ex}(n; P|M)$. Using Theorem 9 one can verify that for $n \geq 8$ we have $\text{ex}(n; P|M) > 3n - 8$. Consequently,

$$\text{ex}^{(2)}(n; P) = \max\{\text{ex}^{(2)}(n; M), \text{ex}(n; P|M)\} = \text{ex}(n; P|M)$$

and Theorem 6 for $n \geq 8$ follows by Theorem 9. □

Proof of Theorem 7. By Theorems 4 and 6,

$$\text{ex}^{(2)}(12; P) = 40 \quad \text{and} \quad \text{Ex}^{(1)}(12; P) \cup \text{Ex}^{(2)}(12; P) = \{S_{12}^3, K_6^3 \cup K_6^3\}.$$

Therefore, to determine $\text{ex}^{(3)}(12; P)$ we have to find the largest number of edges in a 12-vertex P -free 3-graph H such that $H \not\subseteq S_{12}^3$ and $H \not\subseteq K_6^3 \cup K_6^3$. The comet $\text{Co}(12)$

satisfies all the above conditions and has 32 edges. Let H be a 12-vertex P -free 3-graph satisfying the above conditions but $H \neq \text{Co}(12)$. We will show that $|H| < 32$. Since $H \not\subseteq S_{12}^3$, either H forms a nontrivial intersecting family and, by Theorem 5,

$$|H| \leq 3 \times 12 - 8 = 28 < 32,$$

or $H \supset M$. We may thus consider the latter case only. If H is disconnected, then, since $H \not\subseteq K_6^3 \cup K_6^3$, by Theorems 4 and 9,

$$\begin{aligned} |H| &\leq \max\{\text{ex}(7; P) + \text{ex}(5; P), \text{ex}(8; P) + \text{ex}(4; P), \\ &\quad \text{ex}(9; P) + \text{ex}(3; P), \text{ex}(10; P|M), \text{ex}(11; P|M)\} = \\ &\quad \max\{20 + 10, 21 + 4, 28 + 1, 24, 30\} = 30 < 32. \end{aligned}$$

Assume, finally, that H is connected and $H \supseteq M$. If, in addition, $H \supseteq C$, then, by Lemma 1, we have

$$|H| \leq 3 \times 12 - 8 = 28 < 32.$$

Otherwise, H is a $\{P, C\}$ -free 3-graph containing M . Therefore, by Theorem 10,

$$|H| < \text{ex}(12; \{P, C\}|M) = 4 + \binom{12-4}{2} = 32,$$

as the comet $\text{Co}(12)$ is the only M -extremal 3-graph for $\{P, C\}$. □

Proof of Theorem 9. Recall, that we want to determine the conditional Turán number $\text{ex}(n; P|M)$. By considering whether or not a 3-graph contains a triangle, we infer that

$$\text{ex}(n; P|M) = \max\{\text{ex}(n; P|\{M, C\}), \text{ex}(n; \{P, C\}|M)\}.$$

The number $\text{ex}(n; \{P, C\}|M)$ is given by Theorem 10, whereas

$$\text{ex}(n; P|\{M, C\}) = \text{ex}(n; P|C),$$

since the unique extremal graph from Corollary 1 contains M . One can easily check that for $6 \leq n \leq 12$,

$$\text{ex}(n; P|\{M, C\}) > \text{ex}(n; \{P, C\}|M),$$

for $n = 13$,

$$\text{ex}(n; P|\{M, C\}) = \text{ex}(n; \{P, C\}|M) = 4 + \binom{13-4}{2} = 40,$$

while for $n \geq 14$,

$$\text{ex}(n; P|\{M, C\}) < \text{ex}(n; \{P, C\}|M).$$

Theorem 9 follows now immediately from the respective parts of Corollary 1 and Theorem 10. □

5 Proofs of Lemmas 1 and 2

For a 3-graph F and a vertex $v \in V(F)$ set $F(v) = \{e \in F : v \in e\}$. The degree of v in F is defined as $|F(v)|$.

Proof of Lemma 1. Let H be a P -free, connected 3-graph with $V(H) = V$ and $|V| = n \geq 7$, containing a triangle. With some abuse of notation, we denote by C a fixed copy of the triangle in H . Set

$$U = V(C) = \{x_1, x_2, x_3, y_1, y_2, y_3\},$$

and let, recalling that we identify the edge set of a 3-graph with the 3-graph itself,

$$C = \{\{x_i, y_j, x_k\} : \{i, j, k\} = \{1, 2, 3\}\}.$$

Thus, the vertices x_1, x_2, x_3 are of degree two in C , while y_1, y_2, y_3 are of degree one. Further, let

$$W = V \setminus U, \quad |W| = n - 6$$

and let $H(U, W)$ denote the set of all edges of H which intersect both U and W .

It was observed in [11], Fact 1, that

$$H(U, W) = H \cap T,$$

where

$$T = T_1 \cup T_2$$

and

$$T_1 = \{\{x_i, y_i, w\} : 1 \leq i \leq 3, 1 \leq l \leq n - 6, w \in W\},$$

$$T_2 = \{\{x_i, x_j, w\} : 1 \leq i < j \leq 3, 1 \leq l \leq n - 6, w \in W\}.$$

Moreover, no edge of $H(U, W)$ may intersect an edge of $H[W]$, since otherwise there would be a copy of P in H ([11], Fact 2). This and the connectivity assumption imply that $H[W] = \emptyset$. Thus,

$$H = H[U] \cup H(U, W),$$

and, clearly $H(U, W) \neq \emptyset$, as $W \neq \emptyset$ (see Figure 3).

If H is an intersecting family (non-trivial due to the presence of C), then, by Theorem 5, $|H| \leq 3n - 8$. We will show that if, on the other hand, $H \supseteq M$, then, in fact, $|H|$ is even smaller. We begin with a simple observation.

Fact 2. $H(U, W)$ is an intersecting family.

Proof. Recall that $H(U, W) \subseteq T_1 \cup T_2$ and note that T_2 is intersecting by definition. On the other hand, if $e \in T_1$, $f \in T$, and $e \cap f = \emptyset$, then $C \cup \{e\} \cup \{f\} \supset P$, so either e or f cannot be in H . \square

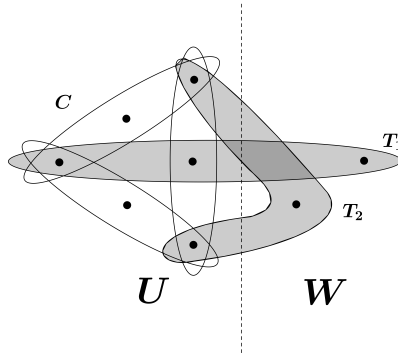


Figure 3: Set-up for the proof of Lemma 1

Let $f, h \in H$ satisfy $f \cap h = \emptyset$. By Fact 2, at least one of f and h belongs to $H[U]$. If both of them were in $H[U]$ then, clearly, $f \cup h = U$ and, by the P -freeness of H , each $e \in H(U, W)$ would need to be disjoint from one of them. In summary, if $H \supseteq M$, then there exist two disjoint edges $e, f \in H$ such that $e \in H(U, W)$ and $f \in H[U]$.

If $e \in T_1$, then one can easily check by inspection that $C \cup \{e\} \cup \{f\} \supset P$. Thus, $e \in T_2$, say $e \cap U = \{x_1, x_2\}$. The only edge in $H[U]$ disjoint from e which does not create a copy of the path P with $C \cup \{e\}$ is $f = \{x_3, y_1, y_2\}$ (see Figure 4). Further, observe that all triples in T , except those of the type $\{x_1, x_2, w\}$, $w \in W$, form a copy of P with f and some edge of C .

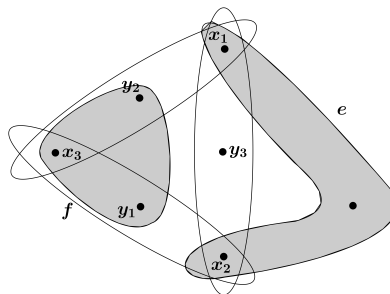


Figure 4: Illustration to the proof of Lemma 1

Hence,

$$H(U, W) \subseteq \{\{x_1, x_2, w\} : w \in W\}.$$

and, consequently, $|H(U, W)| \leq |W| = n - 6$. Let

$$X = \{\{y_1, y_2, y_3\}, \{x_j, y_i, y_3\}, \{x_i, x_3, y_3\}, i \in \{1, 2\}, j \in \{1, 2, 3\}\},$$

Notice that $|X| = 9$ and, for each $h \in X$, we have $C \cup \{e, f, h\} \supset P$. Thus, $H[U] \subseteq \binom{U}{3} \setminus X$, so that $|H[U]| \leq 20 - 9 = 11$. Consequently, for $n \geq 7$,

$$|H| = |H[U]| + |H(U, W)| \leq 11 + n - 6 < 3n - 8. \quad \square$$

Proof of Lemma 2. Let V be a set with $|V| = n \geq 6$. Fix four distinct vertices $v_1, v_2, v_3, v_4 \in V$ and define a 3-graph $H_n^{(0)}$ on V as

$$H_n^{(0)} = \left\{ h \in \binom{V}{3} : \{v_i, v_{i+1}\} \subset h, \quad i \in \{1, 3\} \right\},$$

Note that $H_n^{(0)} \supset M$ and $|H_n^{(0)}| = 2n - 4$. Moreover, since every edge contains one of the pairs $\{v_1, v_2\}$ or $\{v_3, v_4\}$, among any three edges at least two share two vertices. Therefore, $H_n^{(0)}$ is $\{P, C, P_2^3 \cup K_3^3\}$ -free and, thus,

$$\text{ex}(n; \{P, C, P_2^3 \cup K_3^3\} | M) \geq 2n - 4.$$

To show the opposite inequality, consider a $\{P, C, P_2^3 \cup K_3^3\}$ -free 3-graph H containing $M = \{e, f\}$, with $V(H) = V$, $|V| = n \geq 6$. Since H is $P_2^3 \cup K_3^3$ -free, $H[V \setminus e]$ and $H[V \setminus f]$ are P_2^3 -free, and by Fact 1,

$$|H[V \setminus e]| \leq n - 3 \quad \text{and} \quad |H[V \setminus f]| \leq n - 3.$$

Also, since H is P -free, there is no edge $h \in H$ with $|h \cap e| = |h \cap f| = 1$. Hence, if $|H[e \cup f]| = 2$, then $|H| \leq 2(n - 3) = 2n - 6$.

On the other hand, if there exists an edge $h \in H[e \cup f] \setminus \{e, f\}$, then, since H is $P_2^3 \cup K_3^3$ -free, all edges of H intersect one of e or f on at least two vertices. Let

$$F_e = \{h \in H : |h \cap e| = 2\}, \quad F_f = \{h \in H : |h \cap f| = 2\}.$$

If there existed $h_1, h_2 \in F_e$ with $|h_1 \cap h_2| = 1$, then, depending on whether $|(h_1 \cup h_2) \cap f| = 0, 1$, or 2 , the edges $\{h_1, h_2, f\}$ would form, respectively, a copy of $P_2^3 \cup K_3^3$, P , or C (see Figure 5).

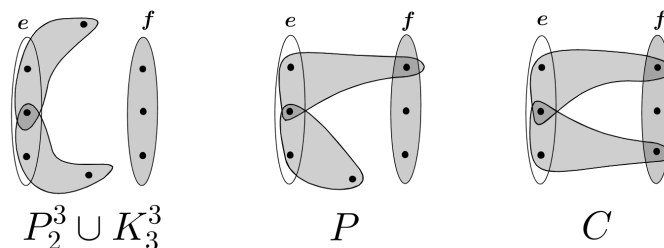


Figure 5: Illustration to the proof of Lemma 2

Thus,

$$\forall h_1, h_2 \in F_e, \quad |h_1 \cap h_2| = 2,$$

so, either all pairs $h_1, h_2 \in F_e$ share two vertices of e or all pairs $h_1, h_2 \in F_e$ share one vertex of $V \setminus e$ (and another in e)

This implies that

$$|F_e| \leq \max\{n - 3, 3\} = n - 3.$$

Similarly, $|F_f| \leq n - 3$ and, consequently,

$$|H| = |\{e, f\}| + |F_e| + |F_f| \leq 2 + (n - 3) + (n - 3) = 2n - 4. \quad \square$$

6 Proof of Theorem 10

This section is entirely devoted to proving Theorem 10, that is, to determining the largest number of edges in an n -vertex 3-graph which is P -free and C -free but is not an intersecting family.

First note that since $|V(P_2^3 \cup K_3^3)| = 8$, no n -vertex 3-graph, $n = 6, 7$, contains a copy of $P_2^3 \cup K_3^3$ and therefore, by Lemma 2,

$$\text{ex}(n; \{P, C\} | M) = \text{ex}(n; \{P, C, P_2^3 \cup K_3^3\} | M) = 2n - 4.$$

Thus, from now on we will be assuming that $n \geq 8$. Define a sequence of 3-graphs

$$H_n = \begin{cases} H_n^{(0)} & \text{for } 8 \leq n \leq 9, \\ K_5^3 \cup K_5^3 & \text{for } n = 10, \\ \text{Co}(n) & \text{for } n \geq 11, \end{cases}$$

where $H_n^{(0)}$ is the 3-graph introduced in the proof of Lemma 2. By simple inspection one can see that H_n is $\{P, C\}$ -free and contains M . Hence

$$\text{ex}(n; \{P, C\} | M) \geq |H_n| = \begin{cases} 2n - 4 & \text{for } 8 \leq n \leq 9, \\ 20 & \text{for } n = 10, \\ 4 + \binom{n-4}{2} & \text{for } n \geq 11. \end{cases}$$

The main difficulty lies in showing the reverse inequality, namely, that any $\{P, C\}$ -free 3-graph H on $n \geq 8$ vertices, containing M , satisfies $|H| \leq |H_n|$. Moreover, for $n \geq 11$, we want to show that the equality is reached by the extremal 3-graph $H_n = \text{Co}(n)$ only. We may assume that H contains a copy of $P_2^3 \cup K_3^3$, since otherwise, by Lemma 2,

$$|H| \leq 2n - 4 \leq |H_n|,$$

where the last inequality is strict for $n \geq 10$. Before we turn to the actual proof of Theorem 10, we need to introduce some notation and prove preliminary results about the structure of H .

6.1 Preparations for the proof

We assume that H is $\{P, C\}$ -free and contains a copy of $P_2^3 \cup K_3^3$. Let $e_1, e_2 \in H$ and $x \in V = V(H)$ be such that $e_1 \cap e_2 = \{x\}$ and there is an edge in H disjoint from $e_1 \cup e_2$. We know that such a choice of e_1, e_2, x exists, because $H \supseteq P_2^3 \cup K_3^3$. We split $V = U \cup W$, where

$$U = e_1 \cup e_2, \quad \text{and} \quad W = V \setminus U.$$

Note that $|U| = 5$ and $|W| = n - 5$. Further set

$$H(U, W) = H \setminus (H[U] \cup H[W])$$

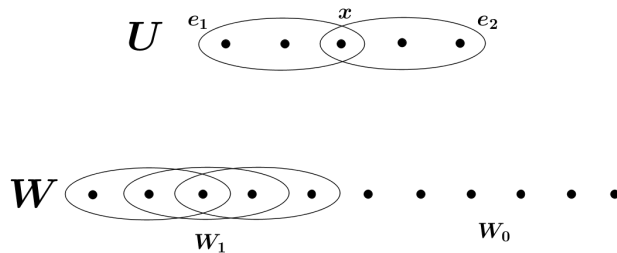


Figure 6: Set-up for the proof of Theorem 10

for the sub-3-graph of H consisting of all edges intersecting both, U and W . Notice that $H[W] \neq \emptyset$, and thus the set W_0 of vertices of degree 0 in $H[W]$ has size

$$|W_0| \leq n - 8. \tag{1}$$

Set also $W_1 = W \setminus W_0$ (see Figure 6).

Let us split

$$H[U] = \{e_1, e_2\} \cup E(x) \cup E(\bar{x}),$$

where $E(x)$ contains all edges of $H[U]$ which contain vertex x , except for e_1 and e_2 , while $E(\bar{x})$ contains all other edges of $H[U]$. Note that

$$\max\{|E(x)|, |E(\bar{x})|\} \leq 4. \tag{2}$$

We also split the set of edges of $H(U, W)$. First, notice that if for some $h \in H(U, W)$ we have $|h \cap U| = 1$, then $h \cap U = \{x\}$, since otherwise h together with e_1 and e_2 would form a copy of P in H . We let

$$F^0 = \{h \in H(U, W) : h \cap U = \{x\}\}.$$

The edges $h \in H(U, W)$ with $|h \cap U| = 2$ must satisfy $h \cap U \subset e_1$ or $h \cap U \subset e_2$, since otherwise h together with e_1 and e_2 would form a copy of C in H . For $k = 1, 2$ define

$$F^k = \{h \in H(U, W) : |h \cap U \setminus \{x\}| = k\}.$$

We have $H(U, W) = F^0 \cup F^1 \cup F^2$. (Note that in each case $k = 0, 1, 2$, the superscript k stands for the common size of the set $h \cap U \setminus \{x\}$ – see Figure 7.)

For a sub-3-graph $F \subseteq H(U, W)$ and $i = 0, 1$, set

$$F_i = \{h \in F : h \cap W \subset W_i\},$$

which in the important case of $F = H(U, W)$ will be abbreviated to H_i . In particular, for $i = 0, 1$, $H_i = F_i^0 \cup F_i^1 \cup F_i^2$, where F_i^0 is the subset of edges $h \in F^0$ with $|h \cap W_i| = 2$, while F_i^k , $k = 1, 2$, is the subset of edges of F^k whose unique vertex in W lies in W_i .

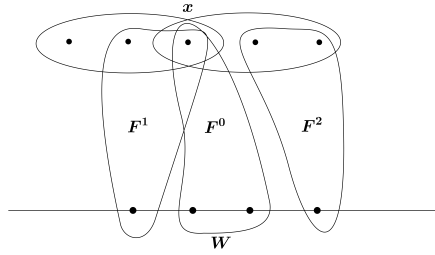


Figure 7: Three types of edges in $H(U, W)$

A simple but crucial observation is that, since H is P -free, for every two disjoint edges in H , no edge may intersect each of them in exactly one vertex. Thus, there is no edge in H with one vertex in each of the sets, U , W_0 and W_1 . Therefore,

$$H(U, W) = H_0 \cup H_1, \quad (3)$$

and consequently,

$$\begin{aligned} H &= H[U] \cup H(U, W) \cup H[W] = H[U] \cup H_0 \cup H_1 \cup H[W] \\ &= H[U \cup W_0] \cup H_1 \cup H[W]. \end{aligned} \quad (4)$$

Furthermore, by the same principle, if $e \in F_1^0$, then the pair $e \cap W_1$ must be *nonseparable* in $H[W_1]$, that is, every edge of $H[W_1]$ must contain both these vertices or none. Since, as it can be easily proved, there are at most $|W_1|$ nonseparable pairs in W_1 ,

$$|F_1^0| \leq |W_1|. \quad (5)$$

Another consequence of the above observation is that $F_1^1 = \emptyset$. Thus,

$$H_1 = F_1^0 \cup F_1^2. \quad (6)$$

To make use of (6), in addition to (5), we need to bound $|F_1^2|$ which, however, requires a detailed analysis of the degrees of vertices $v \in W$ in the 3-graphs F^k , $k = 0, 1, 2$. For $v \in W$ and $F \subseteq H$, recall that $|F(v)|$ is the degree of v in F .

It can be easily checked that, since H is P -free, for every $v \in W$ either

$$F^0(v) = \emptyset \quad \text{or} \quad F^2(v) = \emptyset. \quad (7)$$

Moreover, by the definitions of F^1 and F^2 ,

$$|F^1(v)| \leq 4 \quad \text{and} \quad |F^2(v)| \leq 2. \quad (8)$$

For $v \in W_0$, by the remark preceding (3), $|F^0(v)| \leq |W_0| - 1$, and thus, by (7), (8), and (1),

$$|H(v)| = |F^0(v)| + |F^1(v)| + |F^2(v)| \leq 4 + \max\{2, n - 9\}.$$

In particular, for $n = 10$,

$$\forall v \in W_0, \quad |H(v)| \leq 6, \quad (9)$$

while for $n \geq 11$,

$$\forall v \in W_0, \quad |H(v)| \leq n - 5, \quad (10)$$

where the equality for $n \geq 12$ is achieved only when $|F^0(v)| = n - 9$, $|F^1(v)| = 4$, and $F^2(v) = \emptyset$.

Consider now $v \in W_1$. For each $e \in F^0$, the pair $e \cap W$ must be nonseparable and v belongs to at most two nonseparable pairs. Thus, $|F^0(v)| \leq 2$ and, consequently, by (6), (7), and (8),

$$\forall v \in W_1, \quad |H_1(v)| = |F^0(v)| + |F^2(v)| \leq 2. \quad (11)$$

One can also show, that

$$|F_1^2| \leq \max\{|W_1|, 2|W_1| - 4\}. \quad (12)$$

Indeed, if for all $v \in W_1$ we have $|F^2(v)| = |F_1^2(v)| = 1$, then $|F_1^2| \leq |W_1|$. Otherwise, let $v \in W_1$ have, by (8), $|F^2(v)| = 2$ and let $\{v, v', v''\} \in H[W]$. Since H is P -free, $F^2(v') = F^2(v'') = \emptyset$, and therefore, again by (8),

$$|F_1^2| \leq 2(|W_1| - 2) = 2|W_1| - 4.$$

Now we are ready to set bounds on the number of edges in H_1 , as well as in $H[U] \cup H_1$, which will be repeatedly used in the proof of Theorem 10. Recall that $|W_1| \geq 3$.

Fact 3. *We have*

$$|H_1| \leq 2|W_1| - 3 \quad (13)$$

and, for $|W_1| \geq 4$,

$$|H[U]| + |H_1| \leq 2|W_1| + 2. \quad (14)$$

Proof. Let $h \in H[W]$. It is easy to check by inspection that $\sum_{v \in h} |H_1(v)| \leq 3$, while for $v \in W_1 \setminus h$, by (11), $|H_1(v)| \leq 2$. This yields $|H_1| \leq 3 + 2(|W_1| - 3)$ and takes care of (13).

If $H_1 = \emptyset$ then (14) holds, as $|H[U]| \leq 10$. To prove (14) also when $H_1 \neq \emptyset$, we need a better bound on $|H[U]|$. To this end, note that if $F^0 \neq \emptyset$ then $E(\bar{x}) = \emptyset$, while if $F_1^2 \neq \emptyset$ then $E(x) = \emptyset$. Hence, by (6) and (2),

$$H_1 \neq \emptyset \quad \Rightarrow \quad |H[U]| \leq 6. \quad (15)$$

So, if one of the sets, F_1^0 or F_1^2 , is empty, then we get (14) by (15), (5), and (12). If both these sets are nonempty, then $E(\bar{x}) = E(x) = \emptyset$, and thus $|H[U]| = 2$. In this case (14) follows by (13) with a margin. \square

Since H is C -free, on several occasions our proof relies on two instances of Theorem 3. Namely, if $|W_0| \geq 1$ then

$$|H[U \cup W_0]| \leq \binom{|W_0| + 4}{2}, \quad (16)$$

while if $|W| = n - 5 \geq 6$ then

$$|H[W]| \leq \binom{n-6}{2}. \quad (17)$$

Finally, there cannot be too many edges between U and the vertex set of a copy of P_2^3 in $H[W]$ if there happens to be one. For a subset $W' \subset W$, we denote by $H(U, W')$ the bipartite, induced sub-3-graph of H with bipartition (U, W') .

Fact 4. *If $H[W]$ contains a copy Q of P_2^3 with $V(Q) \subseteq W$, then*

$$|H(U, V(Q))| \leq 4. \quad (18)$$

Proof. Note that, due to P -freeness of H , the only edges allowed in $H(U, V(Q))$ with one vertex in U must belong to F_0 (there are at most two such edges). By symmetry, there are also at most two edges in $H(U, V(Q))$ with one vertex in W , which yields (18). \square

6.2 The proof

The structure of the proof is as follows. We first settle the three smallest cases, $n = 8, 9, 10$, one by one. Then we turn to the main case of $n \geq 11$. Here, after quickly taking care of the easy subcase $W_0 = \emptyset$, we assume that $W_0 \neq \emptyset$ and proceed by induction on n with $n = 11$ being the base case. This part is a bit pedestrian, but afterwards, the induction step is almost immediate.

Let H be a $\{P, C\}$ -free n -vertex 3-graph which contains a copy of $P_2^3 \cup K_3^3$. We adopt the notation and terminology from Subsection 6.1. In addition, for $v \in V$, we will write $H - v$ for $H[V \setminus \{v\}]$.

$n = 8$. We have $|W_1| = 3$, $|H[W]| = 1$, and $W_0 = \emptyset$. If $H(U, W) = H_1 = \emptyset$, then

$$|H| = |H[U]| + |H[W]| \leq 10 + 1 < 12 = |H_8|,$$

Otherwise, by (15), $|H[U]| \leq 6$ and, therefore, by (13),

$$|H| = |H[U]| + |H_1| + |H[W]| \leq 6 + 3 + 1 < 12.$$

$n = 9$. We have $|W| = 4$ and $|H[W]| \leq \binom{4}{3} = 4$. If $W_0 = \emptyset$ then, by (14),

$$|H| = |H[U]| + |H_1| + |H[W]| \leq 2|W_1| + 2 + 4 = 14 = |H_9|.$$

If $W_0 \neq \emptyset$ then $|W_0| = 1$, $|W_1| = 3$ and $|H[W]| = 1$. By (13), $|H_1| \leq 3$, and consequently, by (4) and (16),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 10 + 3 + 1 = 14.$$

$n = 10$. We have $|W| = 5$, $|W_0| \leq 2$ and $|H[W]| \leq \binom{5}{3} = 10$. If $W_0 = \emptyset$ then, by (14), $|H[U]| + |H_1| \leq 2|W_1| + 2 = 12$. If, additionally, $|H[W]| \leq 5$, then

$$|H| = |H[U]| + |H_1| + |H[W]| \leq 12 + 5 < 20 = |H_{10}|.$$

Otherwise, by Fact 1, $H[W]$ contains a copy Q of P_2^3 (note that $V(Q) = W_1$), and, by (18), $|H_1| \leq 4$. Hence, using (15) along the way,

$$|H| = |H[U]| + |H_1| + |H[W]| \leq \max\{10 + 0, 6 + 4\} + 10 = 20.$$

Now, let $W_0 \neq \emptyset$. Fix $v \in W_0$ and notice that $H - v$ is $\{P, C\}$ -free and contains M . Since we have already proved that $\text{ex}(9; \{P, C\} | M) = 14$,

$$|H - v| \leq 14.$$

Moreover, by (9), $|H(v)| \leq 6$, and consequently,

$$|H| = |H - v| + |H(v)| \leq 14 + 6 = 20.$$

$n \geq 11$. The proof is by induction on n with $n = 11$ being the base case. First, however, we take care of a simple subcase when $W_0 = \emptyset$, for which, by (14) and (17),

$$|H| = |H[U]| + |H_1| + |H[W]| \leq 2(n - 5) + 2 + \binom{n - 6}{2} = 3 + \binom{n - 4}{2} < |H_n|.$$

Hence, in what follows we will be assuming that $W_0 \neq \emptyset$.

$n = 11$ (base case). Suppose first that $H[W]$ contains a copy Q of P_2^3 . Then $|W_0| = 1$, $|W_1| = 5$, $V(Q) = W_1$, and by (18), $|H_1| \leq 4$. Consequently, by (4), (16), and (17),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 10 + 4 + 10 < 25 = |H_{11}|.$$

In the remainder of this part of the proof, besides the assumption that $W_0 \neq \emptyset$, we will be also assuming that $H[W]$ is P_2^3 -free and thus, by Fact 1, $|H[W]| \leq 6$. We consider three cases with respect to the size of $|W_0|$.

$|W_0| = 1$. We have $|W_1| = 5$ and, by (13), $|H_1| \leq 7$. Consequently, by (4) and (16),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 10 + 7 + 6 < 25.$$

$|W_0| = 2$. We have $|W_1| = 4$ and therefore $|H[W]| \leq \binom{4}{3} = 4$. Moreover, by (13), $|H_1| \leq 5$ and finally, by (4) and (16),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 15 + 5 + 4 < 25.$$

$|W_0| = 3$. We have $|W_1| = 3$ and therefore $|H[W]| = 1$. Moreover, by (13), $|H_1| \leq 3$ and thus, by (4) and (16),

$$|H| = |H[U \cup W_0]| + |H_1| + |H[W]| \leq 21 + 3 + 1 = 25,$$

with equality only when $|H_1| = 3$ and $|H[U \cup W_0]| = 21$. The latter, by the second part of Theorem 3, is possible only when $H[U \cup W_0]$ is a star (with the center at x). This, in

turn, implies that $F^2 = \emptyset$ (otherwise H would not be P -free) and, further, by (6), that $H_1 = F_1^0$. Hence, $H = \text{Co}(11)$ with x at the center and W_1 as the head.

$n \geq 12$ (inductive step). Fix $v \in W_0$. By the induction hypothesis

$$|H - v| \leq 4 + \binom{n-5}{2}$$

with the equality only when $H - v = \text{Co}(n-1)$. Looking at the structure of $H - v$, if it is a comet, then it must have the center at x and the head must be the unique edge of $H[W]$. Moreover, by (10), $|H(v)| \leq n-5$, with the equality only when $|F^0(v)| = n-9$, $|F^1(v)| = 4$, and $F^2(v) = \emptyset$. Consequently,

$$|H| = |H - v| + |H(v)| \leq 4 + \binom{n-5}{2} + n-5 = |H_n|.$$

and this bound is achieved only when both $H - v = \text{Co}(n-1)$ and $|H(v)| = n-5$. This, however, implies that $H = \text{Co}(n)$ (with the same center and head as in $H - v$.) Theorem 10 is proved.

7 Final comments

It would be interesting to decide if $R(P; r) = r+6$ for all r . If not, then what is the largest r_0 such that $R(P; r) = r+6$ for all $r \leq r_0$? To even partially answer these questions, we would need to compute the higher order Turán numbers $\text{ex}^{(s)}(n; P)$ for $s \geq 3$.

For a related problem of computing $R(C; r)$, it is only known that $R(C; r) = r+5$ for $r = 2, 3$ and $R(C; r) \geq r+5$ for all r ([7]). Gyárfás and Raësi conjecture in [7] that $R(C; r) = r+5$ for all r . To facilitate our approach to this problem one would need to compute $\text{ex}^{(s)}(n; C)$ for $s \geq 2$ and some small values of n . This would probably include calculating the conditional Turán numbers $\text{ex}(n; C|M) = \text{ex}(n; C|P)$ which might be of independent interest. (The fact that the two numbers are the same was derived in [11] from Theorem 9 which was conjectured there.) In [11] we showed that $\text{ex}(n; C|M) \geq \binom{n-2}{2} + 1$ and conjectured that this lower bound is the true value of $\text{ex}(n; C|M)$.

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