Promotion of increasing tableaux: Frames and homomesies

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Submitted: Feb 18, 2017; Accepted: Aug 26, 2017; Published: Sep 8, 2017
Mathematics Subject Classifications: 05E18

Abstract

A key fact about M.-P. Schützenberger's (1972) promotion operator on rectangular standard Young tableaux is that iterating promotion once per entry recovers the original tableau. For tableaux with strictly increasing rows and columns, H. Thomas and A. Yong (2009) introduced a theory of K-jeu de taquin with applications to K-theoretic Schubert calculus. The author (2014) studied a K-promotion operator $\mathcal{P}$ derived from this theory, but observed that this key fact does not generally extend to K-promotion of such increasing tableaux.

Here, we show that the key fact holds for labels on the boundary of the rectangle. That is, for T a rectangular increasing tableau with entries bounded by q, we have $\mathsf{Frame}(\mathcal{P}^q(T)) = \mathsf{Frame}(T)$, where $\mathsf{Frame}(U)$ denotes the restriction of U to its first and last row and column. Using this fact, we obtain a family of homomesy results on the average value of certain statistics over K-promotion orbits, extending a 2-row theorem of J. Bloom, D. Saracino, and the author (2016) to arbitrary rectangular shapes.

Keywords: Increasing tableau; promotion; K-theory; homomesy; frame

1 Introduction

An important application of the theory of standard Young tableaux is to the product structure of the cohomology of Grassmannians. Much attention in the modern Schubert calculus has been devoted to the study of analogous problems in K-theory (see [18, §1] for a partial survey of such work). In particular, H. Thomas and A. Yong [28] gave a K-theoretic Littlewood-Richardson rule by developing a combinatorial theory of *increasing tableaux* as a K-theoretic analogue of the classical theory of standard Young tableaux. Their

Littlewood-Richardson rule and the associated combinatorics have since been extended to the other *minuscule flag varieties* [2, 3, 4] and into *torus-equivariant K-theory* [17, 30].

The theory of increasing tableaux is moreover of independent combinatorial interest. Various enumerative combinatorics results have recently been obtained [9, 16, 19]; as well as applications to the studies of *combinatorial Hopf algebras* [15], *longest increasing subsequences of random words* [29], *plane partitions* [5, 11], and *combinatorial representation theory* [23]. This paper continues the study begun in [16] of the K-promotion operator on increasing tableaux, a K-theoretic analogue of M.-P. Schützenberger's [25] classical promotion operator.

We systematically identify a partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k)$ with its Young diagram in English orientation. For example, we will treat the partition $\lambda = (4, 3, 1, 1)$ as interchangable with the Young diagram $\ydiagram{4,3,1,1}$ consisting of λ_i left-justified boxes in row i (from the top).

An **increasing tableau** of shape λ is a filling of λ by positive integers such that entries strictly increase from left to right across rows and from top to bottom down columns. We write $\mathrm{Inc}^q(\lambda)$ for the set of all increasing tableaux of shape λ with entries bounded above by q. Using the K-theoretic jeu de taquin of [28], one has a K-promotion operator $\mathcal{P}$ on increasing tableaux [16] by analogy with M.-P. Schützenberger's classical promotion for standard Young tableaux [25]. We describe this operator in detail in Section 2.

The operation of (K-)promotion is of particular interest for tableaux of rectangular shapes. For a standard Young tableau T of shape $m \times n$, one has the key fact that $\mathcal{P}^{mn}(T) = T$ (cf. [10]); indeed, one can completely enumerate the orbits by size in this case via the *cyclic sieving phenomenon* [22]. For increasing tableaux, on the other hand, orbits can be much larger than the cardinality q of the alphabet [16, Example 3.10]. In general, no upper bound is known on the cardinality of the K-promotion orbit of an increasing tableau, even of rectangular shape.

Example 1. Consider the increasing tableau

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline 1 & 3 & 4 & 5 & 7 & 8 & 11 & 13 & 14 & 17 \\ \hline 2 & 4 & 7 & 10 & 12 & 13 & 15 & 17 & 19 & 21 \\ \hline 3 & 6 & 9 & 12 & 13 & 14 & 16 & 18 & 21 & 24 \\ \hline 6 & 8 & 11 & 15 & 20 & 22 & 23 & 24 & 25 & 26 \\ \hline \end{array} \in \mathrm{Inc}^{26}(4 \times 10).$$

Although one might naively expect the cardinality of its K-promotion orbit to divide 26 by analogy with the standard Young tableau case, in fact the orbit of T has size $1222 = 26 \cdot 47$.

The **frame** of a rectangular Young diagram $m \times n$ is the set $\mathsf{Frame}(m \times n)$ of all boxes in the first or last row, or in the first or last column. For $T \in \mathrm{Inc}^q(m \times n)$, we write $\mathsf{Frame}(T)$ for the labeling of T restricted to $\mathsf{Frame}(m \times n)$.

Our first main result is the following:

Theorem 2. *Let $T \in \mathrm{Inc}^q(m \times n)$. Then*

$$\mathsf{Frame}(T) = \mathsf{Frame}(\mathcal{P}^q(T)).$$

Example 3. Let T be as in Example 1. Then

$$\mathcal{P}^{26}(T) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline \mathbf{1} & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{7} & \mathbf{8} & \mathbf{11} & \mathbf{13} & \mathbf{14} & \mathbf{17} \\ \hline \mathbf{2} & \mathbf{4} & 6 & 7 & 10 & 12 & 14 & 15 & \mathbf{19} & \mathbf{21} \\ \hline \mathbf{3} & \mathbf{6} & \mathbf{9} & \mathbf{12} & \mathbf{13} & \mathbf{14} & \mathbf{16} & \mathbf{18} & \mathbf{21} & \mathbf{24} \\ \hline \mathbf{6} & \mathbf{8} & \mathbf{11} & \mathbf{15} & \mathbf{20} & \mathbf{22} & \mathbf{23} & \mathbf{24} & \mathbf{25} & \mathbf{26} \\ \hline \end{array},$$

where we have **bolded** all entries that coincide with the corresponding entries of T. The shaded boxes are those of $\mathsf{Frame}(\mathcal{P}^{26}(T))$. Note that in accordance with Theorem 2, all 24 entries of $\mathsf{Frame}(\mathcal{P}^{26}(T))$ are bolded.

Remark 4. Since $\mathsf{Frame}(2 \times n) = 2 \times n$, Theorem 2 in particular recovers the author's previous result [16, Theorem 1.3] that $\mathcal{P}^q(T) = T$ for $T \in \mathrm{Inc}^q(2 \times n)$.

The following was conjectured in work with K. Dilks and J. Striker [5, Conjecture 4.12]:

Conjecture 5. Let $T \in \mathrm{Inc}^q(3 \times n)$. Then $T = \mathcal{P}^q(T)$.

Theorem 2 may be interpreted as evidence toward Conjecture 5, since Theorem 2 shows that T and $\mathcal{P}^q(T)$ have the same entries in at least $2n + 2$ out of $3n$ pairs of corresponding boxes.

Example 6. We note that Theorem 2 might suggest that $\mathsf{Frame}(\mathcal{P}(T))$ is determined by $\mathsf{Frame}(T)$. This is not the case. Consider the increasing tableaux

$$T = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 4 & 6 \\ \hline 5 & 7 & 8 \\ \hline \end{array} \text{ and } U = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline 5 & 7 & 8 \\ \hline \end{array}$$

and note that $\mathsf{Frame}(T) = \mathsf{Frame}(U)$. Nonetheless, we have

$$\mathcal{P}(T) = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 3 & 5 & 7 \\ \hline 4 & 6 & 8 \\ \hline \end{array} \text{ and } \mathcal{P}(U) = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 7 \\ \hline 6 & 7 & 8 \\ \hline \end{array},$$

which have distinct frames.

Remark 7. Recall that the promotion order of the tableau $T \in \mathrm{Inc}^{26}(4 \times 10)$ from Example 1 is a multiple of 26. This fact is an instance of the *resonance* phenomenon for increasing tableaux established in [5, Theorem 2.2]. Originally, the author hoped to use Theorem 2 toward a strengthening of [5, Theorem 2.2]. However, this seems difficult in light of Example 6.

A set U of objects with a weight function $\mathrm{wt} : U \to \mathbb{C}$ and a group action $G \curvearrowright U$ with finite orbits is said to be **homomesic** if every G-orbit $\mathcal{O}$ has the same average weight $\frac{\sum_{x\in\mathcal{O}} \mathrm{wt}(x)}{|\mathcal{O}|}$. This notion was isolated by J. Propp and T. Roby [20] in response to observations of D. Panyushev [14], and has since been found to appear in diverse situations [6, 7, 8, 12, 13, 24, 27].

Using Theorem 2, we obtain our second main result, a family of new homomesies for increasing tableaux. For $T \in \mathrm{Inc}^q(\lambda)$ and S a set of boxes in λ, let $\mathrm{wt}_S(T)$ denote the sum of the entries of T in S.

Theorem 8. *Let S be a subset of $\mathsf{Frame}(m \times n)$ that is fixed under $180°$ rotation. Then $(\mathrm{Inc}^q(m \times n), \mathcal{P}, \mathrm{wt}_S)$ exhibits homomesy with orbit average $\frac{(q+1)|S|}{2}$.*

Remark 9. The case $m = 2$ of Theorem 8 was previously proved by J. Bloom, O. Pechenik and D. Saracino [1, Theorem 6.5] using results from [16].

The analogue of Theorem 8 for (semi)standard Young tableaux was conjectured by J. Propp and T. Roby [21] and proved by J. Bloom, O. Pechenik and D. Saracino [1, Theorem 1.1]. In fact, for (semi)standard Young tableaux, S need not be contained in $\mathsf{Frame}(m \times n)$. However, [1, Example 6.4] shows that for increasing tableaux a generalization of Theorem 8 without the condition $S \subseteq \mathsf{Frame}(m \times n)$ would be false.

2 K-jeu de taquin and frames of increasing tableaux

This section culminates in a proof of Theorem 2. First we recall the K-jeu de taquin of H. Thomas and A. Yong [28], the key ingredient in the operation of K-promotion on increasing tableaux. While K-promotion can be defined without a full development of K-jeu de taquin, we will need K-jeu de taquin in the proof of Theorem 2.

If λ, ν are Young diagrams with $\lambda \subseteq \nu$, the **skew (Young) diagram** ν/λ is the set-theoretic difference $\nu\backslash\lambda$. The Young diagram of a partition λ will also be referred to as a **straight shape**, to distinguish it from this more general notion. The Young diagram for the partition λ is the same as the skew diagram $\lambda/\varnothing$, where $\varnothing$ is the Young diagram of the empty partition; hence the set of skew Young diagrams contains, in particular, all Young diagrams of straight shape. An **increasing tableau** of skew shape ν/λ is a filling of the boxes of ν/λ by positive integers such that entries strictly increase from left to right across rows and from top to bottom down columns. We write $\mathrm{Inc}^q(\nu/\lambda)$ for the set of all increasing tableaux of shape ν/λ with entries bounded above by q.

2.1 K-jeu de taquin

Let $\mathsf{BulletTableaux}(\nu/\lambda)$ denote the set of all fillings of the skew diagram ν/λ by positive integers and the symbol $\bullet$. For each positive integer i, we define as follows an operator swap_i on $\mathsf{BulletTableaux}(\nu/\lambda)$. Let $T \in \mathsf{BulletTableaux}(\nu/\lambda)$ and consider the boxes of T that contain either i or $\bullet$. The set of such boxes decomposes into edge-connected components. On each such component that is a single box, swap_i does nothing. On each nontrivial component, swap_i simultaneously replaces each i by $\bullet$ and each $\bullet$ by i. The resulting element of $\mathsf{BulletTableaux}(\nu/\lambda)$ is $\mathsf{swap}_i(T)$.

Example 10. Consider

$$T = \begin{array}{|c|c|c|c|c|c|}\hline 4 & 7 & 3 & \bullet & 2 & 2 \\ \hline 1 & 2 & \bullet & 2 \\ \cline{1-4} \bullet & 3 \\ \cline{1-2} \end{array} \in \mathsf{BulletTableaux}\left(\begin{array}{|c|c|c|c|c|c|}\hline \; & \; & \; & \; & \; & \; \\ \hline \; & \; & \; & \; \\ \cline{1-4} \; & \; \\ \cline{1-2} \end{array} \right).$$

In computing $\mathsf{swap}_2(T)$, one looks at two connected components. The southwest component is a single box containing $\bullet$ and is unchanged by swap_2. The other component has six boxes. Hence

$$\mathsf{swap}_2(T) = \begin{array}{|c|c|c|c|c|c|}\hline 4 & 7 & 3 & 2 & \bullet & \bullet \\ \hline 1 & \bullet & 2 & \bullet \\ \cline{1-4} \bullet & 3 \\ \cline{1-2} \end{array}.$$

For a box b in a skew Young diagram, we write $\mathsf{b}^{\rightarrow}$ for the box immediately right of b in its row, $\mathsf{b}^{\downarrow}$ for the box immediately below b in its column, etc.

Consider a skew diagram ν/λ. An **inner corner** of ν/λ is a box $\mathsf{b} \in \lambda$ such that $\mathsf{b}^{\rightarrow} \notin \lambda$ and $\mathsf{b}^{\downarrow} \notin \lambda$. For I any nonempty set of inner corners of ν/λ and $T \in \mathrm{Inc}^q(\nu/\lambda)$, let $\mathsf{In}_I(T)$ be the extension of T formed by adding a $\bullet$ to each box of I. Note that $\mathsf{In}_I(T) \in \mathsf{BulletTableaux}(\nu/\theta)$ for some $\theta \subset \lambda$.

An **outer corner** of ν/λ is a box $\mathsf{b} \in \nu/\lambda$ such that $\mathsf{b}^{\rightarrow} \notin \nu/\lambda$ and $\mathsf{b}^{\downarrow} \notin \nu/\lambda$. If every $\bullet$ in $T \in \mathsf{BulletTableaux}(\nu/\lambda)$ is in an outer corner of ν/λ, then we define $\mathsf{Out}^{\bullet}(T)$ to be the filling obtained by deleting every $\bullet$ from T; otherwise $\mathsf{Out}^{\bullet}(T)$ is undefined. Note that if $\mathsf{Out}^{\bullet}(T)$ is defined, then it has shape δ/λ for some $\delta \subseteq \nu$.

Let $T \in \mathrm{Inc}^q(\nu/\lambda)$ and let I be any nonempty set of inner corners of ν/λ. Then the ***K*-jeu de taquin slide** of T at I is the result of the following composition of operations

$$\mathsf{slide}_I(T) := \mathsf{Out}^{\bullet} \circ \mathsf{swap}_q \circ \cdots \circ \mathsf{swap}_2 \circ \mathsf{swap}_1 \circ \mathsf{In}_I(T).$$

Observe that $\mathsf{Out}^{\bullet}$ is always defined in this context and that $\mathsf{slide}_I(T) \in \mathrm{Inc}^q(\delta/\rho)$ for some $\rho \subset \lambda$ and $\delta \subset \nu$.

Iterating this process for successive nonempty sets of inner corners $I_1, I_2, \ldots$, one eventually obtains an increasing tableau $R \in \mathrm{Inc}^q(\kappa)$ of some straight shape κ. Such a tableau is called a **rectification** of T.

Remark 11. Unlike in the classical standard tableau setting, an increasing tableau $T \in \mathrm{Inc}^q(\nu/\lambda)$ may have more than one rectification and these rectifications may moreover have different straight shapes. For an example of this phenomenon, see [28, Example 1.3].

2.2 *K*-promotion

For $T \in \mathsf{BulletTableaux}(\nu/\lambda)$, we define an operation $\mathsf{Rep}_{1\to\bullet}$ that replaces each instance of 1 by $\bullet$, as well as, for each $n \in \mathbb{Z}_{>0}$, an operation $\mathsf{Rep}_{\bullet\to n}$ that replaces each instance of

$\bullet$ by n. Let Decr be the operator that decrements each numerical entry by 1 (and ignores $\bullet$'s).

***K*-promotion** on $\mathrm{Inc}^q(\nu/\lambda) \subset \mathsf{BulletTableaux}(\nu/\lambda)$ is the composition

$$\mathcal{P} := \mathsf{Decr} \circ \mathsf{Rep}_{\bullet\to q+1} \circ \mathsf{swap}_q \circ \cdots \circ \mathsf{swap}_3 \circ \mathsf{swap}_2 \circ \mathsf{Rep}_{1\to\bullet}.$$

It is not hard to see that if $T \in \mathrm{Inc}^q(\nu/\lambda)$, then $\mathcal{P}(T) \in \mathrm{Inc}^q(\nu/\lambda)$, and that moreover this operation coincides with M.-P. Schützenberger's definition of promotion [25] in the case that T is a standard Young tableau of straight shape. For more details, see [16].

Example 12. Let $T = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&4&6\\\hline\end{array} \in \mathrm{Inc}^6(2\times 3)$. Then one computes $\mathcal{P}(T)$ as follows:

$$T = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&4&6\\\hline\end{array} \xrightarrow{\mathsf{Rep}_{1\to\bullet}} \begin{array}{|c|c|c|}\hline \bullet&2&4\\\hline 3&4&6\\\hline\end{array} \xrightarrow{\mathsf{swap}_2} \begin{array}{|c|c|c|}\hline 2&\bullet&4\\\hline 3&4&6\\\hline\end{array} \xrightarrow{\mathsf{swap}_3} \begin{array}{|c|c|c|}\hline 2&\bullet&4\\\hline 3&4&6\\\hline\end{array}$$

$$\xrightarrow{\mathsf{swap}_4} \begin{array}{|c|c|c|}\hline 2&4&\bullet\\\hline 3&\bullet&6\\\hline\end{array} \xrightarrow{\mathsf{swap}_5} \begin{array}{|c|c|c|}\hline 2&4&\bullet\\\hline 3&\bullet&6\\\hline\end{array} \xrightarrow{\mathsf{swap}_6} \begin{array}{|c|c|c|}\hline 2&4&6\\\hline 3&6&\bullet\\\hline\end{array} \xrightarrow{\mathsf{Rep}_{\bullet\to 7}} \begin{array}{|c|c|c|}\hline 2&4&6\\\hline 3&6&7\\\hline\end{array}$$

$$\xrightarrow{\mathsf{Decr}} \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&5&6\\\hline\end{array} = \mathcal{P}(T).$$

2.3 *K*-evacuation and its dual

To prove Theorem 2 on K-promotion, we will need the related notion of (dual) K-evacuation. If T is an increasing tableau of straight shape with $T \in \mathrm{Inc}^q(\lambda)$ for some λ and q, we write $\mathsf{sh}(T) = \lambda$ to denote the shape of T. Write $T_{\leqslant a}$ for the subtableau of T given by deleting all entries greater than a and removing all empty boxes. In analogous fashion, define $T_{<a}, T_{\geqslant a}$, and $T_{>a}$, where $T_{\geqslant a}$ and $T_{>a}$ will generally be of skew shape. Note that the straight-shaped tableau $T \in \mathrm{Inc}^q(\lambda)$ is uniquely determined by the vector of Young diagrams

$$\Big(\mathsf{sh}(T_{\leqslant 0}), \mathsf{sh}(T_{\leqslant 1}), \ldots, \mathsf{sh}(T_{\leqslant q})\Big).$$

An illustration of this correspondence appears in Example 13.

For an increasing tableau $T \in \mathrm{Inc}^q(\lambda)$ of straight shape, we define the ***K*-evacuation** of T to be the tableau $\mathcal{E}(T)$ encoded by the vector

$$\Big(\mathsf{sh}(\mathcal{P}^q(T)_{\leqslant 0}), \mathsf{sh}(\mathcal{P}^{q-1}(T)_{\leqslant 1}), \ldots, \mathsf{sh}(\mathcal{P}^0(T)_{\leqslant q})\Big).$$

Similarly, the **dual K-evacuation** of T is $\mathcal{E}^*(T)$ encoded by the vector

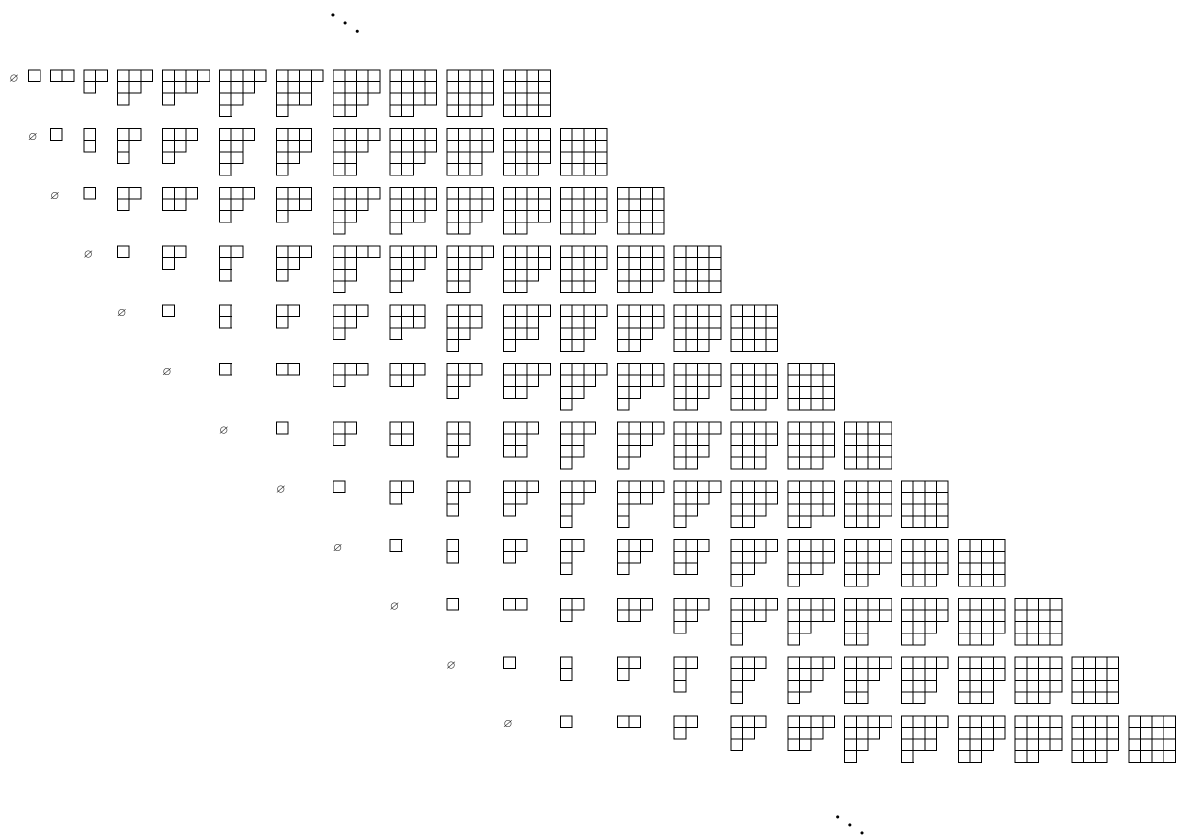
$$\Big(\mathsf{sh}(\mathcal{P}^0(T)_{\leqslant 0}), \mathsf{sh}(\mathcal{P}^{-1}(T)_{\leqslant 1}), \ldots, \mathsf{sh}(\mathcal{P}^{-q}(T)_{\leqslant q})\Big).$$

It is useful to encode all these data in a **K-theoretic growth diagram** as in [28], using ideas that originate in work of S. Fomin (cf. [26, Appendix 1]); the K-theoretic growth diagram for $T \in \mathrm{Inc}^q(\lambda)$ is a semi-infinite 2-dimensional array formed by placing the Young diagram $\mathsf{sh}(\mathcal{P}^j(T)_{\leqslant i})$ in position $(i+j, -j) \in \mathbb{Z} \times \mathbb{Z}$, where $0 \leqslant i \leqslant q$ and $j \in \mathbb{Z}$.

Example 13. Let

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 & 4 & 5 & 8 \\ \hline 4 & 6 & 7 & 9 \\ \hline 6 & 8 & 10 & 11 \\ \hline \end{array}\,.$$

Then the K-theoretic growth diagram of T is



Here the top illustrated row encodes T, the bottom row encodes $\mathcal{P}^{11}(T)$, and the central column encodes $\mathcal{E}(T)$ (which is also $\mathcal{E}^*(\mathcal{P}^{11}(T))$). The $\varnothing$ at the left of the top row is located at the origin $(0,0) \in \mathbb{Z} \times \mathbb{Z}$ in our Cartesian coordinate system.

Using the K-theoretic growth diagram, it is not hard to uncover various relations between the operators under consideration. Together [28, Theorem 4.1] and [16, Lemma 3.1] give the following facts that we will need:

Lemma 14. *The following relations hold for operations on the set* $\mathrm{Inc}^q(\lambda)$ *of straight-shaped increasing tableaux:*

(a) $\mathcal{E}^2 = (\mathcal{E}^*)^2 = \mathrm{id}$*;*

(b) $\mathcal{E}^* \circ \mathcal{E} = \mathcal{P}^q$*;*

(c) $\mathcal{P} \circ \mathcal{E} = \mathcal{E} \circ \mathcal{P}^{-1}$*;*

(d) if $\lambda = m \times n$ *is rectangular, then* $\mathcal{E}^* = \mathsf{rot} \circ \mathcal{E} \circ \mathsf{rot}$*, where* $\mathsf{rot}(T)$ *is given by rotating* T *by* $180°$ *and replacing* i *by* $q+1-i$*.*

2.4 Proof of Theorem 2

Fix $T \in \mathrm{Inc}^q(m \times n)$, a tableau of rectangular straight shape. Let w be the **reading word** of T, given by reading the entries of T by rows from left to right and from bottom to top, i.e. in "reverse Arabic fashion." Let $\mathsf{rot}(w) := w_0 \cdot w \cdot w_0$, where w_0 is the longest element of the symmetric group S_q. Since $\mathsf{rot}(w)$ is obtained from w by reversing the order of the letters of w and then replacing i by $q+1-i$, we see that $\mathsf{rot}(w)$ is the reading word of $\mathsf{rot}(T)$.

Define $w_{\leqslant a}$ to be the subword of w obtained by deleting all letters greater than a, with analogous definitions of $w_{<a}$, $w_{\geqslant a}$, and $w_{>a}$.

Lemma 15. *The tableaux* $\mathsf{rot}(T)$ *and* $\mathcal{E}(T)$ *have the same first row.*

Proof. The reading word of $\mathsf{rot}(T)_{\leqslant a}$ is $\mathsf{rot}(w)_{\leqslant a}$. Hence by [28, Theorem 6.1], the length of the first row of $\mathsf{rot}(T)_{\leqslant a}$ is $\mathsf{LIS}(\mathsf{rot}(w)_{\leqslant a})$, where $\mathsf{LIS}(u)$ denotes the length of the longest strictly increasing subsequence of the word u. By the definition of rot, we have $\mathsf{LIS}(\mathsf{rot}(w)_{\leqslant a}) = \mathsf{LIS}(w_{>n-a})$. But by [28, Theorem 6.1], $\mathsf{LIS}(w_{>n-a})$ is the length of the first row of any K-rectification of $T_{>n-a}$. By definition, the shape of $\mathcal{E}(T)_{\leqslant a}$ is the shape of a particular K-rectification of $T_{>n-a}$. Thus the length of the first row of $\mathcal{E}(T)_{\leqslant a}$ is also the length of the first row of $\mathsf{rot}(T)_{\leqslant a}$. The lemma follows. ☐

Lemma 16. *The tableaux* $\mathsf{rot}(T)$ *and* $\mathcal{E}(T)$ *have the same first column.*

Proof. The proof is the same as for Lemma 15, except that one should replace the use of [28, Theorem 6.1] on the relation between first rows and longest increasing subsequences with the use of the analogous relation between first *columns* and longest *decreasing* subsequences (see [29] or [3, Corollary 6.8]). ☐

The following proposition is of independent interest. It extends [16, Proposition 3.3], which is the special case where $T \in \mathrm{Inc}^q(2 \times n)$.

Proposition 17. *The tableaux* $\mathsf{rot}(T)$ *and* $\mathcal{E}(T)$ *have the same frame.*

Proof. By Lemmas 15 and 16, it remains to show that $\mathsf{rot}(T)$ and $\mathcal{E}(T)$ have the same last row and column.

Let $T' = \mathcal{E}(T)$. Then by Lemmas 15 and 16, $\mathsf{rot}(T')$ and $\mathcal{E}(T')$ have the same first row and column. But by Lemma 14(a), $\mathcal{E}(T') = T$, so $\mathsf{rot}(T')$ and T have the same first row and column. Hence, $\mathsf{rot}(\mathsf{rot}(T'))$ and $\mathsf{rot}(T)$ have the same last row and column. Since $\mathsf{rot}(\mathsf{rot}(T')) = \mathcal{E}(T)$, we are done. $\square$

Clearly, $\mathsf{rot}(T)$ and $\mathsf{rot}(\mathsf{rot}(\mathsf{rot}(T)))$ have the same frame. Since by Lemma 14(d), we have $\mathcal{E}^* = \mathsf{rot} \circ \mathcal{E} \circ \mathsf{rot}$, it thereby follows from Proposition 17 that $\mathsf{rot}(T)$ and $\mathcal{E}^*(T)$ also have the same frame. Thus $\mathcal{E}^*(\mathcal{E}(T))$ has the same frame as $\mathsf{rot}(\mathsf{rot}(T)) = T$. But by Lemma 14(b), $\mathcal{P}^q(T) = \mathcal{E}^*(\mathcal{E}(T))$, so

$$\mathsf{Frame}(T) = \mathsf{Frame}(\mathcal{P}^q(T)).$$

This concludes the proof of Theorem 2. $\square$

3 Homomesy

In this section, we prove a family of new homomesy results for increasing tableaux. We will obtain these by imitating the proof of [1, Theorem 1.1] and using Theorem 2.

For a rectangular tableau $T \in \mathrm{Inc}^q(m \times n)$ and b a box in $\mathsf{Frame}(m \times n)$, let $\mathrm{Dist}(T, \mathsf{b})$ be the *multiset*

$$\mathrm{Dist}(T, \mathsf{b}) := \{\mathrm{wt}_{\{\mathsf{b}\}}(\mathcal{P}^k(T)) : 0 \leqslant k < q\}.$$

Proposition 18. *For a rectangular increasing tableau $T \in \mathrm{Inc}^q(m \times n)$ and a box $\mathsf{b} \in \mathsf{Frame}(m \times n)$,*

$$\mathrm{Dist}(T, \mathsf{b}) = \mathrm{Dist}(\mathcal{E}(T), \mathsf{b}).$$

Proof. This proof is perhaps best understood by following along with the succeeding Example 19.

Consider the K-theoretic growth diagram G for T. A fixed row r of G encodes an increasing tableau R. The row immediately below this encodes $\mathcal{P}(R)$. The column that intersects r at its rightmost Young diagram encodes, by definition, $\mathcal{E}(R)$. The column immediately left of this then encodes $\mathcal{P}(\mathcal{E}(R))$ by Lemma 14(c). Say the **rank** of a Young diagram π in G is the number $\mathrm{rank}(\pi)$ of Young diagrams that are strictly left of π and in its row. Note that the rank is also the number of Young diagrams strictly below π in its column.

Shade each Young diagram in G that contains the box b. For any set of q consecutive rows $\{r_i : 0 < i \leqslant q\}$, we have by Theorem 2 the equality of multisets

$$\mathrm{Dist}(T, \mathsf{b}) = \{\mathrm{rank}(\rho_i) : 0 < i \leqslant q\},$$

where ρ_i is the leftmost shaded Young diagram in row r_i. In the same way, for any set of q consecutive columns $\{c_j : 0 \leqslant j < q\}$, we have

$$\mathrm{Dist}(\mathcal{E}(T), \mathsf{b}) = \{\mathrm{rank}(\gamma_j) : 0 \leqslant j < q\},$$

where γ_j is the bottommost shaded Young diagram in column c_j.

Fix $C \in \mathbb{Z}$. For $1 \leqslant k \leqslant 2q+1$, let d_k denote the diagonal line of slope one through G given by

$$y = x - k + C.$$

(Recall that the Cartesian coordinate system underlying G has the $\varnothing$ representing $\mathsf{sh}(T_{\leqslant 0})$ at the origin, and the remaining Young diagrams for subtableaux of T arrayed along the x-axis; in general, G has $\mathsf{sh}(\mathcal{P}^b(T)_{\leqslant a})$ in position $(a+b, -b)$.) For each diagonal d_k, let δ_k be the smallest shaded Young diagram that lies on d_k. Observe that the Young diagrams δ_k are restricted to $q+1$ consecutive rows $\{r_i : 0 \leqslant i \leqslant q\}$ of G and to $q+1$ consecutive columns $\{c_j : 0 \leqslant j \leqslant q\}$ of G. We have the equalities of multisets

$$\{\operatorname{rank}(\rho_i) : 0 < i \leqslant q\} = \{\operatorname{rank}(\delta_k) : 1 < k \leqslant 2q+1, \operatorname{rank}(\delta_k) = \operatorname{rank}(\delta_{k-1}) - 1\}$$

and

$$\{\operatorname{rank}(\gamma_j) : 0 \leqslant j < q\} = \{\operatorname{rank}(\delta_k) : 1 \leqslant k < 2q+1, \operatorname{rank}(\delta_k) = \operatorname{rank}(\delta_{k+1}) - 1\}.$$

Now construct a lattice path P in the first quadrant of the plane by plotting the points $(k, \operatorname{rank}(\delta_k))$ for $1 \leqslant k \leqslant 2q+1$ and connecting the vertex $(k, \operatorname{rank}(\delta_k))$ to the vertex $(k+1, \operatorname{rank}(\delta_{k+1}))$ by a line segment. Note that

$$\operatorname{rank}(\delta_1) = \operatorname{rank}(\delta_{2q+1})$$

by Theorem 2. Moreover,

$$\operatorname{rank}(\delta_{k+1}) = \operatorname{rank}(\delta_k) \pm 1$$

for all k. Hence for any positive integer h, the number of k in the interval $[2, 2q+1]$ with

$$\operatorname{rank}(\delta_k) = h = \operatorname{rank}(\delta_{k-1}) - 1$$

equals the number of k in the interval $[1, 2q]$ with

$$\operatorname{rank}(\delta_k) = h = \operatorname{rank}(\delta_{k+1}) - 1.$$

This proves that

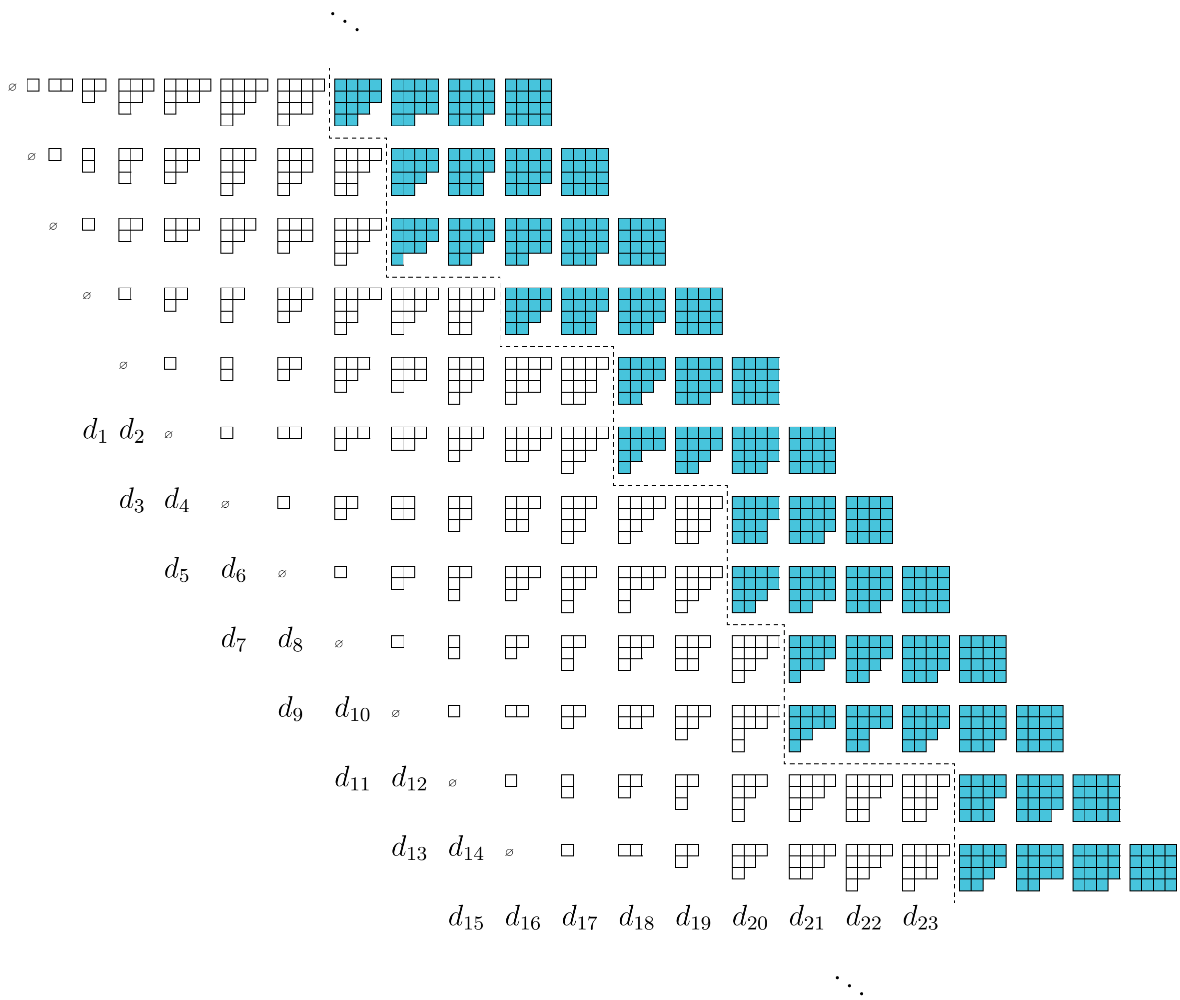
$$\{\operatorname{rank}(\rho_i) : 0 < i \leqslant q\} = \{\operatorname{rank}(\gamma_j) : 0 \leqslant j < q\}$$

and the proposition follows. $\square$

Example 19. Extending Example 13, let b be the shaded box in the increasing tableau

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 4 & 5 \\ \hline 3 & 4 & 5 & 8 \\ \hline 4 & 6 & 7 & 9 \\ \hline 6 & 8 & 10 & 11 \\ \hline \end{array}\,.$$

Then, we shade the K-theoretic growth diagram of T as



where we have also labeled 23 consecutive diagonals. Here, the top illustrated row encodes the tableau T and hence lies on the x-axis of the plane. Thus, our choice of consecutive diagonals comes from setting $C = -7$. (This choice is entirely arbitrary; choosing $C = -7$ is convenient merely because it yields diagonals that fit well on the page.)

We then plot the following lattice path. (Compare this to the dashed piecewise-linear curve separating shaded from unshaded Young diagrams in the K-theoretic growth

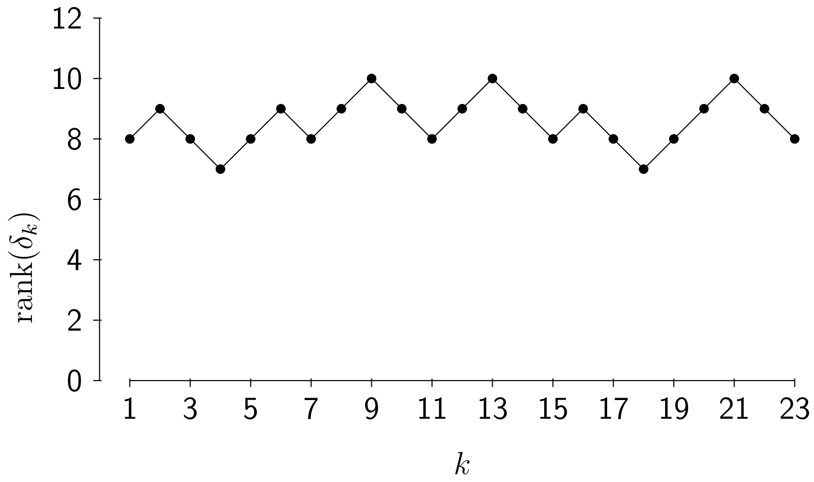


diagram above.)

Now notice that the number of right-hand endpoints of steps equals the number of left-hand endpoints of steps at each height, the final key observation in the proof of Proposition 18.

3.1 Proof of Theorem 8

Consider $\mathsf{b} \in \mathsf{Frame}(m \times n)$ and let $T \in \mathrm{Inc}^q(m \times n)$. For $\mathcal{O}$ the K-promotion orbit of T, we have by Theorem 2 that

$$\frac{\sum_{U \in \mathcal{O}} \mathrm{wt}_{\{\mathsf{b}\}}(U)}{|\mathcal{O}|} = \frac{\sum_{i=0}^{q-1} \mathrm{wt}_{\{\mathsf{b}\}}(\mathcal{P}^i(T))}{q}.$$

By Proposition 18,

$$\frac{\sum_{i=0}^{q-1} \mathrm{wt}_{\{\mathsf{b}\}}(\mathcal{P}^i(T))}{q} = \frac{\sum_{i=0}^{q-1} \mathrm{wt}_{\{\mathsf{b}\}}(\mathcal{P}^i(\mathcal{E}(T)))}{q}.$$

But by Theorem 2 and Lemma 14(c),

$$\frac{\sum_{i=0}^{q-1} \mathrm{wt}_{\{\mathsf{b}\}}(\mathcal{P}^i(\mathcal{E}(T)))}{q} = \frac{\sum_{i=0}^{q-1} \mathrm{wt}_{\{\mathsf{b}\}}(\mathcal{P}^{-i}(\mathcal{E}(T)))}{q} = \frac{\sum_{i=0}^{q-1} \mathrm{wt}_{\{\mathsf{b}\}}(\mathcal{E}(\mathcal{P}^i(T)))}{q}.$$

Finally by Proposition 17,

$$\frac{\sum_{i=0}^{q-1} \mathrm{wt}_{\{\mathsf{b}\}}(\mathcal{E}(\mathcal{P}^i(T)))}{q} = \frac{\sum_{i=0}^{q-1} \mathrm{wt}_{\{\mathsf{b}\}}(\mathsf{rot}(\mathcal{P}^i(T)))}{q} = \frac{\sum_{i=0}^{q-1} \left(q+1-\mathrm{wt}_{\{\mathsf{b}^*\}}(\mathcal{P}^i(T))\right)}{q},$$

where b^* is the image of b under rotating $m \times n$ by $180°$.

Hence, putting these facts together, we have

$$\begin{aligned}\frac{\sum_{U \in \mathcal{O}} \mathrm{wt}_{\{\mathsf{b},\mathsf{b}^*\}}(U)}{|\mathcal{O}|} &= \frac{\sum_{i=0}^{q-1} \left(q+1-\mathrm{wt}_{\{\mathsf{b}^*\}}(\mathcal{P}^i(T))\right)}{q} + \frac{\sum_{i=0}^{q-1} \mathrm{wt}_{\{\mathsf{b}^*\}}(\mathcal{P}^i(T))}{q} \\ &= \frac{\sum_{i=0}^{q-1}(q+1)}{q} = q+1.\end{aligned}$$

Thus, for S any set of boxes in $\mathsf{Frame}(m \times n)$ that is fixed under $180°$ rotation, we have

$$\frac{\sum_{U \in \mathcal{O}} \mathrm{wt}_S(U)}{|\mathcal{O}|} = \frac{(q+1)|S|}{2},$$

as desired. $\square$

Acknowledgements

Thanks to the anonymous referee for their careful reading and helpful suggestions.

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