Internally fair factorizations and internally fair holey factorizations with prescribed regularity

Aras Erzurumluoğlu

Department of Mathematics and Statistics
University of Ottawa
Ottawa, ON, Canada

aerzurum@uottawa.ca

Chris A. Rodger

Department of Mathematics and Statistics
Auburn University
Auburn, AL, United States

rodgec1@auburn.edu

Abstract

Let G be a multipartite multigraph without loops. Then G is said to be *internally fair* if its edges are shared as evenly as possible among all pairs of its partite sets. An *internally fair factorization* of G is an edge-decomposition of G into internally fair regular spanning subgraphs. A *holey factor* of G is a regular subgraph spanning all vertices but one partite set. An *internally fair holey factorization* is an edge-decomposition of G into internally fair holey factors. In this paper, we settle the existence of internally fair (respectively, internally fair holey) factorizations of the complete equipartite multigraph into factors (respectively, holey factors) with prescribed regularity.

1   Introduction

This paper addresses the existence of a general type of balance in edge-colorings of graphs that have an accompanying partition of the vertex set, namely the existence of internally fair edge-colorings. When considering different criteria for balance in edge-colorings, the goal usually is to share edges of each color as evenly as possible among certain substructures. Since such edge-colorings frequently give rise to interesting combinatorial structures which are useful as ingredients in the construction of combinatorial designs, balance in edge-colorings has been of great interest to a wide combinatorics community. Very well-known results on balance in edge-colorings include de Werra's work where he proved (see [11, 12, 13, 14]) that the edges of any bipartite graph G can be colored with k colors such that (i) for each pair of vertices the edges joining these vertices are shared as evenly as possible among the k colors, (ii) at each vertex the incident edges are shared as evenly as possible among the k colors, and (iii) overall, the edges in $E(G)$ are shared as evenly as possible among the k colors. In a more general setting, Hilton [7] proved that each

graph where all degrees are even, has an edge-coloring with k colors (for each k) with the property that (i) each color appears on an even number of incident edges at each vertex, and (ii) the number of times a color appears on incident edges at a vertex v differs from the number of times another color appears on incident edges at vertex v by 0 or 2. (This result was recently extended in [5], where existence results are given for edge-colorings of graphs that satisfy the aforementioned notion of balance together with other notions of balance.)

The most natural graphs to consider in the setting where there is an accompanying partition of the vertex set are the complete multipartite graphs. Indeed, it is on this family of graphs that we focus in this paper. The particular notion of balance that we consider in this paper is defined next.

Let $K(n, p)$ be the complete multipartite graph with p parts and n vertices in each part (note that $K(1, p) = K_p$). Let $\lambda K(n, p)$ denote the λ-fold complete multigraph, in which if two vertices are joined by ϵ edges ($\epsilon \in \mathbb{Z}_2$) in $K(n, p)$ then they are joined by $\lambda \epsilon$ edges in $\lambda K(n, p)$. An edge-coloring of $\lambda K(n, p)$ is said to be *internally fair* if within each color class the edges are shared as evenly as possible among the pairs of parts. As described in [6], if each color class in an edge-coloring of $\lambda K(n, p)$ induces a k-regular spanning subgraph for the same value of k, then the notion of internally fair is identical to the notion of fairness in [6], where an edge-coloring is said to be *fair* if the edges between each pair of parts are shared out as evenly as possible among the color classes.

An *r-factor* of $\lambda K(n, p)$ is an r-regular spanning subgraph of $\lambda K(n, p)$. Suppose that $R_t = \{r_0, r_1, \ldots, r_{t-1}\}$ is a multiset of positive integers. The graph $\lambda K(n, p)$ is said to have an *R_t-factorization* if the edge set of $\lambda K(n, p)$ can be partitioned into subsets $F_0, \ldots, F_{t-1}$ such that for each $i \in \mathbb{Z}_t$, F_i is an r_i-factor of $\lambda K(n, p)$. If $r_i = r$ for all $i \in \mathbb{Z}_t$, then $\lambda K(n, p)$ is said to have an *r-factorization*. A 2-factorization of $\lambda K(n, p)$ where all the factors are connected is said to be a *hamiltonian decomposition*.

It is useful to represent factorizations of $\lambda K(n, p)$ as edge-colorings: two edges are colored the same if and only if they are in the same factor of $\lambda K(n, p)$. In this context, if $\lambda K(n, p)$ has partition $P = \{V_0, \ldots, V_{p-1}\}$ of the vertex set, then an internally fair R_t-factorization of $\lambda K(n, p)$ has the property that for all $p_1, p_2, p_3, p_4 \in \mathbb{Z}_p$ with $p_1 < p_2$, $p_3 < p_4$ and for all $i \in \mathbb{Z}_t$, $||F_i(p_1, p_2)| - |F_i(p_3, p_4)|| \leqslant 1$ where for $0 \leqslant j < k \leqslant p - 1$, $F_i(j, k)$ is the set of edges in the r_i-factor F_i joining a vertex in V_j to a vertex in V_k. In 2002 Leach and Rodger [8] completely settled the existence problem for fair hamiltonian decompositions of $K(n, p)$ (so each factor is not only 2-regular, but also connected) showing that they exist if and only if $n(p - 1)$ is even. More recently the authors [6] settled the existence problem for fair 1-factorizations of $K(n, p)$ showing that they exist if and only if np is even.

It is not difficult to see that a 1-factorization of $K(n, p)$ corresponds to a quasigroup of order np with diagonal holes of size n. The discovery of holey quasigroups has allowed big breakthroughs in combinatorial design theory during the 1980's. Holey quasigroups with certain properties have been used in many instances as substitutes for complete quasigroups when the required complete quasigroups with the imposed set of properties cannot exist. For example, an easy observation shows that there are no idempotent

symmetric quasigroups of even order; however, symmetric quasigroups with holes of size 2 do exist, and these structures are often usefully employed to construct certain designs. Therefore from a design theoretic perspective, it is also of interest to consider partitions of the edges of $\lambda K(n,p)$, each element inducing a factor of $\lambda K(n,p) - V_i$ for some part V_i in the natural partition of $\lambda K(n,p)$. For each part $V_i \in P$, an r-factor of $\lambda K(n,p) - V_i$ is said to be a *holey r-factor* (or *holey parallel class*) of $\lambda K(n,p) - V_i$ (V_i is said to be the *deficiency* of the holey parallel class). A *holey r-factorization* of $\lambda K(n,p)$ is a partition of $E(\lambda K(n,p))$ into holey parallel classes. When $n = 1$, a holey r-factor (or a holey parallel class) is said to be a *near r-factor* (or an *almost parallel class*). The corresponding holey r-factorization is then said to be a *near r-factorization* (or an *almost resolvable r-factorization*). Suppose that $R(f_p) = \{r_{i,j} | r_{i,j} \in \mathbb{Z}^+, i \in \mathbb{Z}_p, j \in \mathbb{Z}_{f_p(i)}\}$ is a multiset of positive integers where $f_p : \mathbb{Z}_p \to \mathbb{Z}^+$ is a function. Then $\lambda K(n,p)$ with partition $P = \{V_0, \ldots, V_{p-1}\}$ of the vertex set is said to have a *holey $R(f_p)$-factorization* if the edge set of $\lambda K(n,p)$ can be partitioned into subsets $F_{i,j}$ where for each $i \in \mathbb{Z}_p$ and for each $j \in \mathbb{Z}_{f_p(i)}$, $F_{i,j}$ is an $r_{i,j}$-factor of $\lambda K(n,p) - V_i$. If $r_{i,j} = r$ for each $i \in \mathbb{Z}_p$ and for each $j \in \mathbb{Z}_{f_p(i)}$, then $\lambda K(n,p)$ is said to have a *holey r-factorization.* To see the design theoretic connection, it is easy to observe that, for example, a holey 1-factorization of $K(n,p)$ is the same as a $(2,1)$-frame of type n^p (for a definition of frames, see for example page 261 of [3]), and a holey hamiltonian decomposition of $K(n,p)$ is the same as an (l,m)-cycle frame of type n^p where $l = n(p-1)$ is the length of each cycle and $m = 1$ is the number of cycles containing each pair of vertices (for a definition of cycle frames, see [2] for example).

A holey $R(f_p)$-factorization is said to be *internally fair* if for all $p_1, p_2, p_3, p_4 \in \mathbb{Z}_p$ with $p_1 < p_2$, $p_3 < p_4$, for all $i \in \mathbb{Z}_p \setminus \{p_1, p_2, p_3, p_4\}$, and for all $j \in \mathbb{Z}_{f_p(i)}$, $||F_{i,j}(p_1, p_2)| - |F_{i,j}(p_3, p_4)|| \leqslant 1$ where for $0 \leqslant m < n \leqslant p - 1$, $F_{i,j}(m,n)$ is the set of edges in the $r_{i,j}$-factor $F_{i,j}$ joining a vertex in V_m to a vertex in V_n. For each holey factor F with deficiency V_i ($i \in \mathbb{Z}_p$), a part V_k in P is said to be *permitted* for F if $k \neq i$. Informally, the internally fair condition requires that the edges in each holey factor F be shared as evenly as possible among the pairs of parts permitted for F in P. In [6] it was shown that if $r_{i,j} = 1$ for all $i \in \mathbb{Z}_p$, and for all $j \in \mathbb{Z}_{f_p(i)}$, then this factorization can also be made fair, namely that $||F_{i_1,j_1}(p_1, p_2)| - |F_{i_2,j_2}(p_1, p_2)|| \leqslant 1$ for all $i_1, i_2 \in \mathbb{Z}_p \setminus \{p_1, p_2\}$, and for all $j_k \in \mathbb{Z}_{f_p(i_k)}$ ($k = 1, 2$). In [6] the authors settled the existence problem for fair holey 1-factorizations of $K(n,p)$ showing that they exist if and only if $n(p-1)$ is even and $p \neq 2$. In [4] a complete characterization was also found for when fair holey hamiltonian decompositions of $K(n,p)$ exist, which simultaneously settles the existence of cycle frames of type n^p for cycles of the longest length.

As one might expect, holey factorizations of $\lambda K(n,p)$ can be represented as edge-colorings as well: an edge-coloring of $\lambda K(n,p)$ is said to be a *holey edge-coloring* if each color class induces a holey factor of $\lambda K(n,p)$. In this context, a holey edge-coloring of $\lambda K(n,p)$ is said to be *internally fair* if the corresponding holey factorization of $\lambda K(n,p)$ is internally fair.

The main aim of this paper is to extend results on fair factorizations and fair holey factorizations of $K(n,p)$ to the much more general setting where

(i) color classes corresponding to the factors (or holey factors) are not required to have the same regularity, and

(ii) $K(n, p)$ is generalized to the λ-fold graph $\lambda K(n, p)$.

Note that in this more general setting the relevant fairness notion to study is the internally fairness notion defined in this paper; the original fairness notion in [6] cannot be satisfied in most cases where the regularities of the color classes are not all the same. Theorem 8 and Theorem 10 prove that such internally fair factorizations and internally fair holey factorizations exist for any prescribed regularity of the factors (or holey factors) as long as the (holey) factors satisfy some obvious necessary conditions. The main idea in proving these theorems is to apply the technique of amalgamations (which is described in the next section) to some carefully constructed edge-colorings of complete multigraphs with high multiplicity. The related edge-colorings are obtained by ordering a certain number of factors of the complete multigraph and coloring consecutive factors with the same color until the associated color class reaches the desired regularity.

Let G be a graph on the vertex set $\mathbb{Z}_g$. The *difference* of the edge $e = \{u, v\}$ in G, named so that $u < v$, is defined as $d = D_g(e) = D_g(u, v) = min\{v-u, g-(v-u)\}$; and then the vertices u and v are said to be *distance d apart*. For any subset $D \subseteq \{1, 2, \ldots, \lfloor g/2 \rfloor\}$, let $G(D, g)$ be the graph with $V(G(D, g)) = \mathbb{Z}_g$ and $E(G(D, g)) = \{\{u, v\} | D_g(u, v) \in D\}$.

The following result, known as Stern-Lenz Lemma, characterizes when the graph $G(D, g)$ has a 1-factorization. (For an in-depth study of Stern-Lenz Lemma, see for example [9]).

Lemma 1. $G(D, g)$ *has a 1-factorization if and only if D contains an element d where $g/gcd(\{d, g\})$ is even.*

2 Amalgamations

A graph H is said to be an *amalgamation* of a graph G if there exists a function ψ from $V(G)$ onto $V(H)$ and a bijection $\psi' : E(G) \to E(H)$ such that $e = \{u, v\} \in E(G)$ if and only if $\psi'(e) = \{\psi(u), \psi(v)\} \in E(H)$. The function ψ is called an amalgamation function. We say that G is a *detachment* of H, where each vertex v of H splits into the vertices of $\psi^{-1}(\{v\})$. An *n-detachment* of H is a detachment in which each vertex of H splits into n vertices. Amalgamating a finite graph G to form the corresponding amalgamated graph H can be thought of as grouping the vertices of G and forming one supervertex for each such group by squashing together the original vertices in the same group. An edge incident with a vertex in G is then incident with the corresponding new vertex in H; in particular an edge joining two vertices from the same group becomes a loop on the corresponding new vertex in H.

In what follows, $G(j)$ denotes the subgraph of G induced by the edges colored j, $d_G(u)$ denotes the degree of vertex u in G, and $m_G(u, v)$ denotes the number of edges between u and v in G. The following theorem was proved in much more generality by Bahmanian and Rodger in [1], but this is sufficient for our purposes.

Theorem 2. *Let H be a k-edge-colored loopless graph. Then there exists an n-detachment G of H with amalgamation function $\psi : V(G) \to V(H)$ such that G satisfies the following properties:*

(i) *$d_G(u) \in \{\lfloor d_H(w)/n \rfloor, \lceil d_H(w)/n \rceil\}$ for each $w \in V(H)$ and each $u \in \psi^{-1}(w)$,*

(ii) *$d_{G(j)}(u) \in \{\lfloor d_{H(j)}(w)/n \rfloor, \lceil d_{H(j)}(w)/n \rceil\}$ for each $w \in V(H)$ and each $u \in \psi^{-1}(w)$ and each $j \in \mathbb{Z}_k$,*

(iii) *$m_G(u,v) \in \{\lfloor m_H(w,z)/n^2 \rfloor, \lceil m_H(w,z)/n^2 \rceil\}$ for every pair of distinct vertices $w, z \in V(H)$, each $u \in \psi^{-1}(w)$ and each $v \in \psi^{-1}(z)$.*

Having the internally fairness notion defined in the Introduction for (holey) edge-colorings of $\lambda K(n,p)$ which result from (holey) factorizations of $\lambda K(n,p)$, we derive the following two corollaries of Theorem 2.

Corollary 3. *Let $R_t = \{r_0, r_1, \ldots, r_{t-1}\}$ be a multiset of positive integers. If there exists an internally fair t-edge-coloring of $\lambda n^2 K_p$ in which each color class i is nr_i-regular (for each $i \in \mathbb{Z}_t$), then there exists an internally fair R_t-factorization of $\lambda K(n,p)$.*

Proof. Using the notation in Theorem 2, let $H = \lambda n^2 K_p$. Then by Theorem 2 there exists an n-detachment G of H such that

(i) the degree of each vertex in G is $\lambda n(p-1)$,

(ii) each color class in G induces a spanning r_i-regular subgraph (since we are given that $d_{H(j)}(w) = nr_i$ for each $w \in V(H)$ and each $j \in \mathbb{Z}_t$), and

(iii) there are exactly λ edges between each pair of vertices u and v of G for which $\psi(u) \neq \psi(v)$, and no edges otherwise.

So, by (i) and (iii) it is clear that $G = \lambda K(n,p)$ with partition $\{\psi^{-1}(w) \mid w \in V(H)\}$ of the vertex set, and by (ii) each color class $i \in \mathbb{Z}_t$ induces an r_i-factor of $\lambda K(n,p)$. This yields an R_t-factorization of $\lambda K(n,p)$. There is a one-to-one correspondence between the edges joining any pair of vertices w and z in H and the edges between the two corresponding parts of $G = \lambda K(n,p)$, one of which contains the vertices in $\psi^{-1}(w)$ and the other contains the vertices in $\psi^{-1}(z)$. Hence, since the edge-coloring of $\lambda n^2 K_p$ is internally fair, the edge-coloring of $\lambda K(n,p)$ is also internally fair. $\qquad\square$

Corollary 4. *Let $R(f_p) = \{r_{i,j} \mid r_{i,j} \in \mathbb{Z}^+, i \in \mathbb{Z}_p, j \in \mathbb{Z}_{f_p(i)}\}$ be a multiset of positive integers where $f_p : \mathbb{Z}_p \to \mathbb{Z}^+$ is a function. If there exists an internally fair holey $\sum_{i=0}^{p-1} f_p(i)$-edge-coloring of $\lambda n^2 K_p$ in which each color class (i,j) with $i \in \mathbb{Z}_p = V(\lambda n^2 K_p)$ and $j \in \mathbb{Z}_{f_p(i)}$ is an $nr_{i,j}$-regular subgraph of $\lambda n^2 K_p - i$ then there exists an internally fair holey $R(f_p)$-factorization of $\lambda K(n,p)$.*

Proof. Using the notation in Theorem 2, let $H = \lambda n^2 K_p$. Then by Theorem 2 there exists an n-detachment G of H such that

(i) the degree of each vertex in G is $\lambda n(p-1)$,

(ii) each color class (i,j) with $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_{f_p(i)}$ in G induces a spanning $r_{i,j}$-regular subgraph of $G - \psi^{-1}(i)$ (since we are given that $d_{H((i,j))}(w) = nr_{i,j}$ for each $i \in V(\lambda n^2 K_p) = \mathbb{Z}_p$, for each $w \in V(H) \setminus \{i\}$ and for $j \in \mathbb{Z}_{f_p(i)}$), and

(iii) there are exactly λ edges between each pair of vertices u and w of G for which $\psi(u) \neq \psi(w)$, and no edges otherwise.

So, by (i) and (iii) it is clear that $G = \lambda K(n,p)$ with partition $\{\psi^{-1}(w) \mid w \in V(H)\}$ of the vertex set, and by (ii) that each color class (i,j) with $i \in \mathbb{Z}_p$ and $j \in \mathbb{Z}_{f_p(i)}$ in G induces a holey $r_{i,j}$-factor of $\lambda K(n,p)$. So this is a holey $R(f_p)$-factorization of $\lambda K(n,p)$. There is a one-to-one correspondence between the edges colored c joining any pair of vertices u and w in H and the edges colored c between the two corresponding parts $\psi^{-1}(u)$ and $\psi^{-1}(w)$ of $G = \lambda K(n,p)$. Hence, since the holey edge-coloring of $\lambda n^2 K_p$ is internally fair, the holey edge-coloring of $\lambda K(n,p)$ is also internally fair. □

3 Coloring Results

In this section we obtain edge-colorings of $\lambda n^2 K_p$ that can then be used in conjunction with Corollary 3 and Corollary 4 to produce Theorem 8 and Theorem 10.

Proposition 5. *Suppose that $\sum_{i=0}^{t-1} r_i = \lambda n(p-1)$, and for each $i \in \mathbb{Z}_t$ r_i is even whenever np is odd. Then there exists an edge-coloring of $\lambda n^2 K_p$ with t colors such that*

(i) the edge-coloring is internally fair, and

(ii) each color class induces an nr_i-regular subgraph.

Proof. Suppose that $\sum_{i=0}^{t-1} r_i = \lambda n(p-1)$, and for each $i \in \mathbb{Z}_t$, r_i is even whenever np is odd. First suppose that p is even. Then clearly, K_p has a 1-factorization consisting of $p-1$ 1-factors $F_0, F_1, \ldots, F_{p-2}$. Let $F = (F_0, F_1, \ldots, F_{\lambda n^2(p-1)-1})$ be a sequence of $\lambda n^2(p-1)$ 1-factors of K_p where $F_i = F_j$ if $i \equiv j$ (modulo $p-1$). For each $i \in \mathbb{Z}_t$, let $s(i) = \sum_{j=0}^{i} nr_j$ and let $G(i)$ be the subgraph induced by the edges in $\bigcup_{k=s(i-1)}^{s(i)-1} F_k$. Color all edges in $G(i)$ with i. Then $\{E(G(i)) \mid i \in \mathbb{Z}_t\}$ is a partition of the edge set of $\lambda n^2 K_p$. Clearly this coloring of $\lambda n^2 K_p$ satisfies condition (ii). To see that it also satisfies condition (i), first note that condition (i) is equivalent to requiring that for each $i \in \mathbb{Z}_t$, color i appears on $\lfloor (nr_i p/2)/(p(p-1)/2) \rfloor = \lfloor nr_i/(p-1) \rfloor$ or $\lceil (nr_i p/2)/(p(p-1)/2) \rceil = \lceil nr_i/(p-1) \rceil$ edges between each pair of vertices. Also note that for each $i \in \mathbb{Z}_t$, $\{F_{i+l} \mid l \in \mathbb{Z}_{p-1}\}$ is a 1-factorization of K_p. So for each $i \in \mathbb{Z}_t$, there are $\lfloor (nr_i(p/2))/(p(p-1)/2) \rfloor = \lfloor nr_i/(p-1) \rfloor$ or $\lceil (nr_i(p/2))/(p(p-1)/2) \rceil = \lceil nr_i/(p-1) \rceil$ edges colored i between each pair of vertices, as required.

Finally, suppose that p is odd. Then either n is even or for each $i \in \mathbb{Z}_t$, r_i is even. K_p has a 2-factorization consisting of $(p-1)/2$ 2-factors $F_0, F_1, \ldots, F_{(p-3)/2}$ (by Petersen's Theorem [10]). Let $F = (F_0, F_1, \ldots, F_{\lambda n^2(p-1)/2-1})$ be a sequence of $\lambda n^2(p-1)/2$ 2-factors

of K_p where $F_i = F_j$ if $i \equiv j$ (modulo $(p-1)/2$). For each $i \in \mathbb{Z}_t$, let $s'(i) = \sum_{j=0}^{i}(n/2)r_j$ and let $G(i)$ be the subgraph induced by the edges in $\bigcup_{k=s'(i-1)}^{s'(i)-1} F_k$. Color all edges in $G(i)$ with i. Then $\{E(G(i)) \mid i \in \mathbb{Z}_t\}$ is a partition of the edge set of $\lambda n^2 K_p$. Clearly this coloring of $\lambda n^2 K_p$ satisfies condition (ii). To see that it also satisfies condition (i), note that for each $i \in \mathbb{Z}_t$, $\{F_{i+l} \mid l \in \mathbb{Z}_{(p-1)/2}\}$ is a 2-factorization of K_p. So for each $i \in \mathbb{Z}_t$, there are $\lfloor (nr_i(p/2))/(p(p-1)/2) \rfloor = \lfloor nr_i/(p-1) \rfloor$ or $\lceil (nr_i(p/2))/(p(p-1)/2) \rceil = \lceil nr_i/(p-1) \rceil$ edges colored i between each pair of vertices, as required. $\qquad \square$

The following result was proved in [6].

Proposition 6. *Suppose $p > 2$ is even. There exists a set $\mathcal{F} = \{F_d \mid d \in \mathbb{Z}_{(p-2)/2}\}$ of $(p-2)/2$ almost resolvable 2-factorizations of $2K_p$ such that for each vertex $v \in V(2K_p)$,*

(i) *the almost parallel classes in $\bigcup_{F \in \mathcal{F}} F$ with deficiency v form a 2-factorization of K_{p-1}.*

The next result follows directly from the constructions in the proof of Proposition 3.3 in [6].

Proposition 7. *Suppose $p > 1$ is odd and $V = \mathbb{Z}_{p-1} \cup \{\infty\}$. There exists a set $\mathcal{F}' = \{F'_d \mid d \in \mathbb{Z}_{(p-3)/2}\}$ of $(p-3)/2$ almost resolvable 2-factorizations of $2K_p$ and an almost resolvable 1-factorization $F'_{(p-3)/2}$ of K_p on the vertex set V such that*

(i) *for each vertex $v \in V$ the almost parallel classes with deficiency v in $\mathcal{F} = \mathcal{F}' \cup \{F'_{(p-3)/2}\}$ are edge-disjoint (therefore each edge in K_{p-1} on the vertex set $V \setminus \{v\}$ occurs in an almost parallel class in $\mathcal{F}$ with deficiency v),*

(ii) *for each $v \neq \infty$ and for each $d \in \mathbb{Z}_{(p-3)/2}$, $F'_d(v)$ is a hamilton cycle on $V \setminus \{v\}$, and*

(iii) *$F'_{(p-3)/2}(\infty) = \{\{i, i+(p-1)/2\} \mid i \in \mathbb{Z}_{(p-1)/2}\}$, and for each $d \in \mathbb{Z}_{(p-3)/2}$ $F'_d(\infty) = \{\{i, i+d+1\} \mid i \in \mathbb{Z}_{p-1}\}$.*

Note that by (iii) $F'_{(p-3)/2}(\infty)$ is a holey 1-factor with deficiency ∞ consisting of edges that are distance $(p-1)/2$ apart, and for each $d \in \mathbb{Z}_{(p-3)/2}$ $F'_d(\infty)$ is a holey 2-factor with deficiency ∞ consisting of edges that are distance $d+1$ apart. $\qquad (*)$

4 Main Results

Throughout the rest of the paper we assume that the partition of the vertices of $K(n,p)$ is naturally the parts of $K(n,p)$.

Theorem 8. *There exists an internally fair R_t-factorization of $\lambda K(n,p)$ if and only if*

(i) *$\sum_{i=0}^{t-1} r_i = \lambda n(p-1)$, and*

(ii) *for each $i \in \mathbb{Z}_t$, r_i is even whenever np is odd.*

Proof. Suppose that there exists an internally fair R_t-factorization of $\lambda K(n,p)$. For each $i \in \mathbb{Z}_t$, $j \in \mathbb{Z}_p$ and each $v \in V_j$, the degree of v in the r_i-factor of $\lambda K(n,p)$ contributes r_i to the degree of v in $\lambda K(n,p)$. The degree of v in $\lambda K(n,p)$ is $\lambda n(p-1)$. So, condition (i) must hold. For each $i \in \mathbb{Z}_t$, an r_i-factor of $\lambda K(n,p)$ contains $npr_i/2$ edges. Hence r_i must be even whenever np is odd, which shows the necessity of condition (ii).

The sufficiency of this set of conditions follows immediately by Proposition 5 and Corollary 3. □

The following lemma gives a set of necessary conditions for an internally fair holey $R(f_p)$-factorization of $\lambda K(n,p)$ to exist.

Lemma 9. *If there exists an internally fair holey $R(f_p)$-factorization of $\lambda K(n,p)$ then*

(i) $p \neq 2$,

(ii) *for each* $i \in \mathbb{Z}_p$, $\sum_{j=0}^{f_p(i)-1} r_{i,j} = \lambda n$, *and*

(iii) *for each* $i \in \mathbb{Z}_p$ *and for each* $j \in \mathbb{Z}_{f_p(i)}$, $r_{i,j}$ *is even whenever* $n(p-1)$ *is odd.*

Proof. Suppose that there exists an internally fair holey $R(f_p)$-factorization of $\lambda K(n,p)$. There are no holey factors in $\lambda K(n,2)$, so $p \neq 2$, which shows the necessity of condition (i). For each $i,k \in \mathbb{Z}_p$, each $v \in V_i$ and each $j \in \mathbb{Z}_{f_p(k)}$, the degree of v in the holey $r_{k,j}$-factor of $\lambda K(n,p)$ contributes zero to the degree of v in $\lambda K(n,p)$ if $k = i$ (since it has deficiency $V_k = V_i$), and is $r_{i,j}$ if $k \neq i$. Hence for each $i \in \mathbb{Z}_p$, if $v \in V_i$ then

$$deg_{\lambda K(n,p)}(v) = \sum_{\substack{0 \leqslant k \leqslant p-1 \\ k \neq i}} \sum_{j=0}^{f_p(k)-1} r_{k,j} = \lambda n(p-1). \qquad (1)$$

Summing (1) over all p we get

$$\sum_{i=0}^{p-1} \sum_{\substack{0 \leqslant k \leqslant p-1 \\ k \neq i}} \sum_{j=0}^{f_p(k)-1} r_{k,j} = p\lambda n(p-1). \qquad (2)$$

In (2), for each $i \in \mathbb{Z}_p$ and for each $j \in \mathbb{Z}_{f_p(i)}$, $r_{i,j}$ appears $p-1$ times in the summation on the left. So we get

$$\sum_{i=0}^{p-1} \sum_{j=0}^{f_p(i)-1} r_{i,j} = p\lambda n. \qquad (3)$$

Finally, from (1) and (3) we see that for each $i \in \mathbb{Z}_p$,

$$\sum_{j=0}^{f_p(i)-1} r_{i,j} = \sum_{i=0}^{p-1} \sum_{j=0}^{f_p(i)-1} r_{i,j} - \sum_{\substack{0 \leqslant k \leqslant p-1 \\ k \neq i}} \sum_{j=0}^{f_p(k)-1} r_{k,j} = \lambda n,$$

which shows the necessity of condition (ii). For each $i \in \mathbb{Z}_p$, for each $j \in \mathbb{Z}_{f_p(i)}$ a holey $r_{i,j}$-factor of $\lambda K(n, p)$ contains $n(p-1)r_i/2$ edges. Hence r_i must be even whenever $n(p-1)$ is odd, which shows the necessity of condition (iii). $\square$

Now we prove that the necessary conditions in Lemma 9 are also sufficient.

Theorem 10. *There exists an internally fair holey $R(f_p)$-factorization of $\lambda K(n, p)$ if and only if*

(i) $p \neq 2$,

(ii) *for each $i \in \mathbb{Z}_p$, $\sum_{j=0}^{f_p(i)-1} r_{i,j} = \lambda n$, and*

(iii) *for each $i \in \mathbb{Z}_p$ and for each $j \in \mathbb{Z}_{f_p(i)}$, $r_{i,j}$ is even whenever $n(p-1)$ is odd.*

Proof. The necessity of these conditions is proved in Lemma 9, so we turn to a proof of the sufficiency. The result is trivial if $p = 1$, so to prove the sufficiency we can assume that $p \geqslant 3$.

First suppose that p is even. Then by condition (iii), if n is odd then $r_{i,j}$ is even for each $i \in \mathbb{Z}_p$ and for each $j \in \mathbb{Z}_{f_p(i)}$. By Proposition 6, there exist $(p-2)/2$ almost resolvable 2-factorizations of $2K_p$, $F_0, \ldots, F_{(p-4)/2}$, on the vertex set $\mathbb{Z}_p$ such that for each $v \in \mathbb{Z}_p$ the almost parallel classes with deficiency v in $F_0, \ldots, F_{(p-4)/2}$ form a 2-factorization of K_{p-1} on the vertex set $\mathbb{Z}_p \setminus \{v\}$. Extend this list for all $i \geqslant 0$ by defining $F_i = F_j$ if and only if $i \equiv j$ (modulo $(p-2)/2$). From this extended list, form a sequence $T = (T_0, \ldots, T_{(\lambda n^2 p/2)-1})$ of $\lambda n^2 p/2$ almost parallel classes of $\lambda n^2 K_p$ where for $i \in \mathbb{Z}_p$ and for $k \in \mathbb{Z}_{\lambda n^2/2}$, T_{i+kp} is the almost parallel class in F_k with deficiency i. For each $i \in \mathbb{Z}_p$ and for each $l \in \mathbb{Z}_{f_p(i)}$, let $s(i, l) = \sum_{j=0}^{l} (n/2)r_{i,j}$, and let $G(i, l)$ be the subgraph of $\lambda n^2 K_p$ induced by the edges in $\bigcup_{k=s(i,l-1)}^{s(i,l)-1} T_{i+kp}$. Color all edges in $G(i, l)$ with (i, l). Then for each $i \in \mathbb{Z}_p$, and for each $l \in \mathbb{Z}_{f_p(i)}$, color class (i, l) consists of the $r_{i,l}n(p-1)/2$ edges of $r_{i,l}n/2$ almost parallel classes which all have the same deficiency $i \in V(\lambda n^2 K_p)$, and hence each color class is an $r_{i,l}n$-regular subgraph of $\lambda n^2 K_p - i$. Furthermore, by condition (i) of Proposition 6, for each $i \in V(2K_p)$ the almost parallel classes with deficiency i in any set of $(p-2)/2$ consecutive almost resolvable 2-factorizations of $2K_p$ from the extended list $F_0, \ldots, F_{\lambda n^2/2-1}$ form a 2-factorization of $K_p - i$, which ensures that each color class induces a balanced multigraph (a multigraph G is said to be *balanced* if the multiplicity between any two pairs of vertices differs by at most 1). Hence, the $r_{i,l}n(p-1)/2$ edges in each color class are shared out evenly among the $(p-1)(p-2)/2$ pairs of vertices in $\mathbb{Z}_p \setminus \{i\}$, so between any such pair of vertices there are $\lceil (r_{i,l}n(p-1)/2)/((p-1)(p-2)/2) \rceil = \lceil r_{i,l}n/(p-2) \rceil$ or $\lfloor (r_{i,l}n(p-1)/2)/((p-1)(p-2)/2) \rfloor = \lfloor r_{i,l}n/(p-2) \rfloor$ edges colored (i, l). Therefore this $(\sum_{i=0}^{p-1} \sum_{l=0}^{f_p(l)-1} r_{i,l})$-edge-coloring is internally fair. By Corollary 4 we conclude that there exists an internally fair holey $R(f_p)$-factorization of $\lambda K(n, p)$.

Next suppose that p is odd. It is important to note that $(p-1)/gcd(\{d, p-1\})$ is even when d is odd. So by Lemma 1,

$G(D, p-1)$ has a 1-factorization whenever D contains an odd number. $(**)$

Throughout this case we will focus on the ordering $F'_0, F'_1, \ldots, F'_{(p-5)/2}, F'_{(p-3)/2}$ and the reverse ordering $F'_{(p-3)/2}, F'_{(p-5)/2}, \ldots, F'_1, F'_0$ of the $(p-3)/2$ almost resolvable 2-factorizations of $2K_p$ in $\mathcal{F}'$ and one almost resolvable 1-factorization $F'_{(p-3)/2}$ of K_p from Proposition 7 on the vertex set $V = \mathbb{Z}_{p-1} \cup \{\infty\}$. In order to reduce redundancies we limit our attention to the first ordering and make a few remarks (noting that similar observations hold for the reverse ordering as well).

Notice that by $(**)$,

for $0 \leqslant i \leqslant (p-5)/2$ there exists a 1-factorization of the graph consisting of the edges in $E(F'_i(\infty)) \cup E(F'_{i+1}(\infty))$ on the vertex set $\mathbb{Z}_{p-1}$ (into four 1-factors if $i < (p-5)/2$ and three 1-factors if $i = (p-5)/2$), and notice that $F'_{(p-3)/2}(\infty)$ is a 1-factor on the vertex set $\mathbb{Z}_{p-1}$. $\hspace{2cm} (***)$

Also notice that by (ii) of Proposition 7,

for each $v \neq \infty$ and for each $d \in \mathbb{Z}_{(p-3)/2}$, $F'_d(v)$ is a hamilton cycle on $V \setminus \{v\}$, and since $|V \setminus \{v\}|$ is even there exists a 1-factorization of each of these hamilton cycles into two 1-factors on the vertex set $V \setminus \{v\}$, each 1-factor being formed by taking every second edge on the hamilton cycle. $\hspace{2cm} (****)$

As in the previous case where p is even, the plan is to proceed by finding a certain coloring of $\lambda n^2 K_p$ and then applying Corollary 4 to get the required internally fair holey $R(f_p)$-factorization of $\lambda K(n,p)$. Let $\lambda n^2 = s(p-2) + t$ where $0 \leqslant t \leqslant p - 3$. We will consider eight cases depending on both the value of t (modulo 4) and whether $p \equiv 1$ or 3 (modulo 4). A rough indication as to why these subcases arise is as follows. The critical problem to address is ensuring that each color class induces a balanced multigraph. For each vertex v we select s near 1-factorizations of K_p with deficiency v, and are then left to find t more near 1-factors with deficiency v. This is easy to do in all cases except when $v = \infty$, where we regularly need to join pairs of near 2-factors in order to construct the near 1-factors, as described in $(***)$. Dealing with this gives rise to the eight situations which are considered to be two cases, each with four subcases. Let $O = O_0, \ldots, O_{(p-3)/2}$ be an arbitrary ordering of the list $L = F'_0, F'_1, \ldots, F'_{(p-5)/2}, F'_{(p-3)/2}$. Extend the list O to a list $O^* = O^*_0, O^*_1, \ldots, O^*_{s(p-1)/2-1}$ of $s(p-3)/2$ almost resolvable 2-factorizations and s almost resolvable 1-factorizations by defining $O^*_i = O_j$ if $i \equiv j$ (modulo $(p-1)/2$). Additionally, define $t/2$ (if t is even) or $(t+1)/2$ (if t is odd) almost resolvable factorizations as follows: If t is even, then $O^*_{s(p-1)/2} = F'_0, O^*_{s(p-1)/2+1} = F'_1, \ldots, O^*_{s(p-1)/2+t/2-1} = F'_{t/2-1}$. If t is odd, then $O^*_{s(p-1)/2} = F'_{(p-3)/2}, O^*_{s(p-1)/2+1} = F'_{(p-5)/2}, \ldots, O^*_{s(p-1)/2+(t-3)/2} = F'_{(p-t)/2}, O^*_{s(p-1)/2+(t-1)/2} = F'_{(p-t-2)/2}$. For each $v \in \mathbb{Z}_{p-1} \cup \{\infty\}$ form a sequence $T(v) = (T(v,0), T(v,1), \ldots, T(v,q))$ (where $q = (s(p-1)+t)/2 - 1$ if t is even, and $q = (s(p-1)+t-1)/2$ if t is odd) of almost parallel classes of K_p, by defining $T(v,k)$ for each $v \in \mathbb{Z}_{p-1} \cup \{\infty\}$ and for each $k \in \mathbb{Z}_{q+1}$ to be the almost parallel class (a near 1-factor or a near 2-factor) in O^*_k with deficiency v.

Now for each $v \in \mathbb{Z}_{p-1} \cup \{\infty\}$ form a longer new sequence $W(v) = (W(v)_0, W(v)_1, \ldots)$ from $T(v)$ as follows. First suppose $v \in \mathbb{Z}_{p-1}$. For each $v \in \mathbb{Z}_{p-1}$ and for $0 \leqslant k \leqslant (s(p-1)+t)/2 - 1$, if $O^*_k \neq F'_{(p-3)/2}$ then replace $T(v,k)$ in $T(v)$ with the two

near 1-factors with deficiency v, say $W_v'(k)$ and $W_v''(k)$, formed from $T(v, k)$ (using $(\ast \ast \ast)$). (Note that for $v \in \mathbb{Z}_{p-1}$ if $O_k^* = F_{(p-3)/2}'$, then $T(v, k)$ is already a near 1-factor with deficiency v, which we also name with $W_v(k)$.) We now use two specific orderings of the list L to define $W(\infty)$ depending on the parity of t.

Case 1: Suppose t is even (modulo 4).

Let O be the ordering $F_0', F_1', \ldots, F_{(p-5)/2}', F_{(p-3)/2}'$.

Subcase 1.1: Suppose $t \equiv 0$ (modulo 4) and $p \equiv 1$ (modulo 4).

For each k such that k is even (modulo $(p-1)/2$) and $0 \leqslant k \leqslant (p-9)/2$ (modulo $(p-1)/2$), replace $T(\infty, k)$ and $T(\infty, k+1)$ with the four near 1-factors with deficiency ∞, say $W_\infty'(k, k+1), W_\infty''(k, k+1), W_\infty'''(k, k+1)$ and $W_\infty''''(k, k+1)$, formed from $T(\infty, k)$ and $T(\infty, k+1)$ using $(\ast \ast \ast)$. For $k \equiv (p-5)/2$ (modulo $(p-1)/2$) replace $T(\infty, k)$ and $T(\infty, k+1)$ with the three near 1-factors with deficiency ∞, say $W_\infty'(k, k+1), W_\infty''(k, k+1)$ and $W_\infty'''(k, k+1)$, formed from $T(\infty, k)$ and $T(\infty, k+1)$ using $(\ast \ast \ast)$. See Example 11.

Subcase 1.2: Suppose $t \equiv 2$ (modulo 4) and $p \equiv 1$ (modulo 4).

For each $k \equiv 0$ (modulo $(p-1)/2$) replace $T(\infty, k)$ with the two near 1-factors with deficiency ∞, say $W_\infty'(k)$ and $W_\infty''(k)$, formed from $T(\infty, k)$ (notice that for each $k \equiv 0$ (modulo $(p-1)/2$) $T(\infty, k) = \{\{i, i+1\} \mid i \in \mathbb{Z}_{p-1}\}$, which can be factored into two near 1-factors on $\mathbb{Z}_{p-1}$ using $(\ast \ast)$). For each k such that k is odd (modulo $(p-1)/2$) and $1 \leqslant k \leqslant (p-7)/2$ (modulo $(p-1)/2$), replace $T(\infty, k)$ and $T(\infty, k+1)$ with the four near 1-factors with deficiency ∞, say $W_\infty'(k, k+1), W_\infty''(k, k+1), W_\infty'''(k, k+1)$ and $W_\infty''''(k, k+1)$, formed from $T(\infty, k)$ and $T(\infty, k+1)$ using $(\ast \ast \ast)$. Note that for $k \equiv (p-3)/2$ (modulo $(p-1)/2$) $T(\infty, k)$ is already a near 1-factor with deficiency ∞, which we also name with $W_\infty(k)$. See Example 12.

Subcase 1.3: Suppose $t \equiv 0$ (modulo 4) and $p \equiv 3$ (modulo 4).

For each k such that k is even (modulo $(p-1)/2$) and $0 \leqslant k \leqslant (p-7)/2$ (modulo $(p-1)/2$), replace $T(\infty, k)$ and $T(\infty, k+1)$ with the four near 1-factors with deficiency ∞, say $W_\infty'(k, k+1), W_\infty''(k, k+1), W_\infty'''(k, k+1)$ and $W_\infty''''(k, k+1)$, formed from $T(\infty, k)$ and $T(\infty, k+1)$ using $(\ast \ast \ast)$. (Note that for $k \equiv (p-3)/2$ (modulo $(p-1)/2$) $T(\infty, k)$ is already a near 1-factor with deficiency ∞, which we also name with $W_\infty(k)$.) See Example 13.

Subcase 1.4: Suppose $t \equiv 2$ (modulo 4) and $p \equiv 3$ (modulo 4).

For each $k \equiv 0$ (modulo $(p-1)/2$) replace $T(\infty, k)$ with the two near 1-factors with deficiency ∞, say $W_\infty'(k)$ and $W_\infty''(k)$, formed from $T(\infty, k)$ (notice that for each $k \equiv 0$ (modulo $(p-1)/2$) $T(\infty, k) = \{\{i, i+1\} \mid i \in \mathbb{Z}_{p-1}\}$, which can be factored into two near 1-factors on $\mathbb{Z}_{p-1}$ using $(\ast \ast)$). For each k such that k is odd (modulo $(p-1)/2$) and $1 \leqslant k \leqslant (p-9)/2$ (modulo $(p-1)/2$), replace $T(\infty, k)$ and $T(\infty, k+1)$ with the four near 1-factors with deficiency ∞, say $W_\infty'(k, k+1), W_\infty''(k, k+1), W_\infty'''(k, k+1)$ and $W_\infty''''(k, k+1)$, formed from $T(\infty, k)$ and $T(\infty, k+1)$ using $(\ast \ast \ast)$. Finally, for $k \equiv (p-5)/2$ (modulo $(p-1)/2$) replace $T(\infty, k)$ and $T(\infty, k+1)$ with the three near 1-factors with

deficiency ∞, say $W'_\infty(k, k+1), W''_\infty(k, k+1)$ and $W'''_\infty(k, k+1)$, formed from $T(\infty, k)$ and $T(\infty, k+1)$ using $(**)$. See Example 14.

Case 2: Suppose t is odd (modulo 4).

Let O be the ordering $F'_{(p-3)/2}, F'_{(p-5)/2}, \dots, F'_1, F'_0$.

Subcase 2.1: Suppose $t \equiv 1$ (modulo 4) and $p \equiv 1$ (modulo 4).

Note that for $k \equiv 0$ (modulo $(p-1)/2$) $T(\infty, k)$ is already a near 1-factor with deficiency ∞, which we now name with $W_\infty(k)$. For each k such that k is odd (modulo $(p-1)/2$) and $1 \leqslant k \leqslant (p-7)/2$ (modulo $(p-1)/2$), replace $T(\infty, k)$ and $T(\infty, k+1)$ with the four near 1-factors with deficiency ∞, say $W'_\infty(k, k+1), W''_\infty(k, k+1), W'''_\infty(k, k+1)$ and $W''''_\infty(k, k+1)$, formed from $T(\infty, k)$ and $T(\infty, k+1)$ using $(**)$. Finally for each $k \equiv (p-3)/2$ (modulo $(p-1)/2$) replace $T(\infty, k)$ with the two near 1-factors with deficiency ∞, say $W'_\infty(k)$ and $W''_\infty(k)$, formed from $T(\infty, k)$ (notice that for each $k \equiv (p-3)/2$ (modulo $(p-1)/2$) $T(\infty, k) = \{\{i, i+1\} \mid i \in \mathbb{Z}_{p-1}\}$, which can be factored into two near 1-factors on $\mathbb{Z}_{p-1}$ using $(**)$). See Example 15.

Subcase 2.2: Suppose $t \equiv 3$ (modulo 4) and $p \equiv 1$ (modulo 4).

For $k \equiv 0$ (modulo $(p-1)/2$) replace $T(\infty, k)$ and $T(\infty, k+1)$ with the three near 1-factors with deficiency ∞, say $W'_\infty(k, k+1), W''_\infty(k, k+1)$ and $W'''_\infty(k, k+1)$, formed from $T(\infty, k)$ and $T(\infty, k+1)$ using $(**)$. For each k such that k is even (modulo $(p-1)/2$) and $2 \leqslant k \leqslant (p-5)/2$ (modulo $(p-1)/2$), replace $T(\infty, k)$ and $T(\infty, k+1)$ with the four near 1-factors with deficiency ∞, say $W'_\infty(k, k+1), W''_\infty(k, k+1), W'''_\infty(k, k+1)$ and $W''''_\infty(k, k+1)$, formed from $T(\infty, k)$ and $T(\infty, k+1)$ using $(**)$. See Example 16.

Subcase 2.3: Suppose $t \equiv 1$ (modulo 4) and $p \equiv 3$ (modulo 4).

Note that for $k \equiv 0$ (modulo $(p-1)/2$) $T(\infty, k)$ is already a near 1-factor with deficiency ∞, which we now name with $W_\infty(k)$. For each k such that k is odd (modulo $(p-1)/2$) and $1 \leqslant k \leqslant (p-5)/2$ (modulo $(p-1)/2$), replace $T(\infty, k)$ and $T(\infty, k+1)$ with the four near 1-factors with deficiency ∞, say $W'_\infty(k, k+1), W''_\infty(k, k+1), W'''_\infty(k, k+1)$ and $W''''_\infty(k, k+1)$, formed from $T(\infty, k)$ and $T(\infty, k+1)$ using $(**)$. See Example 17.

Subcase 2.4: Suppose $t \equiv 3$ (modulo 4) and $p \equiv 3$ (modulo 4).

For $k \equiv 0$ (modulo $(p-1)/2$) replace $T(\infty, k)$ and $T(\infty, k+1)$ with the three near 1-factors with deficiency ∞, say $W'_\infty(k, k+1), W''_\infty(k, k+1)$ and $W'''_\infty(k, k+1)$, formed from $T(\infty, k)$ and $T(\infty, k+1)$ using $(**)$. For each k such that k is even (modulo $(p-1)/2$) and $2 \leqslant k \leqslant (p-7)/2$ (modulo $(p-1)/2$), replace $T(\infty, k)$ and $T(\infty, k+1)$ with the four near 1-factors with deficiency ∞, say $W'_\infty(k, k+1), W''_\infty(k, k+1), W'''_\infty(k, k+1)$ and $W''''_\infty(k, k+1)$, formed from $T(\infty, k)$ and $T(\infty, k+1)$ using $(**)$. Finally for each $k \equiv (p-3)/2$ (modulo $(p-1)/2$) replace $T(\infty, k)$ with the two near 1-factors with deficiency ∞, say $W'_\infty(k)$ and $W''_\infty(k)$, formed from $T(\infty, k)$ (notice that for each $k \equiv (p-3)/2$ (modulo $(p-1)/2$) $T(\infty, k) = \{\{i, i+1\} \mid i \in \mathbb{Z}_{p-1}\}$, which can be factored into two near 1-factors on $\mathbb{Z}_{p-1}$ using $(**)$). See Example 18.

$W(v)$ has been defined for all $v \in \mathbb{Z}_{p-1} \cup \{\infty\}$ in all possible cases. This is used to define the color classes as follows. For each $v \in \mathbb{Z}_{p-1} \cup \{\infty\}$ and for each $l \in \mathbb{Z}_{f_p(v)}$, let $s(v,l) = \sum_{j=0}^{l} r_{v,j} n$, and let $G(v,l)$ be the subgraph of $\lambda n^2 K_p$ induced by the edges in $\bigcup_{z=s(v,l-1)}^{s(v,l)-1} W(v)_z$. Color all edges in $G(v,l)$ with (v,l). Then for each $v \in \mathbb{Z}_{p-1} \cup \{\infty\}$, and for each $l \in \mathbb{Z}_{f_p(v)}$, color class (v,l) consists of the $r_{v,l} n (p-1)/2$ edges of $r_{v,l} n$ near 1-factors which all have the same deficiency v, and hence each color class is an $r_{v,l} n$-regular subgraph of $\lambda n^2 K_p - v$. Furthermore, by condition (i) of Proposition 7, each color class (v,l) is balanced. Hence, the $r_{v,l} n (p-1)/2$ edges in each color class are shared out evenly among the $(p-1)(p-2)/2$ pairs of vertices in $\mathbb{Z}_{p-1} \cup \{\infty\} \setminus \{v\}$, so between any such pair of vertices there are $\lceil (r_{v,l} n (p-1)/2)/((p-1)(p-2)/2) \rceil = \lceil r_{v,l} n/(p-2) \rceil$ or $\lfloor (r_{v,l} n (p-1)/2)/((p-1)(p-2)/2) \rfloor = \lfloor r_{v,l} n/(p-2) \rfloor$ edges colored (v,l). Therefore this edge-coloring is internally fair. By Corollary 4 we conclude that there exists an internally fair holey $R(f_p)$-factorization of $\lambda K(n,p)$. $\qquad\square$

As was suggested in the last proof, examples are now provided of the orderings that are a crucial part of the proof of Theorem 10.

Example 11. If $p = 13$ and $\lambda n^2 = 26$, then the ordering of the 26 near 1-factors with deficiency $i \in \mathbb{Z}_{p-1}$ is as follows:
$W(i) = (W(i)_0, W(i)_1, \ldots, W(i)_{25}) = (W_i'(0), W_i''(0), W_i'(1), W_i''(1), W_i'(2), W_i''(2),$
$W_i'(3), W_i''(3), W_i'(4), W_i''(4), W_i(5), W_i'(0), W_i''(0), W_i'(1), W_i''(1), W_i'(2), W_i''(2), W_i'(3),$
$W_i''(3), W_i'(4), W_i''(4), W_i(5), W_i'(0), W_i''(0), W_i'(1), W_i''(1)).$
The ordering of the 26 near 1-factors with deficiency ∞ is as follows:
$W(\infty) = (W(\infty)_0, W(\infty)_1, \ldots, W(\infty)_{25}) = (W_\infty'(0,1), W_\infty''(0,1), W_\infty'''(0,1),$
$W_\infty''''(0,1), W_\infty'(2,3), W_\infty''(2,3), W_\infty'''(2,3), W_\infty''''(2,3), W_\infty'(4,5), W_\infty''(4,5), W_\infty'''(4,5),$
$W_\infty'(0,1), W_\infty''(0,1), W_\infty'''(0,1), W_\infty''''(0,1), W_\infty'(2,3), W_\infty''(2,3), W_\infty'''(2,3), W_\infty''''(2,3),$
$W_\infty'(4,5), W_\infty''(4,5), W_\infty'''(4,5), W_\infty'(0,1), W_\infty''(0,1), W_\infty'''(0,1), W_\infty''''(0,1)).$

Example 12. If $p = 13$ and $\lambda n^2 = 24$, then the ordering of the 24 near 1-factors with deficiency $i \in \mathbb{Z}_{p-1}$ is as follows:
$W(i) = (W(i)_0, W(i)_1, \ldots, W(i)_{23}) = (W_i'(0), W_i''(0), W_i'(1), W_i''(1), W_i'(2), W_i''(2),$
$W_i'(3), W_i''(3), W_i'(4), W_i''(4), W_i(5), W_i'(0), W_i''(0), W_i'(1), W_i''(1), W_i'(2), W_i''(2), W_i'(3),$
$W_i''(3), W_i'(4), W_i''(4), W_i(5), W_i'(0), W_i''(0)).$
The ordering of the 24 near 1-factors with deficiency ∞ is as follows:
$W(\infty) = (W(\infty)_0, W(\infty)_1, \ldots, W(\infty)_{23}) = (W_\infty'(0), W_\infty''(0), W_\infty'(1,2), W_\infty''(1,2),$
$W_\infty'''(1,2), W_\infty''''(1,2), W_\infty'(3,4), W_\infty''(3,4), W_\infty'''(3,4), W_\infty''''(3,4), W_\infty(5), W_\infty'(0), W_\infty''(0),$
$W_\infty'(1,2), W_\infty''(1,2), W_\infty'''(1,2), W_\infty''''(1,2), W_\infty'(3,4), W_\infty''(3,4), W_\infty'''(3,4), W_\infty''''(3,4),$
$W_\infty(5), W_\infty'(0), W_\infty''(0)).$

Example 13. If $p = 11$ and $\lambda n^2 = 22$, then the ordering of the 22 near 1-factors with deficiency $i \in \mathbb{Z}_{p-1}$ is as follows:
$W(i) = (W(i)_0, W(i)_1, \ldots, W(i)_{21}) = (W_i'(0), W_i''(0), W_i'(1), W_i''(1), W_i'(2), W_i''(2),$
$W_i'(3), W_i''(3), W_i(4), W_i'(0), W_i''(0), W_i'(1), W_i''(1), W_i'(2), W_i''(2), W_i'(3), W_i''(3), W_i(4),$
$W_i'(0), W_i''(0), W_i'(1), W_i''(1)).$

The ordering of the 22 near 1-factors with deficiency ∞ is as follows:
$$W(\infty) = (W(\infty)_0, W(\infty)_1, \ldots, W(\infty)_{21}) = (W'_\infty(0,1), W''_\infty(0,1), W'''_\infty(0,1),$$
$W''''_\infty(0,1), W'_\infty(2,3), W''_\infty(2,3), W'''_\infty(2,3), W''''_\infty(2,3), W_\infty(4), W'_\infty(0,1), W''_\infty(0,1),$
$W'''_\infty(0,1), W''''_\infty(0,1), W'_\infty(2,3), W''_\infty(2,3), W'''_\infty(2,3), W''''_\infty(2,3), W_\infty(4), W'_\infty(0,1),$
$W''_\infty(0,1), W'''_\infty(0,1), W''''_\infty(0,1))$.

Example 14. If $p = 11$ and $\lambda n^2 = 20$, then the ordering of the 20 near 1-factors with deficiency $i \in \mathbb{Z}_{p-1}$ is as follows:
$$W(i) = (W(i)_0, W(i)_1, \ldots, W(i)_{19}) = (W'_i(0), W''_i(0), W'_i(1), W''_i(1), W'_i(2), W''_i(2),$$
$W'_i(3), W''_i(3), W_i(4), W'_i(0), W''_i(0), W'_i(1), W''_i(1), W'_i(2), W''_i(2), W'_i(3), W''_i(3), W_i(4),$
$W'_i(0), W''_i(0))$.

The ordering of the 20 near 1-factors with deficiency ∞ is as follows:
$$W(\infty) = (W(\infty)_0, W(\infty)_1, \ldots, W(\infty)_{19}) = (W'_\infty(0), W''_\infty(0), W'_\infty(1,2), W''_\infty(1,2),$$
$W'''_\infty(1,2), W''''_\infty(1,2), W'_\infty(3,4), W''_\infty(3,4), W'''_\infty(3,4), W'_\infty(0), W''_\infty(0), W'_\infty(1,2), W''_\infty(1,2),$
$W'''_\infty(1,2), W''''_\infty(1,2), W'_\infty(3,4), W''_\infty(3,4), W'''_\infty(3,4), W'_\infty(0), W''_\infty(0))$.

Example 15. If $p = 13$ and $\lambda n^2 = 27$, then the ordering of the 27 near 1-factors with deficiency $i \in \mathbb{Z}_{p-1}$ is as follows:
$$W(i) = (W(i)_0, W(i)_1, \ldots, W(i)_{26}) = (W_i(0), W'_i(1), W''_i(1), W'_i(2), W''_i(2), W'_i(3),$$
$W''_i(3), W'_i(4), W''_i(4), W'_i(5), W''_i(5), W_i(0), W'_i(1), W''_i(1), W'_i(2), W''_i(2), W'_i(3), W''_i(3),$
$W'_i(4), W''_i(4), W'_i(5), W''_i(5), W_i(0), W'_i(1), W''_i(1), W'_i(2), W''_i(2))$.

The ordering of the 27 near 1-factors with deficiency ∞ is as follows:
$$W(\infty) = (W(\infty)_0, W(\infty)_1, \ldots, W(\infty)_{26}) = (W_\infty(0), W'_\infty(1,2), W''_\infty(1,2), W'''_\infty(1,2),$$
$W''''_\infty(1,2), W'_\infty(3,4), W''_\infty(3,4), W'''_\infty(3,4), W''''_\infty(3,4), W'_\infty(5), W''_\infty(5), W_\infty(0), W'_\infty(1,2),$
$W''_\infty(1,2), W'''_\infty(1,2), W''''_\infty(1,2), W'_\infty(3,4), W''_\infty(3,4), W'''_\infty(3,4), W''''_\infty(3,4), W'_\infty(5),$
$W''_\infty(5), W_\infty(0), W'_\infty(1,2), W''_\infty(1,2), W'''_\infty(1,2), W''''_\infty(1,2))$.

Example 16. If $p = 13$ and $\lambda n^2 = 25$, then the ordering of the 25 near 1-factors with deficiency $i \in \mathbb{Z}_{p-1}$ is as follows:
$$W(i) = (W(i)_0, W(i)_1, \ldots, W(i)_{24}) = (W_i(0), W'_i(1), W''_i(1), W'_i(2), W''_i(2), W'_i(3),$$
$W''_i(3), W'_i(4), W''_i(4), W'_i(5), W''_i(5), W_i(0), W'_i(1), W''_i(1), W'_i(2), W''_i(2), W'_i(3), W''_i(3),$
$W'_i(4), W''_i(4), W'_i(5), W''_i(5), W_i(0), W'_i(1), W''_i(1))$.

The ordering of the 25 near 1-factors with deficiency ∞ is as follows:
$$W(\infty) = (W(\infty)_0, W(\infty)_1, \ldots, W(\infty)_{24}) = (W'_\infty(0,1), W''_\infty(0,1), W'''_\infty(0,1), W'_\infty(2,3),$$
$W''_\infty(2,3), W'''_\infty(2,3), W''''_\infty(2,3), W'_\infty(4,5), W''_\infty(4,5), W'''_\infty(4,5), W''''_\infty(4,5), W'_\infty(0,1),$
$W''_\infty(0,1), W'''_\infty(0,1), W'_\infty(2,3), W''_\infty(2,3), W'''_\infty(2,3), W''''_\infty(2,3), W'_\infty(4,5), W''_\infty(4,5),$
$W'''_\infty(4,5), W''''_\infty(4,5), W'_\infty(0,1), W''_\infty(0,1), W'''_\infty(0,1))$.

Example 17. If $p = 11$ and $\lambda n^2 = 23$, then the ordering of the 23 near 1-factors with deficiency $i \in \mathbb{Z}_{p-1}$ is as follows:
$$W(i) = (W(i)_0, W(i)_1, \ldots, W(i)_{22}) = (W_i(0), W'_i(1), W''_i(1), W'_i(2), W''_i(2), W'_i(3),$$
$W''_i(3), W'_i(4), W''_i(4), W_i(0), W'_i(1), W''_i(1), W'_i(2), W''_i(2), W'_i(3), W''_i(3), W'_i(4), W''_i(4),$
$W_i(0), W'_i(1), W''_i(1), W'_i(2), W''_i(2))$.

The ordering of the 23 near 1-factors with deficiency ∞ is as follows:

$W(\infty) = (W(\infty)_0, W(\infty)_1, \ldots, W(\infty)_{22}) = (W_\infty(0), W_\infty'(1,2), W_\infty''(1,2), W_\infty'''(1,2),$
$W_\infty''''(1,2), W_\infty'(3,4), W_\infty''(3,4), W_\infty'''(3,4), W_\infty''''(3,4), W_\infty(0), W_\infty'(1,2), W_\infty''(1,2),$
$W_\infty'''(1,2), W_\infty''''(1,2), W_\infty'(3,4), W_\infty''(3,4), W_\infty'''(3,4), W_\infty''''(3,4), W_\infty(0), W_\infty'(1,2),$
$W_\infty''(1,2), W_\infty'''(1,2), W_\infty''''(1,2)).$

Example 18. If $p = 11$ and $\lambda n^2 = 21$, then the ordering of the 21 near 1-factors with deficiency $i \in \mathbb{Z}_{p-1}$ is as follows:

$W(i) = (W(i)_0, W(i)_1, \ldots, W(i)_{20}) = (W_i(0), W_i'(1), W_i''(1), W_i'(2), W_i''(2), W_i'(3),$
$W_i''(3), W_i'(4), W_i''(4), W_i(0), W_i'(1), W_i''(1), W_i'(2), W_i''(2), W_i'(3), W_i''(3), W_i'(4), W_i''(4),$
$W_i(0), W_i'(1), W_i''(1)).$

The ordering of the 21 near 1-factors with deficiency ∞ is as follows:

$W(\infty) = (W(\infty)_0, W(\infty)_1, \ldots, W(\infty)_{20}) = (W_\infty'(0,1), W_\infty''(0,1), W_\infty'''(0,1), W_\infty'(2,3),$
$W_\infty''(2,3), W_\infty'''(2,3), W_\infty''''(2,3), W_\infty'(4), W_\infty''(4), W_\infty'(0,1), W_\infty''(0,1), W_\infty'''(0,1), W_\infty'(2,3),$
$W_\infty''(2,3), W_\infty'''(2,3), W_\infty''''(2,3), W_\infty'(4), W_\infty''(4), W_\infty'(0,1), W_\infty''(0,1), W_\infty'''(0,1)).$

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