Note on the union-closed sets conjecture

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Abstract

The union-closed sets conjecture states that if a finite family of sets $\mathcal{A} \neq \{\varnothing\}$ is union-closed, then there is an element which belongs to at least half the sets in $\mathcal{A}$. In 2001, D. Reimer showed that the average set size of a union-closed family, $\mathcal{A}$, is at least $\frac{1}{2}\log_2|\mathcal{A}|$. In order to do so, he showed that all union-closed families satisfy a particular condition, which in turn implies the preceding bound. Here, answering a question raised in the context of T. Gowers' polymath project on the union-closed sets conjecture, we show that Reimer's condition alone is not enough to imply that there is an element in at least half the sets.

1 Introduction

Given the set $[n] = \{1, \ldots, n\}$ and a family $\mathcal{A} \subseteq 2^{[n]}$ we say $\mathcal{A}$ is union-closed if for $A, B \in \mathcal{A}$ we have $A \cup B \in \mathcal{A}$. The union-closed sets conjecture, due to P. Frankl [3], states that if $\mathcal{A} \subseteq 2^{[n]}$ is union-closed and $\mathcal{A} \neq \{\varnothing\}$ then there is some element of $[n]$ which belongs to at least half the sets in $\mathcal{A}$. One method of approaching this conjecture is to look at the average frequency of an element or, equivalently, the average set size. The following theorem of D. Reimer [2] was thus motivated by and can be shown to follow from, the union-closed sets conjecture.

Theorem 1. *If $\mathcal{A} \subseteq 2^{[n]}$ and is union-closed, then*

$$\frac{\sum_{A\in\mathcal{A}}|A|}{|\mathcal{A}|} \geqslant \frac{\log_2|\mathcal{A}|}{2} \tag{1}$$

We will say that $\mathcal{F} \subseteq 2^{[n]}$ is a filter if $G \supseteq F$ and $F \in \mathcal{F}$ implies $G \in \mathcal{F}$. Additionally, for $A \subseteq B \subseteq [n]$ define $[A, B] \coloneqq \{C : A \subseteq C \subseteq B\}$. In order to prove Theorem 1, Reimer introduced the following criterion for a family $\mathcal{A} \subseteq 2^{[n]}$:

Definition 2. We say $\mathcal{A} \subseteq 2^{[n]}$ satisfies *Condition 1* if there exists a filter $\mathcal{F} \subseteq 2^{[n]}$ and a bijection $A \mapsto F_A$ from $\mathcal{A}$ to $\mathcal{F}$ satisfying:

1. $A \subseteq F_A$ for all $A \in \mathcal{A}$
2. For distinct $A, B \in \mathcal{A}$ we have $[A, F_A] \cap [B, F_B] = \varnothing$.

Reimer's proof of Theorem 1 consists of two steps. He first shows that every union-closed family $\mathcal{A}$ satisfies Condition 1. He then shows that Condition 1 implies Theorem 1.

In 2016, T. Gowers began a polymath project focused on the union-closed sets conjecture. In the comments on the initial post I. Balla first proposed:

Conjecture 3. Assume $\mathcal{A} \subseteq 2^{[n]}$ satisfies Condition 1. Then there is an element $x \in [n]$ in at least half the sets of $\mathcal{A}$.

Gowers reiterates Conjecture 3 in his second post focused on strengthenings of the union-closed sets conjecture. In the comments there is a discussion of a possible counterexample, and it is stated that all families with ground set at most 5 and a random sampling of families with ground set at most 12 have been confirmed to satisfy Conjecture 3 [1].

The conjecture is certainly a natural one to consider: Reimer's work has been perhaps the most successful in finding a way to exploit the union-closed hypothesis, and one would like to decide whether more can be gotten from his approach, particularly as finding a way into the problem has proved so difficult. The polymath project's lack of recent progress, after much initial enthusiasm, may be considered further evidence of this difficulty.

As Reimer showed that all union-closed families satisfy Condition 1, Conjecture 3 is clearly a strengthening of the union-closed sets conjecture. The purpose of this note is to show that Conjecture 3 is false.

2 Counterexample

In what follows we will always have $\mathcal{A}$ and $\mathcal{F}$ as in Definition 2.

Note 4. *An equivalent way of stating the second part of Condition 1 is that at least one of $A \setminus F_B$ or $B \setminus F_A$ is non-empty.*

We will use the following notation:

- $\mathcal{A}_x = \{A \in \mathcal{A} : x \in A\}$
- A_0 is the set for which $F_{A_0} = [n]$
- A_i is the set for which $F_{A_i} = [n] \setminus \{i\}$ for $i \in [n]$
- $B_{i,j}$ is the set for which $F_{B_{i,j}} = [n] \setminus \{i, j\}$ for $i \neq j \in [n]$

Before giving the counterexample we will briefly describe how we found it and indicate why no smaller example is possible. The following observation was our starting point.

Fact 5. *Assume $\mathcal{A}$ satisfies Condition 1. If every set in $\mathcal{F}$ has size at least $n-1$ then there is an element in at least half of the sets of $\mathcal{A}$.*

Proof. Without loss of generality assume $\mathcal{F} = \{[n]\} \cup \{[n] \setminus \{i\} : i \in [k]\}$. Hence, $|\mathcal{F}| = |\mathcal{A}| = k+1$. By Note 4 we know that $[k] \subseteq A_0$. Now we will view each A_i as a vertex labelled i in a digraph, D, on vertex set $[k]$, with (i, j) an edge exactly when $i \in A_j$. Again by Note 4 we know that D must contain a tournament (an orientation of K_n). Furthermore, the number of sets containing i is simply the out-degree of i plus 1 (since $i \in A_0$). Since D has k vertices and contains a tournament it has maximum out-degree at least $\frac{k-1}{2}$. Hence there is always an element in at least $\frac{k+1}{2}$ members of $\mathcal{A}$. $\square$

Assume n is the smallest integer such that there is a counterexample to Conjecture 3 on $[n]$ and $\mathcal{A}$ is such a counterexample with corresponding filter $\mathcal{F}$. We will use the following three observations to show that $n \geqslant 8$, and then exhibit a counterexample when $n = 8$.

Note 6. *$\mathcal{F}$ must contain all sets of size $n-1$.*

Proof. Suppose instead that the elements of $\mathcal{F}$ of size $n-1$ are $[n] \setminus \{i\}$ for $i \in [k]$ with $k < n$. Since $\mathcal{F}$ is a filter we have $\{k+1, \ldots, n\} \subseteq F$ for all $F \in \mathcal{F}$, implying that the condition in Note 4 is not affected if we replace each $X \in \mathcal{A} \cup \mathcal{F}$ by $X \setminus \{k+1, \ldots, n\}$. This produces a counterexample on a smaller set, contradicting the minimality of n. $\square$

Restrict $\mathcal{A}$ to $\mathcal{A}' \coloneqq \{A_i\}_{i=0}^{n}$. If n is even then there exists $x \in [n]$ with $|\mathcal{A}'_x| \geqslant \frac{n+2}{2}$. Hence we need at least two sets in $\mathcal{A} \setminus \mathcal{A}'$. (If n is odd similar reasoning shows that there must be at least three sets in $\mathcal{A} \setminus \mathcal{A}'$.)

In our example we will take n to be even and $\mathcal{F}$ to consist of $[n] \setminus \{1, 2\}$ and $[n] \setminus \{3, 4\}$ along with all sets of size at least $n-1$. Thus $|\mathcal{F}| = |\mathcal{A}| = n+3$, $A_0 = [n]$, and we want to arrange that $|\mathcal{A}_x| \leqslant \frac{n}{2} + 1$ for all $x \in [n]$. We will use the same digraph, D, as in the proof of Fact 5 (with (i, j) an edge if and only if $i \in A_j$). Note that the $B_{i,j}$'s do not directly affect the digraph.

Note 7. *The sum of the out-degrees in D must be at least $\frac{n^2-n}{2} + 2$.*

Proof. Recall that by Note 4 D must contain a tournament. Additionally, by Note 4 if $B_{i,j} \in \mathcal{A}$ then $i \in A_j$ and $j \in A_i$. Thus we must have at least one additional out-degree for every $B_{i,j}$. $\square$

Note 8. *$B_{1,2}$ and $B_{3,4}$ must both be non-empty,*

Proof. Without loss of generality $1 \in B_{3,4}$, since $B_{1,2}$ and $B_{3,4}$ must satisfy the condition of Note 4. Additionally, if $B_{1,2} = \varnothing$ then to satisfy Note 4 all other sets in $\mathcal{A}$ must contain 1 or 2. However, A_0 contains both 1 and 2, so one of 1 or 2 must appear in at least half the sets, contradicting that $\mathcal{A}$ is a counterexample. $\square$

By Note 8 we must have at least 2 vertices with out-degree no more than $\frac{n}{2} - 1$. The remaining out-degrees must still be no more than $\frac{n}{2}$. Combining this with Note 7 we have the inequality $2(\frac{n}{2} - 1) + (n-2)(\frac{n}{2}) \geqslant \frac{n^2-n}{2} + 2$, i.e. $n \geqslant 8$. (If $|\mathcal{A} \setminus \mathcal{A}'| > 2$ then we get even more "extra" degrees and the lower bound on n increases.) When n is odd similar consideration gives $n \geqslant 13$; so, since our example does indeed use $n = 8$ it is of the smallest possible size.

Counterexample 9. Here we will take our universe to be $[8]$. Our family $\mathcal{A}$ consists of the following 11 sets:

- $A_0 = [8]$
- $A_1 = \{2, 4, 6, 7, 8\}$
- $A_2 = \{1, 3, 5, 8\}$
- $A_3 = \{1, 4, 7, 8\}$
- $A_4 = \{2, 3, 5, 6\}$
- $A_5 = \{1, 3, 7\}$
- $A_6 = \{2, 3, 5\}$
- $A_7 = \{2, 4, 6\}$
- $A_8 = \{4, 5, 6, 7\}$
- $B_{1,2} = \{8\}$
- $B_{3,4} = \{1\}$

We (or our computers) can easily check that the requirement in Note 4 is satisfied (a short maple script can be found at http://sites.math.rutgers.edu/~ajr224/counterexample-check.txt.) and that each element appears in at most 5 sets. The bijection between $\mathcal{A}$ and $\mathcal{F}$ is given explicitly in the appendix.

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References

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[2] David Reimer. An average set size theorem. *Combinatorics, Probability, and Computing*, 12(1):89–93, 2003.

[3] Ivan Rival, editor. *Graphs and Order*, volume 147 of *Nato Science Series C:* Springer, 1st edition, 1985.

3 Appendix

Below is the complete bijection between $\mathcal{A}$ and $\mathcal{F}$ in our counterexample. All the sets are represented by their indicator vectors:

$$A_0 \mapsto F_{A_0} \quad \begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix} \mapsto \begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix} \qquad A_1 \mapsto F_{A_1} \quad \begin{bmatrix}0\\1\\0\\1\\0\\1\\1\\1\end{bmatrix} \mapsto \begin{bmatrix}0\\1\\1\\1\\1\\1\\1\\1\end{bmatrix} \qquad A_2 \mapsto F_{A_2} \quad \begin{bmatrix}1\\0\\1\\0\\1\\0\\0\\1\end{bmatrix} \mapsto \begin{bmatrix}1\\0\\1\\1\\1\\1\\1\\1\end{bmatrix} \qquad A_3 \mapsto F_{A_3} \quad \begin{bmatrix}1\\0\\0\\1\\0\\0\\1\\1\end{bmatrix} \mapsto \begin{bmatrix}1\\1\\0\\1\\1\\1\\1\\1\end{bmatrix} \qquad A_4 \mapsto F_{A_4} \quad \begin{bmatrix}0\\1\\1\\0\\1\\1\\0\\0\end{bmatrix} \mapsto \begin{bmatrix}1\\1\\1\\0\\1\\1\\1\\1\end{bmatrix}$$

$$A_5 \mapsto F_{A_5} \quad \begin{bmatrix}1\\0\\1\\0\\0\\0\\1\\0\end{bmatrix} \mapsto \begin{bmatrix}1\\1\\1\\1\\0\\1\\1\\1\end{bmatrix} \qquad A_6 \mapsto F_{A_6} \quad \begin{bmatrix}0\\1\\1\\0\\1\\0\\0\\0\end{bmatrix} \mapsto \begin{bmatrix}1\\1\\1\\1\\1\\0\\1\\1\end{bmatrix} \qquad A_7 \mapsto F_{A_7} \quad \begin{bmatrix}0\\1\\0\\1\\0\\1\\0\\0\end{bmatrix} \mapsto \begin{bmatrix}1\\1\\1\\1\\1\\1\\0\\1\end{bmatrix} \qquad A_8 \mapsto F_{A_8} \quad \begin{bmatrix}0\\0\\0\\1\\1\\1\\1\\0\end{bmatrix} \mapsto \begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\0\end{bmatrix} \qquad B_{1,2} \mapsto F_{B_{1,2}} \quad \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{bmatrix} \mapsto \begin{bmatrix}0\\0\\1\\1\\1\\1\\1\\1\end{bmatrix} \qquad B_{3,4} \mapsto F_{B_{3,4}} \quad \begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix} \mapsto \begin{bmatrix}1\\1\\0\\0\\1\\1\\1\\1\end{bmatrix}$$