Majority choosability of digraphs

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Abstract

A *majority coloring* of a digraph is a coloring of its vertices such that for each vertex v, at most half of the out-neighbors of v have the same color as v. A digraph D is *majority k-choosable* if for any assignment of lists of colors of size k to the vertices there is a majority coloring of D from these lists. We prove that every digraph is majority 4-choosable. This gives a positive answer to a question posed recently by Kreutzer, Oum, Seymour, van der Zypen, and Wood (2017). We obtain this result as a consequence of a more general theorem, in which majority condition is profitably extended. For instance, the theorem implies also that every digraph has a coloring from arbitrary lists of size three, in which at most 2/3 of the out-neighbors of any vertex share its color. This solves another problem posed by the same authors, and supports an intriguing conjecture stating that every digraph is majority 3-colorable.

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1 Introduction

Let D be a directed graph. Let $d^+(v)$ denote the number of out-neighbors of vertex v. A coloring c of the vertices of D is called *majority coloring* if for every vertex v the number of its out-neighbors in color $c(v)$ is at most $\frac{1}{2}d^+(v)$. This concept was introduced recently by van der Zypen [7], in connection to neural networks, and studied by Kreutzer, Oum, Seymour, van der Zypen, and Wood in [3]. It is proved there, among other results, that every digraph is majority 4-colorable. The proof is very simple: first, notice that every digraph with no directed cycles is majority 2-colorable (just apply greedy coloring), next, split the edges of a given digraph into two acyclic digraphs, and take the product of the two colorings. It is conjectured in [3] that actually three colors are sufficient for majority coloring of any digraph. This would be best possible since a majority coloring of an odd directed cycle must be a proper coloring of the underlying undirected graph.

Another interesting problem posed in [3] concerns *list* version of the majority coloring. Suppose that each vertex v of a digraph D is assigned with a list of colors $L(v)$. Then D is majority colorable *from* these lists if there is a majority coloring c of D with $c(v) \in L(v)$. If D is majority colorable from any lists of size k, then we say that D is *majority k-choosable*. The authors of [3] asked if there is a finite number k such that every digraph is majority k-choosable. We answer this question in the affirmative by proving a more general theorem which implies that actually every digraph is majority 4-choosable. As another consequence we infer that every digraph is 3-choosable so that at most $\frac{2}{3}d^+(v)$ of the out-neighbors of any vertex v have the same color as v. This solves another problem posed in [3], and extends a result of Seymour from [5], asserting that every digraph has 3-coloring in which at least one out-neighbor of each vertex (of positive out-degree) is colored differently.

There are many variants of majority coloring that may be studied in a variety of contexts (see [3]). Perhaps our approach might be useful in some of these situations. We shall discuss briefly these issues in the final section.

2 The results

Our main result reads as follows.

Theorem 1. *Let D be a directed graph. Suppose that each vertex v is assigned with a list $L(v)$ of four colors. Suppose further that for each vertex v, each color x in $L(v)$ is assigned a real number $r_v(x)$, the* rank *of color x in $L(v)$. Assume that for every vertex v, the color ranks $r_v(x)$ satisfy the following condition:*

$$\sum_{x \in L(v)} r_v(x) \geqslant 2d^+(v). \qquad (*)$$

Then there is a vertex coloring of D from lists $L(v)$ satisfying the following constraint: If x is a color assigned to v, then the number of out-neighbors of v in color x is at most $r_v(x)$.

Proof. Let us remark first that we do not impose any restrictions on color ranks, except condition $(*)$. These ranks may be positive, negative, or zero. If $r_v(x) = 0$ and v is colored with x, then to satisfy the assertion of the theorem, none of the out-neighbors of v may be colored with x. If $r_v(x)$ is strictly negative, then actually v cannot be colored with x at all (no set may have negative cardinality).

The proof goes by induction on the number of vertices in D. It is not hard to check that the theorem is true for one-vertex digraph. Indeed, by condition $(*)$, at least one color rank in the list must be non-negative, and we may use it to color the only vertex in the digraph. So, let $n \geqslant 2$, and assume that the assertion of the theorem is true for all digraphs with at most $n-1$ vertices. Let D be a digraph on n vertices satisfying the assumptions of the theorem, and let v be any vertex of D. Consider a new digraph D' obtained by deleting vertex v with color ranks modified as follows. Let a and b be the two colors with highest ranks, $r_v(a)$ and $r_v(b)$, in the list $L(v)$. For each in-neighbor u of vertex v, decrease the ranks $r_u(a)$ and $r_u(b)$ by one, provided these colors are contained in the list $L(u)$. All the remaining color ranks in these or other lists are left unchanged.

We claim that digraph D' with modified color ranks still satisfies condition $(*)$. Indeed, for each in-neighbor u of v, the left hand side of $(*)$ decreased by at most two, while the right-hand side of $(*)$ decreased by exactly two (since the out-degree $d^+(u)$ decreased by exactly one). So, by the inductive assumption there is a coloring of D' satisfying the assertion of the theorem.

We now extend this coloring to the deleted vertex v in the following way. First notice that

$$r_v(a) + r_v(b) \geqslant d^+(v). \tag{1}$$

Indeed, by the maximality of ranks of colors a and b in the list $L(v)$, the inequality $r_v(a) + r_v(b) < d^+(v)$ would imply $\sum_{x \in L(v)} r_v(x) < 2d^+(v)$, contrary to the assumption. Let n_a and n_b denote the number of out-neighbors of v colored with colors a and b, respectively. Obviously, $n_a + n_b \leqslant d^+(v)$. Hence, by (1), at least one of the following inequalities must be satisfied:

$$r_v(a) \geqslant n_a \quad \text{or} \quad r_v(b) \geqslant n_b. \tag{2}$$

We chose a color whose rank satisfies one of these inequalities, and assign that color to v.

We claim that the extended coloring satisfies the assertion of the theorem. First, let u be arbitrary in-neighbor of v. Let x denote the color assigned to u in coloring of D'. If x is one of the colors a or b, then the number of out-neighbors of u in D' colored with x is at most $r_u(x) - 1$, by inductive assumption. Thus, their number in D after coloring the vertex v is still bounded by $r_u(x)$. If x is neither equal to a nor to b, then the constraint is fulfilled even more. If u is an arbitrary out-neighbor of v, or any other vertex of D', then the corresponding constraint holds by induction, since out-neighborhoods and color ranks for such vertices remained unchanged in D'. Finally, for the vertex v we have chosen color a or b so that the corresponding inequality of (2) is satisfied. This completes the proof. □

We obtain now easily the aforementioned consequences for majority choosability of digraphs.

Corollary 2. *Every digraph is majority 4-choosable.*

Proof. Put $r_v(x) = \frac{1}{2}d^+(v)$ for each vertex v and for every color x from its list $L(v)$, and apply the theorem. $\square$

Corollary 3. *Let D be a digraph with color lists of size three assigned to the vertices. Then there is a coloring from these lists such that for each vertex v, at most $\frac{2}{3}$ of its out-neighbors have the color of v.*

Proof. Let $0 < \varepsilon < \frac{1}{3}$ be a real number. Let v be a vertex in D, and let $L(v)$ denote its list with three colors. For each color x in $L(v)$ assign the rank $r_v(x) = \frac{2}{3}d^+(v) + \varepsilon$. Now, add a new fictitious color f with the rank $r_v(f) = -3\varepsilon$ to each list $L(v)$. The assertion of the corollary follows now directly from Theorem 1. $\square$

3 Discussion

There are many variants of majority coloring that may be studied for various combinatorial structures (see [3]). For instance, in a multi-color version considered in [3], the majority constraint is strengthened to $\frac{1}{k}d^+(v)$, where $k \geqslant 2$ is a fixed integer. It is easy to see that k colors are sufficient for acyclic digraphs, and thus k^2 colors suffice for arbitrary digraph (by taking the product of the colorings). It is conjectured in [3] that $k+1$ colors are actually enough. As noted by David Wood (personal communication), the proof of Theorem 1 can be easily extended to the multi-color setting, however, it only gives the same quadratic bound in the list version of the problem.

The situation looks much simpler for undirected graphs. An old result of Lovász [4] asserts that every graph is majority 2-colorable, and more generally, it is k-colorable so that at most $1/k$ neighbors of each vertex share its color, for every $k \geqslant 2$. The proof is very simple: just take a coloring that minimizes the total number of monochromatic edges. The same argument works in the list version, and after slight modification it gives a result similar to Theorem 1 (with color ranks in each list summing up to at least the degree of the corresponding vertex).

Majority coloring may be studied for infinite graphs as well. For undirected graphs it is known as the problem of *unfriendly partitions* (see [1], [2]). As proved by Shelah and Milner [6], every infinite graph is majority 3-colorable, but there are graphs on uncountably many vertices that are not majority 2-colorable. Whether every countably infinite graph has a majority 2-coloring remains a mystery. Perhaps it would be interesting to consider similar questions for infinite directed graphs.

We conclude the paper with the following strengthening of the majority coloring conjecture from [3].

Conjecture 4. Every digraph is majority 3-choosable.

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