Group Actions on Partitions

Byungchan Kim[*]
School of Liberal Arts
Seoul National University of Science and Technology
Seoul, Republic of Korea
bkim4@seoultech.ac.kr

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Abstract

We introduce group actions on the integer partitions and their variances. Using generating functions and Burnside's lemma, we study arithmetic properties of the counting functions arising from group actions. In particular, we find a modulo 4 congruence involving the number of ordinary partitions and the number of partitions into distinct parts.

Keywords: Partitions, unimodal sequences, group action, partition congruence, Bailey pairs, Bailey Lemma

1 Introduction

In the theory of integer partitions, the conjugation is one of the most important operations on partitions. The Durfee square of size $d \times d$ in the partition λ is defined by the largest integer d such that the partition λ has at least d parts $\geqslant d$. Using the Durfee square, we can represent the partition λ by (d, π_1, π_2), where d is the size of the Durfee square and π_1 and π_2 are partitions into parts $\leqslant d$. Let D be the set of partitions represented by the Durfee square decomposition

$$D := \{(d, \pi_1, \pi_2) : d \geqslant 0,\ \pi_1 \text{ and } \pi_2 \text{ are partitions into parts } \leqslant d\}.$$

For the symmetric group $G = S_2$, we define a group action on D by

$$\sigma(d, \pi_1, \pi_2) = (d, \pi_{\sigma(1)}, \pi_{\sigma(2)}),$$

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for $\sigma \in G$. When $\sigma = (12)$, this group action corresponds to the conjugation of the partition. In the light of that we can understand the conjugation of partitions as a group action, it is natural to ask whether there are interesting consequences on partitions in the view of group actions. Moreover, once we define a group action on the set X, then it is also natural to examine how many orbits there are. For an element x in an orbit, we define the weight of x as an element of X and we define $|X/G|(n)$ is the number of orbits of weight n. For example, $|D/S_2|(n)$ is the number of partitions of n, where if two partitions are conjugate to each other, then they are considered to be the same partition.

The goal of this paper is investigating properties of counting functions arising from group actions on various partition sets X and groups G. To this end, we will use Burnside's lemma frequently, which says that the number of orbits $|X/G|$ is given by

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where X^g is the invariant subset of X under the action of g.

One of the main results is a modulo 4 congruence involving the number of ordinary partitions and the number of partitions into distinct parts. Before stating the result, we introduce some notation. Let $p(n)$ be the number of ordinary partitions of n and let $d(n)$ be the number of partitions of n into distinct parts. For a given partition λ, $\ell(\lambda)$ is defined by the largest part of λ. If λ is an empty partition, then we define $\ell(\lambda) = 0$.

Theorem 1. *Let $s(n)$ be the number of partitions of n of which Durfee decompositions (d, π_1, π_2) satisfy $\ell(\pi_1) = \ell(\pi_2)$. Then, for all non-negative integers n,*

$$p(n) + s(n) \equiv 2d(n) \pmod 4.$$

Remark 2. By Euler's pentagonal number theorem, we know that

$$2d(n) \equiv \begin{cases} 2 \pmod 4, & \text{if } n = \frac{k(3k-1)}{2} \text{ for an integer } k, \\ 0 \pmod 4, & \text{otherwise.} \end{cases}$$

In the next section we prove Theorems 1, and we consider multi-partitions and unimodal sequences under group actions in the remainder of the paper.

2 Proof of Theorem 1

Let C_2 be the set defined by

$$C_2 := \{(d, \nu_1, \nu_2) : d \geqslant 0,\ \nu_1 \text{ and } \nu_2 \text{ are compositions into non-negative integers} \leqslant d\}.$$

Note that we allow ν_i's can have zeros as parts and recall that two sequences that differ in the order of their terms define different compositions, while they are considered to be the same partition. The weight of an element $(d, \nu_1, \nu_2) \in C_2$ is defined by d^2 plus the sum of parts in ν_1 and the first part of ν_2. For the convenience, the empty composition is considered to be $\{0\}$ so that we can define the first part of an empty composition is zero. For the symmetric group $G = S_2$, we define a group action on C_2 by exchanging the first part of ν_i and $\nu_{\sigma(i)}$ for $i = 1$ or 2. By employing Burnside's lemma, we find that

$$|C_2/S_2|(n) = \frac{1}{2}\left(|C_2^{(1)}|(n) + |C_2^{(12)}|(n)\right).$$

Here and in the sequel, for a given set X, $|X|(n)$ denotes the number of elements in the set X with the weight n. Now we will count how many partitions there are among the elements in the invariant subsets. To this end, we define

$$|C_2/S_2|_D(n) := \frac{1}{2}\left(|C_2^{(1)} \cap D|(n) + |C_2^{(12)} \cap D|(n)\right),$$

where D is the set of partitions represented by Durfee square decompositions. By employing general combinatorial arguments [1], we obtain that

$$\sum_{n=0}^{\infty} |C_2^{(1)} \cap D|(n) q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n^2}$$
$$\sum_{n=0}^{\infty} |C_2^{(12)} \cap D|(n) q^n = \sum_{n=0}^{\infty} q^{n^2} \sum_{k=0}^{n} \frac{q^{2k}}{(q)_k^2},$$

which implies that

$$\sum_{n=0}^{\infty} |C_2/S_2|_D(n) q^n = \frac{1}{2}\left(\frac{1}{(q)_\infty} + \sum_{n=0}^{\infty} q^{n^2} \sum_{k=0}^{n} \frac{q^{2k}}{(q)_k^2}\right). \tag{1}$$

Here we have used the usual q-series notation,

$$(a)_n = (a;q)_n := \prod_{j=0}^{n-1} (1 - aq^j)$$

for $n \in \mathbb{N}_0 \cup \{\infty\}$.

By using the inclusion-exclusion principle, we derive that

$$\frac{1}{(q)_n^2} = 2\sum_{k=0}^{n} \frac{q^k}{(q)_k^2} - \sum_{k=0}^{n} \frac{q^{2k}}{(q)_k^2}. \tag{2}$$

Therefore, we find that

$$f(q) := \sum_{n=0}^{\infty} |C_2/S_2|_D(n) q^n = \sum_{n=0}^{\infty} q^{n^2} \sum_{k=0}^{n} \frac{q^k}{(q)_k^2}.$$

It turns out that we can reformulate $f(q)$ by using a false theta function.

Lemma 3. *We have*

$$f(q) = \frac{1}{(q)_\infty}\left(1 - \sum_{n=1}^{\infty} q^{n(3n-1)}(1-q^{2n})\right).$$

Before proving the above lemma, we show that Lemma 3 implies Theorem 1.

Proof of Theorem 1. We first note that (1) implies that

$$f(q) = \sum_{n=0}^{\infty} |C_2/S_2|_D(n)q^n = \frac{1}{2}\left(\sum_{n=0}^{\infty} p(n)q^n + \sum_{n=0}^{\infty} s(n)q^n\right). \tag{3}$$

From Euler's pentagonal number theorem [1, Cor 1.7], we find that

$$(q^2;q^2)_\infty = 1 + \sum_{n=1}^{\infty}(-1)^n q^{n(3n-1)}(1+q^{2n}).$$

Therefore, by Lemma 3, we obtain that

$$f(q) \equiv \frac{(q^2;q^2)_\infty}{(q)_\infty} \equiv (-q)_\infty \pmod 2.$$

From (3) and the fact that

$$(-q)_\infty = \sum_{n=0}^{\infty} d(n),$$

we conclude Theorem 1. □

Now we turn to prove Lemma 3.

Proof of Lemma 3. First we recall that a Bailey pair relative to a is a pair of sequences $(\alpha_n, \beta_n)_{n\geqslant 0}$ satisfying

$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q)_{n-k}(aq)_{n+k}}. \tag{4}$$

If (α_n, β_n) is a Bailey pair relative to 1, then Bailey Lemma (see [12] for example) implies that

$$\frac{1}{(q)_\infty}\sum_{n=0}^{\infty} q^{n^2}\alpha_n = \sum_{n=0}^{\infty} q^{n^2}\beta_n. \tag{5}$$

From [11], we find that

$$\begin{aligned} \alpha_0 &= 1, \\ \alpha_n &= -q^{n(n-1)}(1-q^{2n}) \text{ for } n > 0, \\ \beta_n &= \frac{q^n}{(q)_n^2} \end{aligned} \tag{6}$$

consists of a Bailey pair relative to 1. Recall (see [4], for example) that if (α_n, β_n) is a Bailey pair relative to a, then

$$\begin{aligned}\alpha'_n &= a^n q^{n^2} \alpha_n,\\ \beta'_n &= \sum_{k=0}^{n} \frac{a^k q^{k^2}}{(q)_{n-k}} \beta_k\end{aligned}$$

is also a Bailey pair relative to a. Using this with (6), we obtain a new Bailey pair relative to 1,

$$\begin{aligned}\alpha'_0 &= 1,\\ \alpha'_n &= -q^{n(2n-1)}(1-q^{2n}) \text{ for } n > 0,\\ \beta'_n &= \sum_{k=0}^{n} \frac{q^{k^2+k}}{(q)_{n-k}(q)_k^2}.\end{aligned} \tag{7}$$

Using Andrews' finite Heine transformation [3, Corollary 4]

$$\sum_{k=0}^{n} \frac{(q^{-n})_k(\alpha)_k(\beta)_k q^k}{(q)_k(\gamma)_k(q^{1-n}/\tau)_k} = \frac{(\alpha\beta\tau/\gamma)_n}{(\tau)_n} \sum_{k=0}^{n} \frac{(q^{-n})_k(\gamma/\alpha)_k(\gamma/\beta)_k q^k}{(q)_k(\gamma)_k(\gamma q^{1-n}/(\alpha\beta\tau))_k}$$

with $\alpha, \beta \to 0$ and $\gamma = \tau = q$, we derive that

$$\beta'_n = \sum_{k=0}^{n} \frac{q^{k^2+k}}{(q)_{n-k}(q)_k^2} = \sum_{k=0}^{n} \frac{q^k}{(q)_k^2}. \tag{8}$$

From (7), (8), and (5), we obtain the desirable identity. □

Note that

$$\sum_{n=0}^{\infty} |C_2/S_2|_D(n) q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} \sum_{k=0}^{n} \frac{q^{k^2+k}}{(q)_k} \begin{bmatrix} n \\ k \end{bmatrix}, \tag{9}$$

where we use the q-binomial coefficient defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k(q)_{n-k}}. \tag{10}$$

Recall that the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ generates the partitions which fit inside a $(n-k) \times k$ rectangle. From (9), we can see that $|C_2/S_2|_D(n)$ is the number of partitions of n with certain Durfee dissections (See [2] for more details on Durfee dissections.).

Corollary 4. *Let $D_2(n)$ be the number of partitions of n with 2 successive Durfee squares of sizes $d_1 \times d_1$ and $d_2 \times d_2$ such that there is a part of size d_2 below the second Durfee square. Then, for all non-negative integers n,*

$$|C_2/S_2|_D(n) = D_2(n) = \frac{1}{2}\left(p(n) + s(n)\right).$$

3 Other Results

In this section, we study the number of orbits in the sets of multi-partitions and in the sets of unimodal sequences under group actions.

3.1 Multi-partitions

Let P_2 be the set of bi-partitions

$$P_2 := \{(\lambda_1, \lambda_2) : \ \lambda_1 \text{ and } \lambda_2 \text{ are ordinary partitions}\}.$$

For the group $G = S_2$, we define a group action on P_2 by

$$\sigma(\lambda_1, \lambda_2) = (\lambda_{\sigma(1)}, \lambda_{\sigma(2)}).$$

Let $|P_2/S_2|(n)$ be the number of orbits with weight n. Then, $|P_2/S_2|(n)$ is the number of bi-partitions of n, where bi-partitions in the same orbit define the same bi-partition. By simple combinatorial arguments and Burnside's formula, we find that

$$\sum_{n=0}^{\infty} |P_2/S_2|(n)q^n = \frac{1}{2}\left(\frac{1}{(q)_\infty^2} + \frac{1}{(q^2;q^2)_\infty}\right).$$

Let $p_{-2}(n)$ be the number of bi-partitions of n. Then, we have

$$\sum_{n=0}^{\infty} p_{-2}(n)q^n = \frac{1}{(q)_\infty^2}.$$

From the famous Ramanujan congruence $p(5n+4) \equiv 0 \pmod 5$ and a bi-partition congruence [6] $p_{-2}(5n+3) \equiv 0 \pmod 5$, we can easily see that

$$|P_2/S_2|(5n+3) \equiv 0 \pmod 5.$$

We define the set of tri-partitions

$$P_3 := \{(\lambda_1, \lambda_2, \lambda_3) : \ \lambda_1,\ \lambda_2, \text{ and } \lambda_3 \text{ are ordinary partitions}\}.$$

For $G = S_3$, we define a group action on P_3 by

$$\sigma(\lambda_1, \lambda_2, \lambda_3) = (\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \lambda_{\sigma(3)}).$$

We may think each partition λ_i is on a vertex of a regular triangle. Then, we can understand $|P_3/S_3|(n)$ is the number of tri-partitions of n, where the partitions are considered to be the same if they are invariant under a reflection or a rotation fixing the triangle. From Burnside's lemma, we observe that

$$\sum_{n=0}^{\infty} |P_3/S_3|(n)q^n = \frac{1}{6}\left(\frac{1}{(q)_\infty^3} + \frac{3}{(q)_\infty(q^2;q^2)_\infty} + \frac{2}{(q^3;q^3)_\infty}\right).$$

Proposition 5. *For all non-negative integers n,*

$$|P_3/S_3|(3n+2) \equiv 0 \pmod 3.$$

Moreover,

$$\sum_{n=0}^{\infty} |P_3/S_3|(3n+2)q^n = \frac{3}{2}\left(\frac{(q^3;q^3)_\infty^9}{(q)_\infty^{12}} + \frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty^4(q^2;q^2)_\infty^4}\right).$$

Proof. H.-C. Chan [5] defined the cubic partition function $c(n)$ by

$$\sum_{n=0}^{\infty} c(n)q^n = \frac{1}{(q)_\infty(q^2;q^2)_\infty},$$

and proved that $c(3n+2) \equiv 0 \pmod 3$. Therefore, it suffices to show that

$$p_{-3}(3n+2) \equiv 0 \pmod 9, \tag{11}$$

where $p_{-3}(n)$ is defined by

$$\sum_{n=0}^{\infty} p_{-3}(n)q^n = \frac{1}{(q)_\infty^3}.$$

We define the Dedekind eta function $\eta(z)$ by

$$\eta(z) = q^{1/24}(q)_\infty \qquad \text{with } q = \exp(2\pi i z).$$

Since $\frac{\eta^9(3z)}{\eta^3(z)}$ is a modular form of weight 3 and level 3 with character $\left(\frac{-3}{\cdot}\right)$ and the dimension of such modular forms space is 2, we find that

$$\frac{\eta^9(3z)}{\eta^3(z)}|U_3 = 9\frac{\eta^9(3z)}{\eta^3(z)} \tag{12}$$

by checking the first two coefficients. Here, U_3 is an operator defined by $\left(\sum a(n)q^n\right)|U_3 := \sum a(3n)q^n$. (For more details on modular forms, see [10] for example.) From (12), we find that

$$\begin{aligned} \frac{\eta^9(3z)}{\eta^3(z)}|U_3 &= \left((q^3;q^3)_\infty^9 \sum_{n=0}^{\infty} p_{-3}(n)q^{n+1}\right)\Big|U_3 \\ &= (q)_\infty^9 \sum_{n=0}^{\infty} p_{-3}(3n+2)q^{n+1} \\ &= 9q\frac{(q^3;q^3)_\infty^9}{(q)_\infty^3}, \end{aligned} \tag{13}$$

which proves $p_{-3}(3n+2) \equiv 0 \pmod 9$ as desired. We also note that $\frac{\eta^3(3z)\eta^3(6z)}{\eta(z)\eta(2z)}$ is a modular form of weight 2 and level 6. Proceeding as before (for this time, we need to check the first three coefficients), we deduce that

$$\frac{\eta^3(3z)\eta^3(6z)}{\eta(z)\eta(2z)}|U_3 = 3\frac{\eta^3(3z)\eta^3(6z)}{\eta(z)\eta(2z)}.$$

In terms of cubic partitions, the above modular identity is equivalent to

$$(q)_\infty^3(q^3;q^3)_\infty \sum_{n=0}^{\infty} c(3n+2)q^n = 3\frac{(q^3;q^3)_\infty^3(q^6;q^6)_\infty^3}{(q)_\infty(q^2;q^2)_\infty}. \tag{14}$$

From (13) and (14), we can obtain the desirable generating function for $|P_3/S_3|(3n+2)$. □

3.2 Unimodal sequences

Recall that a unimodal sequence is a sequence which is weakly increasing up to a point (called the peak), and then weakly decreasing thereafter. The weight of such a sequence is the sum of all of its terms. We define the set of unimodal sequeces

$$U := \{(p, \pi_1, \pi_2) : p \geqslant 0,\ \pi_1 \text{ and } \pi_2 \text{ are partitions into parts} \leqslant p\}.$$

For an element (p, π_1, π_2), its weight is defined by p plus the sum of parts in π_1 and π_2. Recall that C_2 is the set defined by

$$C_2 := \{(d, \nu_1, \nu_2) : d \geqslant 0,\ \nu_1 \text{ and } \nu_2 \text{ are compositions into non-negative integers} \leqslant d\}.$$

To match the weights, in this subsection, the weight of an element $(d, \nu_1, \nu_2) \in C_2$ is redefined by d plus the sum of parts in ν_1 and ν_2. For the symmetric group $G = S_2$, we define a group action on C_2 by exchanging the first part of ν_1 and the first part of ν_2. By employing Burnside's lemma, we find that

$$|C_2/S_2|(n) = \frac{1}{2}\left(|C_2^{(1)}|(n) + |C_2^{(12)}|(n)\right).$$

Here we will count how many unimodal sequences there are among the elements in the invariant subsets. To this end, we define

$$|C_2/S_2|_U(n) := \frac{1}{2}\left(|C_2^{(1)} \cap U|(n) + |C_2^{(12)} \cap U|(n)\right).$$

Then, we can see that

$$\begin{aligned}\sum_{n=0}^{\infty} |C_2/S_2|_U(n)q^n &= \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{q^n}{(q)_n^2} + \sum_{n=0}^{\infty} q^n \sum_{k=0}^{n}\frac{q^{2k}}{(q)_k^2}\right)\\ &= \sum_{n\geqslant k\geqslant 0}\frac{q^{n+k}}{(q)_k^2},\end{aligned}$$

where we have used (2) for the last equality. From this, we find that $|C_2/S_2|_U(n)$ is the number of unimodal sequences of weight n such that the largest part in the partition before the peak is larger than or equal to the largest part in the partition after the peak.

From [9, Theorem 1.1], we find that

$$\sum_{n=0}^{\infty} q^n\beta_n = \frac{1}{(aq)_\infty(q)_\infty}\sum_{n,r\geqslant 0}(-a)^n q^{n(n+1)/2+(2n+1)r}\alpha_r, \tag{15}$$

where (α_n, β_n) is a Bailey pair relative to a. Therefore, by plugging the Bailey pairs (7) and (8) into (15), we derive that

$$\begin{aligned}
\sum_{n\geqslant k\geqslant 0} \frac{q^{n+k}}{(q)_k^2} &= \frac{1}{(q)_\infty^2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \left(1 - \sum_{r=1}^{\infty} q^{(2n+1)r+r(2r-1)}(1-q^{2r})\right) \\
&= \frac{1}{(q)_\infty^2} \sum_{n,r\geqslant 0} (-1)^n q^{n(n+1)/2+(2n+1)r+r(2r+1)} \\
&\quad - \frac{1}{(q)_\infty^2} \sum_{n\geqslant 0, r>0} (-1)^n q^{n(n+1)/2+(2n+1)r+r(2r-1)} \\
&= \frac{1}{(q)_\infty^2} \left(\sum_{n,r\geqslant 0} + \sum_{n,r<0}\right) (-1)^n q^{n(n+1)/2+(2n+1)r+r(2r+1)},
\end{aligned}$$

where we replace n and r by $-n-1$ and $-r$ in the second summation for the last equality. In summary, we have obtained a Hecke-type double sum expression for the $|C_2/S_2|_U(n)$ generating function.

Theorem 6. *We have*

$$\sum_{n=0}^{\infty} |C_2/S_2|_U(n) q^n = \frac{1}{(q)_\infty^2} \left(\sum_{n,r\geqslant 0} + \sum_{n,r<0}\right) (-1)^n q^{n(n+1)/2+2nr+2r(r+1)}. \tag{16}$$

We may think the double sum in the right hand side of (16) as a false theta series in the sense of that it has a wrong sign compare to the theta series

$$f_{a,b,c}(x,y,q) = \left(\sum_{r,s\geqslant 0} - \sum_{r,s<0}\right) (-1)^{r+s} x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}, \tag{17}$$

which is closely related to mock theta functions [7].

In a recent paper [8], the author and J. Lovejoy introduced a unimodal T-sequence. In a unimodal T-sequence, we have a peak and there are partitions into parts $\leqslant$ the peak in the right, in the left, and below of the peak. We define the set of unimodal T-sequences

$$T := \{(p, \pi_1, \pi_2, \pi_3) : p \geqslant 0,\ \pi_1,\ \pi_2, \text{ and } \pi_3 \text{ are partitions into parts } \leqslant p\}.$$

We also define a corresponding composition set C_3 by

$$\begin{aligned}
C_3 := \{(p, \nu_1, \nu_2, \nu_3) : p \geqslant 0,\ &\nu_1, \nu_2 \text{ and } \nu_3 \text{ are compositions} \\
&\text{into non-negative integers } \leqslant d\}.
\end{aligned}$$

For an element in T (resp. C_3), its weight is defined by the size of the peak (i.e. p) plus the sum of parts in the partitions (resp. compositions).

For $G = S_3$, we define a group action on C_3 by switching the first part of ν_i and the first part of $\nu_{\sigma(i)}$. As before, we investigate

$$|C_3/S_3|_T(n) := \frac{1}{6}\sum_{g\in S_3} |C_3^g \cap T|(n).$$

Then, we find that

$$\sum_{n=0}^{\infty} |C_3/S_3|_T(n)q^n = \frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{q^n}{(q)_n^3} + \frac{3q^n}{(q)_n}\sum_{k=0}^{n}\frac{q^{2k}}{(q)_k^2} + 2q^n\sum_{k=0}^{n}\frac{q^{3k}}{(q;q)_k^3}\right).$$

By employing the inclusion-exclusion principle on the largest parts, we find that

$$\frac{1}{(q)_n^3} = 3\sum_{k=0}^{n}\frac{q^k(1-q^k)}{(q)_k^3} + \sum_{k=0}^{n}\frac{q^{3k}}{(q)_k^3}.$$

From (2), (8), and the above, we derive that

$$\begin{aligned}\sum_{n=0}^{\infty} |C_3/S_3|_T(n)q^n &= \sum_{n=0}^{\infty}\frac{q^n}{(q)_n}\sum_{k=0}^{n}\frac{q^k}{(q)_k^2} - \sum_{n=0}^{\infty} q^n \sum_{k=0}^{n}\frac{q^k(1-q^k)}{(q)_k^3}\\ &= \sum_{n=0}^{\infty}\frac{q^n}{(q)_n^2}\sum_{k=0}^{n}\frac{q^{k^2+k}}{(q)_k}\begin{bmatrix} n\\ k\end{bmatrix} - \sum_{n\geqslant k\geqslant 0}\frac{q^{n+k}(1-q^k)}{(q)_k^3}.\end{aligned}$$

Proposition 7. *Let $T_1(n)$ be the number of unimodal T-sequences of weight n such that the partition below the peak has the Durfee rectangle of size $(k+1)\times k$. Let $T_2(n)$ be the number of unimodal T-sequences of weight n such that if the largest part of the partition in the right of the peak is k, then the largest part of the partition in the left (resp. below) of the peak is less than or equal to (resp. less than) k. Then, for all non-negative integers n,*

$$|C_3/S_3|_T(n) = T_1(n) - T_2(n).$$

In particular, $T_1(n) > T_2(n)$ for all positive integers n.

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