Non-flat regular polytopes and restrictions on chiral polytopes

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Abstract

An abstract polytope is *flat* if every facet is incident on every vertex. In this paper, we prove that no chiral polytope has flat finite regular facets and finite regular vertex-figures. We then determine the three smallest non-flat regular polytopes in each rank, and use this to show that for $n \geqslant 8$, a chiral n-polytope has at least $48(n-2)(n-2)!$ flags.

Keywords: abstract regular polytope; chiral polytope; flat polytope; tight polytope.

1 Introduction

In many applications involving convex polytopes, what is most important is the *combinatorial type* of the polytope: how many faces are there in each dimension, and which faces are incident. An *abstract polytope* is essentially a partially ordered set that resembles the incidence relation for a convex polytope or a tiling of a surface or space.

Regular (abstract) polytopes are those that are maximally symmetric. The automorphism group of a regular polytope can be written in a standard form, and in fact the polytope can be recovered from a group presentation in this form. This means that many questions about regular polytopes can be translated to questions in group theory. Furthermore, this makes it possible to collect a large amount of data about regular polytopes, using standard group theory algorithms.

In [4], Conder catalogs the regular polytopes with up to 4000 flags (where a flag is a maximal chain of incidences). He excludes *degenerate* polytopes, such as the digon, which consists of two edges and two vertices, with both edges incident on each vertex. However, many of the listed polytopes possess the minor degeneracy of being *flat*, meaning that

Rank	Flat	Non-flat
3	2292	8186
4	7530	353
5	1561	8
6	52	0

Table 1: Number of non-degenerate polytopes, up to duality, with up to 4000 flags

every facet is incident to every vertex. In ranks 4 and higher, more than 95% of the listed polytopes are flat. See Table 1 for a summary of the counts.

Chiral polytopes are those that are fully symmetric under combinatorial rotations, but without mirror symmetry. Each chiral polytope is built out of regular and chiral polytopes of one dimension lower. One of the fundamental problems in the study of chiral polytopes is the *amalgamation problem*: which polytopes can be assembled together to form a chiral polytope? In Theorem 3.1, we will prove that no finite chiral polytope is built from flat regular polytopes that are arranged in a regular way around each vertex. Using this, we are able to describe several other restrictions on the structure of chiral polytopes in Section 3.

Another important problem is the determination of the smallest chiral polytopes in each rank. Section 5.1 describes what is currently known. In Section 4, we determine the smallest non-flat regular polytopes in each rank. Using this and Theorem 3.1, we prove in Theorem 5.5 that, for $n \geqslant 8$, a chiral n-polytope has at least $48(n-2)(n-2)!$ flags.

2 Background

2.1 Abstract polytopes

Let us start with the definition of an abstract polytope, taken from [13, Sec. 2A]. Consider a partially-ordered set $\mathcal{P}$ with a unique minimal element and a unique maximal element. If the maximal chains of $\mathcal{P}$ all have the same length, then we can endow $\mathcal{P}$ with a rank function, where the minimal element has rank -1, the elements that directly cover it have rank 0, and so on. We then say that $\mathcal{P}$ is an *(abstract) n-polytope* or *polytope of rank n* if its maximal element has rank n and if $\mathcal{P}$ also satisfies the following conditions.

1. (Diamond condition): Whenever $F < G$ and $\text{rank}(G) - \text{rank}(F) = 2$, there are exactly two elements H with $\text{rank}(H) = \text{rank}(F) + 1$ such that $F < H < G$.

2. (Strong connectivity): Suppose $F < G$ and $\text{rank}(G) - \text{rank}(F) \geqslant 3$. If $F < H < G$ and $F < H' < G$, then there is a chain

$$H = H_0 \leqslant H_1 \geqslant H_2 \leqslant H_3 \geqslant H_4 \leqslant \cdots \geqslant H_k = H'$$

such that $F < H_i < G$ for each i.

For example, the face-lattice of any convex n-polytope is an (abstract) n-polytope. In analogy with convex polytopes, we call the elements of $\mathcal{P}$ *faces*, and a face of rank k is a *k-face*. The faces of rank 0, 1, and $n-1$ are called *vertices*, *edges*, and *facets*, respectively. The maximal chains of $\mathcal{P}$ are called *flags*, and two flags that differ in only a single element are said to be *adjacent*.

If $F < G$ are faces of an n-polytope, then the *section* G/F consists of all faces H such that $F \leqslant H \leqslant G$. If F is a facet of the n-polytope $\mathcal{P}$ and F_{-1} is the minimal face, then the section F/F_{-1} is an $(n-1)$-polytope. Usually, when we speak of a facet of $\mathcal{P}$, we have in mind this polytope, rather than just an element of rank $n-1$. If v is a vertex of $\mathcal{P}$ and F_n is the maximal face, then the section F_n/v is also an $(n-1)$-polytope, called the *vertex-figure at v*. Given both a facet F and a vertex v, the section F/v is an $(n-2)$-polytope, called a *medial section* of $\mathcal{P}$; it is both a vertex-figure of the facet F and a facet of the vertex-figure at v.

For each integer $p \geqslant 2$, there is a unique 2-polytope with p vertices, denoted by $\{p\}$. When $p \geqslant 3$, this is simply the face-lattice of a p-gon; the case $p = 2$ yields the *digon*, which has two edges and two vertices, with each edge incident on each vertex. There is also a unique infinite 2-polytope, denoted $\{\infty\}$, which is the face-lattice of the tiling of the real line by unit line segments.

Given faces $F < G$ where $\mathrm{rank}(G) = \mathrm{rank}(F) + 3$, the section G/F is a 2-polytope with some number $p(F, G)$ of vertices. If $\mathcal{P}$ has the property that $p(F, G)$ depends only on the rank of F and G (rather than on the particular choice of faces in those ranks), then we say that $\mathcal{P}$ is *equivelar*. In this case, there are numbers $p_1, \ldots, p_{n-1}$ such that, given any $(i-2)$-face F and $(i+1)$-face G with $F < G$, the section G/F is the polytope $\{p_i\}$. We then say that $\mathcal{P}$ has *Schläfli symbol* (or *type*) $\{p_1, \ldots, p_{n-1}\}$.

If $\mathcal{P}$ and $\mathcal{Q}$ are both n-polytopes, then a *covering* $\pi : \mathcal{Q} \to \mathcal{P}$ is a function that preserves the partial order, the rank of each face, and with the property that if two flags of $\mathcal{Q}$ are adjacent, then their images under π are also adjacent. (Such a function is automatically surjective.) We say then that $\mathcal{Q}$ *covers* $\mathcal{P}$. An *isomorphism* of n-polytopes is a bijection that preserves rank and the partial order.

If the facets of a polytope $\mathcal{P}$ are all isomorphic to $\mathcal{K}$, and the vertex-figures are all isomorphic to $\mathcal{L}$, then we say that $\mathcal{P}$ is of type $\{\mathcal{K}, \mathcal{L}\}$. If $\mathcal{P}$ is of type $\{\mathcal{K}, \mathcal{L}\}$ and it covers all other polytopes of type $\{\mathcal{K}, \mathcal{L}\}$, then we call $\mathcal{P}$ the *universal* polytope of type $\{\mathcal{K}, \mathcal{L}\}$, and often denote it simply by $\{\mathcal{K}, \mathcal{L}\}$. This notation is naturally recursive, so that one may refer to a polytope such as $\{\{\mathcal{K}, \mathcal{L}\}, \{\mathcal{L}, \mathcal{M}\}\}$.

The *dual* of $\mathcal{P}$, denoted $\mathcal{P}^*$, is the polytope with the same underlying set as $\mathcal{P}$ but with the partial order reversed. If $\mathcal{P}$ has Schläfli symbol $\{p_1, \ldots, p_{n-1}\}$, then $\mathcal{P}^*$ has Schläfli symbol $\{p_{n-1}, \ldots, p_1\}$, and if $\mathcal{P}$ is of type $\{\mathcal{K}, \mathcal{L}\}$, then $\mathcal{P}^*$ is of type $\{\mathcal{L}^*, \mathcal{K}^*\}$. When we say that something is true of $\mathcal{P}$ *up to duality*, we mean that it is either true of $\mathcal{P}$ or of $\mathcal{P}^*$.

2.2 Regular and chiral polytopes

The automorphism group of $\mathcal{P}$, denoted $\Gamma(\mathcal{P})$, consists of the isomorphisms from $\mathcal{P}$ to itself. This group acts freely on the flags of $\mathcal{P}$. A polytope is *regular* if $\Gamma(\mathcal{P})$ acts transitively on the flags. The automorphism group of a regular polytope is a *string C-group* (defined below), and every string C-group is the automorphism group of a regular polytope.

We now define string C-groups. Suppose that $\Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, where the generators ρ_i satisfy at least the relations

$$\rho_i^2 = 1, \text{ for } 0 \leqslant i \leqslant n-1, \tag{1}$$

$$(\rho_i \rho_j)^2 = 1, \text{ for } i, j \in \{0, \ldots, n-1\} \text{ with } |i-j| \geqslant 2. \tag{2}$$

Such a group is called a *string group generated by involutions* (sggi). Then Γ is a string C-group if it also satisfies the following *intersection condition* for all subsets I and J of $\{0, \ldots, n-1\}$:

$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle. \tag{3}$$

Regular polytopes are equivelar. If $\mathcal{P}$ is a regular polytope of type $\{p_1, \ldots, p_{n-1}\}$, then $\Gamma(\mathcal{P})$ is a quotient of the string Coxeter group

$$\begin{aligned}[p_1, \ldots, p_{n-1}] := \langle \rho_0, \ldots, \rho_{n-1} \mid & \rho_i^2 = 1 \text{ for } 0 \leqslant i \leqslant n-1, \\ & (\rho_{i-1}\rho_i)^{p_i} = 1 \text{ for } 1 \leqslant i \leqslant n-1, \\ & (\rho_i \rho_j)^2 = 1 \text{ for } i, j \in \{0, \ldots, n-1\} \text{ with } |i-j| \geqslant 2 \rangle. \end{aligned} \tag{4}$$

The facets of a regular n-polytope are all isomorphic to some regular polytope $\mathcal{K}$, and the vertex-figures are isomorphic to some regular polytope $\mathcal{L}$.

A polytope is *chiral* if the flags fall into two orbits under the action of $\Gamma(\mathcal{P})$ with the property that adjacent flags lie in different orbits. Basic information about chiral polytopes can be found in [16], and a survey of important problems can be found in [15]. The facets of a chiral polytope are all isomorphic, as are the vertex-figures. Both the facets and the vertex-figures are either chiral or regular. Furthermore, the facets of the facets and the vertex-figures of the vertex-figures must be regular.

If $\mathcal{P}$ is a chiral polytope of type $\{\mathcal{K}, \mathcal{L}\}$, with at least one of $\mathcal{K}$ and $\mathcal{L}$ regular, then there is a unique minimal regular polytope $\mathcal{R}$ that covers $\mathcal{P}$. The polytope $\mathcal{R}$ is called the *mixed regular cover* of $\mathcal{P}$. Furthermore, if both $\mathcal{K}$ and $\mathcal{L}$ are regular, then $\mathcal{R}$ is also of type $\{\mathcal{K}, \mathcal{L}\}$ (see [14, Sec. 4]).

2.3 Degenerate and flat polytopes

A polytope of type $\{p_1, \ldots, p_{n-1}\}$ is said to be *degenerate* if at least one of the numbers p_i is 2. A polytope is called *flat* if every facet is incident with every vertex. More generally, if $0 \leqslant k < m \leqslant n-1$, then an n-polytope is (k,m)-flat if every k-face is incident with every m-face. We summarize some properties of flatness below (see [13, Prop. 2B16, Section 4E]).

Proposition 2.1. *Let $\mathcal{P}$ be an equivelar n-polytope.*

1. *If $\mathcal{P}$ is degenerate, then it is flat.*

2. *If $0 \leqslant i \leqslant k < m \leqslant j \leqslant n-1$ and $\mathcal{P}$ is (k,m)-flat, then $\mathcal{P}$ is also (i,j)-flat.*

3. *If $m \leqslant n-2$, then $\mathcal{P}$ is (k,m)-flat if and only if the facets of $\mathcal{P}$ are (k,m)-flat.*

4. *If $k \geqslant 1$, then $\mathcal{P}$ is (k,m)-flat if and only if the vertex-figures of $\mathcal{P}$ are $(k-1,m-1)$-flat.*

A polytope of type $\{p_1,\ldots,p_{n-1}\}$ is called *tight* if it has exactly $2p_1\cdots p_{n-1}$ flags, which is the minimum possible for a polytope of that type. Tightness and flatness are related by the following result.

Proposition 2.2 ([8, Theorem 4.4])**.** *For $n \geqslant 2$, an equivelar n-polytope is tight if and only if it is $(i,i+2)$-flat for every i satisfying $0 \leqslant i \leqslant n-3$.*

3 Restrictions on chiral polytopes

The study of chiral polytopes is, in many ways, still in its infancy. A number of general methods for constructing chiral polytopes have been discovered (see [1, 6, 10, 14]), but few structural results are known. Perhaps the most fundamental question is: which regular polytopes can occur as the facets of a chiral polytope? We start with a simple result.

Theorem 3.1. *There are no chiral polytopes with flat, finite, regular facets and finite regular vertex-figures.*

Proof. Suppose that $\mathcal{P}$ is a chiral polytope of type $\{\mathcal{K},\mathcal{L}\}$, where $\mathcal{K}$ and $\mathcal{L}$ are finite regular polytopes, and $\mathcal{K}$ is flat. The mixed regular cover of $\mathcal{P}$ is a regular polytope $\mathcal{R}$ of type $\{\mathcal{K},\mathcal{L}\}$. Now, since $\mathcal{K}$ is flat, so are $\mathcal{P}$ and $\mathcal{R}$, by Proposition 2.1(c). This means that $\mathcal{P}$ and $\mathcal{R}$ both have the same number of vertices; namely, the number of vertices that $\mathcal{K}$ has. Then since $\mathcal{P}$ and $\mathcal{R}$ have isomorphic vertex-figures, and $\mathcal{R}$ covers $\mathcal{P}$, it follows that $\mathcal{R} \cong \mathcal{P}$, which is impossible since $\mathcal{P}$ is chiral and $\mathcal{R}$ is regular. □

Theorem 3.1 leads to several further restrictions on the structure of chiral polytopes.

Theorem 3.2. *If $\mathcal{K}$ is a regular n-polytope that is $(1,n-1)$-flat, then no finite chiral $(n+1)$-polytope has $\mathcal{K}$ as a facet.*

Proof. Let $\mathcal{P}$ be a finite chiral $(n+1)$-polytope of type $\{\mathcal{K},\mathcal{L}\}$, and suppose that $\mathcal{K}$ is a (finite) regular polytope that is $(1,n-1)$-flat. Then by Theorem 3.1, the vertex-figures $\mathcal{L}$ of $\mathcal{P}$ must be chiral. Now, the facets of $\mathcal{L}$ are isomorphic to the vertex-figures of $\mathcal{K}$, which by Proposition 2.1(d) must be isomorphic to a regular, $(0,n-2)$-flat polytope of rank $n-1$. Then Theorem 3.1 implies that the vertex-figures of $\mathcal{L}$ must be chiral. But this is impossible, since the vertex-figures of the vertex-figures of a chiral polytope are always regular. □

For example, let $\mathcal{P}$ be the universal polytope of type $\{\{4,3\},\{3,6\}_{(1,1)}\}$ (denoted by $\{4,3,6\} * 288$ in [12]). Then the vertex-figures of $\mathcal{P}$ are $(0,2)$-flat, and thus $\mathcal{P}$ itself is $(1,3)$-flat. By Theorem 3.2, no finite chiral polytope has $\mathcal{P}$ as a facet. Note that this gives a negative answer to Problem 28 in [15].

As a consequence of Theorem 3.2, we find that finite chiral polytopes cannot be arbitrarily flat.

Corollary 3.3. *There are no finite chiral n-polytopes that are $(1,n-3)$-flat or $(2,n-2)$-flat.*

Proof. Suppose $\mathcal{P}$ is a finite chiral n-polytope that is $(1,n-3)$-flat. Then the facets of $\mathcal{P}$ are $(n-1)$-polytopes that are also $(1,n-3)$-flat, and Theorem 3.2 implies that these facets cannot be regular. So the facets of $\mathcal{P}$ are isomorphic to a finite chiral $(n-1)$-polytope $\mathcal{Q}$ that is $(1,n-3)$-flat. But then $\mathcal{Q}$ itself must have regular facets, and those facets are $(n-2)$-polytopes that are $(1,n-3)$-flat, contradicting Theorem 3.2.

The second half follows since the dual of a $(2,n-2)$-flat n-polytope is $(1,n-3)$-flat. $\square$

By Proposition 2.2, a tight polytope must be $(1,3)$-flat. Thus, Corollary 3.3 implies the following.

Corollary 3.4. *There are no tight chiral n-polytopes with $n \geqslant 6$.*

The Schläfli symbols of tight chiral polyhedra were classified in [9]. Tight chiral 4-polytopes and 5-polytopes are further restricted due to Theorem 3.1.

Theorem 3.5. *If $\mathcal{P}$ is a tight chiral 4-polytope, then it has chiral facets or chiral vertex-figures (or both). If $\mathcal{P}$ is a tight chiral 5-polytope, then it has chiral facets, vertex-figures, and medial sections.*

Proof. Suppose that $\mathcal{P}$ is a tight chiral 4-polytope. Then the facets and vertex-figures of $\mathcal{P}$ are both tight, and thus flat. Then Theorem 3.1 implies that the facets and vertex-figures cannot both be regular, so at least one of them is chiral. If instead $\mathcal{P}$ is a tight chiral 5-polytope, then the same result says that either the facets or the vertex-figures are tight chiral 4-polytopes. In either case, since the facets of the facets and the vertex-figures of the vertex-figures of $\mathcal{P}$ must both be regular, the medial sections of $\mathcal{P}$ must be chiral, which forces the facets and vertex-figures to both be chiral. $\square$

The list of chiral polytopes at [3] includes many tight chiral 4-polytopes. So far, no tight chiral 5-polytopes have been discovered. The obvious candidates, with facets and vertex-figures isomorphic to tight chiral 4-polytopes, seem to always collapse to something regular or something non-polytopal.

Problem 1. Fully classify the tight chiral polyhedra and 4-polytopes.

Problem 2. Determine whether there are any tight chiral 5-polytopes.

Our next goal will be to determine a lower bound for the number of flags of a chiral n-polytope. To do so, we will need to determine the smallest non-flat regular polytopes in each rank.

4 Non-flat regular polytopes

Recall that a polytope is flat if every vertex is incident on every facet. Thus, if a polytope is not flat, then it has at least one more vertex than its facets have. This yields the following simple consequences.

Proposition 4.1. *If $\mathcal{P}$ is an equivelar non-flat n-polytope of type $\{p_1, \ldots, p_{n-1}\}$, with $n \geqslant 3$, then $\mathcal{P}$ has at least $p_{n-1} + n - 2$ facets and at least $p_1 + n - 2$ vertices.*

Proof. If $n = 3$, then the facets are p_1-gons, so in order for $\mathcal{P}$ to be non-flat, it must have at least $p_1 + 1$ vertices. Similarly, the vertex-figures are p_2-gons, so $\mathcal{P}$ must have at least $p_2 + 1$ facets in order to be non-flat. The claim then follows by induction on n. □

Corollary 4.2. *A non-flat equivelar n-polytope has at least $n+1$ facets and $n+1$ vertices.*

Proof. In light of Proposition 4.1, the only way to have fewer than $n + 1$ facets or vertices is for p_1 or p_{n-1} to be 2. But then $\mathcal{P}$ is flat, by Proposition 2.1(a). □

In fact, the fewer vertices that a polytope has (in a fixed rank), the flatter it must be.

Proposition 4.3. *Suppose $\mathcal{P}$ is an equivelar n-polytope of type $\{p_1, \ldots, p_{n-1}\}$ with k vertices, $k \leqslant p_1 + n - 3$. Then $\mathcal{P}$ is $(0, k + 2 - p_1)$-flat.*

Proof. First, suppose that $k = p_1 + n - 3$. Then Proposition 4.1 implies that $\mathcal{P}$ is flat, i.e., $(0, n - 1)$-flat, as desired. For the case $n = 3$, we are done, since $\mathcal{P}$ has at least p_1 vertices.

Now suppose that the claim is true for $(n - 1)$-polytopes with $k' \leqslant p_1 + (n - 1) - 3$, and suppose that $\mathcal{P}$ has $k < p_1 + n - 3$ vertices. Then the facets have $k' \leqslant k < p_1 + n - 3$ vertices. Therefore, $k' \leqslant p_1 + (n - 1) - 3$, and by inductive hypothesis, the facets are $(0, k' + 2 - p_1)$-flat. Then Proposition 2.1(c) shows that $\mathcal{P}$ is $(0, k' + 2 - p_1)$-flat. Since $k' \leqslant k$, this implies that $\mathcal{P}$ is $(0, k + 2 - p_1)$-flat, by Proposition 2.1(b). □

Corollary 4.4. *Suppose $\mathcal{P}$ is an equivelar n-polytope with k vertices, $k \leqslant n$. Then $\mathcal{P}$ is $(0, k - 1)$-flat.*

Proof. If $p_1 = 2$, then [13, Prop. 2B16] says that $\mathcal{P}$ is $(0, 1)$-flat, which implies that it is $(0, k - 1)$-flat. Otherwise, if $p_1 \geqslant 3$, then having $k \leqslant n$ implies that $k \leqslant p_1 + n - 3$, and so Proposition 4.3 implies that $\mathcal{P}$ is $(0, k + 2 - p_1)$-flat. Since $k + 2 - p_1 \leqslant k - 1$, this implies that $\mathcal{P}$ is $(0, k - 1)$-flat, by Proposition 2.1(b). □

The preceding four results apply to any equivelar polytope, most notably regular and chiral polytopes. This is already enough to determine the smallest non-flat regular polytopes in each rank.

Proposition 4.5. *The simplex is the unique smallest non-flat regular n-polytope.*

Proof. We use induction on n. The claim is clearly true for $n = 2$. In general, if $\mathcal{P}$ is a non-flat regular n-polytope, then its facets are non-flat regular $(n-1)$-polytopes. By inductive hypothesis, the facets each are at least as large as simplices, with $n!$ flags. Then since Proposition 4.1 implies that $\mathcal{P}$ has at least $n+1$ facets, it follows that $\mathcal{P}$ has at least $(n+1)!$ flags. Furthermore, the only way for $\mathcal{P}$ to have exactly $(n+1)!$ flags is if it has $n+1$ facets that all have $n!$ flags. By inductive hypothesis, the facets must be simplices, and if there are $n+1$ facets, then $p_{n-1} = 3$ by Proposition 4.1. So $\mathcal{P}$ must be a simplex. □

Before we continue to find small non-flat regular polytopes, let us describe a family of regular polytopes, which we will call *central extensions of simplices*. Consider a sequence $p_1, \ldots, p_{n-1}$, where each p_i is either 3 or 6. Let $\Lambda(p_1, \ldots, p_{n-1})$ be the quotient of $[p_1, \ldots, p_{n-1}]$ by the relations that make each $(\rho_{i-1}\rho_i)^3$ central.

Proposition 4.6. *$\Lambda(p_1, \ldots, p_{n-1})$ is the automorphism group of a regular n-polytope of type $\{p_1, \ldots, p_{n-1}\}$ and with $\dfrac{p_1 \cdots p_{n-1}}{3^{n-1}}(n+1)!$ flags.*

Proof. We start by verifying that the order of each $\rho_{i-1}\rho_i$ is p_i. Let $\Lambda = \Lambda(p_1, \ldots, p_{n-1})$. Clearly Λ covers $[3, \ldots, 3]$, and so the order of each $\rho_{i-1}\rho_i$ is divisible by 3. Now, consider one p_k such that $p_k = 6$. It is straightforward to verify that there is an epimorphism $\pi : \Lambda \to \langle x, y \mid x^2 = y^2 = (xy)^2 = 1 \rangle$ such that

$$\rho_j \pi = \begin{cases} x, & \text{if } j \leqslant k-1 \\ y, & \text{if } j \geqslant k \end{cases}$$

It follows that whenever $p_k = 6$, the order of $\rho_{k-1}\rho_k$ is divisible by 2. It's clear then that the order of each $\rho_{i-1}\rho_i$ is p_i.

The subgroup
$$N = \langle (\rho_0\rho_1)^3, \ldots, (\rho_{n-2}\rho_{n-1})^3 \rangle$$
is central in Λ, and has order $p_1 \cdots p_{n-1}/3^{n-1}$. Furthermore, the quotient of Λ by N is the group of the n-simplex $\{3, \ldots, 3\}$, of order $(n+1)!$. That proves that Λ has the desired order.

It remains to prove that Λ is a string C-group. First, note that $\Lambda(6, p_2, \ldots, p_{n-1})$ covers $\Lambda(3, p_2, \ldots, p_{n-1})$, and this cover is one-to-one on the subgroup $\langle \rho_1, \ldots, \rho_{n-1} \rangle$. Then the quotient criterion (see [13, Thm. 2E17]) implies that the former is a string C-group provided that the latter is. The same argument works with $\Lambda(p_1, \ldots, p_{n-2}, 6)$. So to prove the result, it suffices to prove it for $\Lambda(3, p_2, \ldots, p_{n-2}, 3)$. This already settles the case $n = 3$.

Suppose now that $n \geqslant 4$, that $p_1 = p_{n-1} = 3$, and that the subgroups $\langle \rho_0, \ldots, \rho_{n-2} \rangle$ and $\langle \rho_1, \ldots, \rho_{n-1} \rangle$ are both string C-groups. Let $\varphi \in \langle \rho_0, \ldots, \rho_{n-2} \rangle \cap \langle \rho_1, \ldots, \rho_{n-1} \rangle$. To prove that Λ itself is a string C-group, it suffices to show that $\varphi \in \langle \rho_1, \ldots, \rho_{n-2} \rangle$ (by [13, Prop. 2E16(a)]). Let $\pi : \Lambda \to \Lambda/N \cong [3, \ldots, 3]$, and note that $N \leqslant \langle \rho_1, \ldots, \rho_{n-2} \rangle$ since $p_1 = p_{n-1} = 3$. Denoting the image of ρ_i under π by $\overline{\rho_i}$, we have that $\overline{\varphi}$ lies in $\langle \overline{\rho_0}, \ldots, \overline{\rho_{n-2}} \rangle \cap \langle \overline{\rho_1}, \ldots, \overline{\rho_{n-1}} \rangle$. Since $\Gamma/N = [3, \ldots, 3]$ is a string C-group, it follows that

$\overline{\varphi} \in \langle \overline{\rho_1}, \ldots, \overline{\rho_{n-2}} \rangle$, and thus $\varphi \in \langle \rho_1, \ldots, \rho_{n-2} \rangle N = \langle \rho_1, \ldots, \rho_{n-2} \rangle$. Thus, Λ is a string C-group provided that its facet subgroup and vertex-figure subgroup are string C-groups, and the result follows by induction on the rank of Λ. $\square$

Let $\mathcal{P}(p_1, \ldots, p_{n-1})$ be the polytope (a central extension of a simplex) whose automorphism group is $\Lambda(p_1, \ldots, p_{n-1})$. The group of the vertex-figure is $\Lambda(p_2, \ldots, p_{n-1})$, which has index $(n+1)p_1/3$ in $\Lambda(p_1, \ldots, p_{n-1})$. Thus, the polytope $\mathcal{P}(p_1, \ldots, p_{n-1})$ has $(n+1)p_1/3$ vertices, while its facets $\mathcal{P}(p_1, \ldots, p_{n-2})$ have $np_1/3$ vertices. This shows that these polytopes are not flat.

Next, let us show that any polytope built out of central extensions of simplices is itself a central extension of a simplex.

Proposition 4.7. *Suppose that $n \geqslant 4$, and that $\mathcal{P}$ is an n-polytope with facets isomorphic to $\mathcal{P}(p_1, \ldots, p_{n-2})$ and vertex-figures isomorphic to $\mathcal{P}(p_2, \ldots, p_{n-1})$, then $\mathcal{P} \cong \mathcal{P}(p_1, \ldots, p_{n-1})$.*

Proof. Clearly $\mathcal{P}$ is a quotient of $\mathcal{P}(p_1, \ldots, p_{n-1})$. The facets $\mathcal{K}$ of $\mathcal{P}$ have $np_1/3$ vertices, and so $\mathcal{P}$ itself has at least $np_1/3$ vertices. Since $\mathcal{P}$ is covered by $\mathcal{P}(p_1, \ldots, p_{n-1})$, which has $(n+1)p_1/3$ vertices, the number of vertices of $\mathcal{P}$ must divide $(n+1)p_1/3$. It follows that $\mathcal{P}$ itself has $(n+1)p_1/3$ vertices and that $\mathcal{P} \cong \mathcal{P}(p_1, \ldots, p_{n-1})$. $\square$

Our goal now is to find the several smallest non-flat regular polytopes in each rank.

Proposition 4.8. *Suppose $\mathcal{P}$ is the second smallest non-flat regular n-polytope, and $n \geqslant 3$. Then $\mathcal{P}$ has $2(n+1)!$ flags. Furthermore, if $n \geqslant 4$, then $\mathcal{P}$ is a central extension of a simplex.*

Proof. We prove the claim by induction on n. The claim can be shown to be true for $n = 3$ and $n = 4$, using [12]. Suppose that $n \geqslant 5$. Since $\mathcal{P}$ is the second smallest non-flat regular n-polytope, it is not a simplex. Then up to duality, we may assume that the facets of $\mathcal{P}$ are not simplices. These facets have at least $2n!$ flags, by inductive hypothesis, and there are at least $n+1$ of them (by Corollary 4.2), so $\mathcal{P}$ has at least $2(n+1)!$ flags. On the other hand, the polytopes $\mathcal{P}(p_1, \ldots, p_{n-1})$ with a single $p_i = 6$ have exactly $2(n+1)!$ flags, and so if $\mathcal{P}$ is the second smallest, it must have exactly $2(n+1)!$ flags. It follows that the facets have exactly $2n!$ flags, and by inductive hypothesis, these facets are central extensions of simplices. Similarly, the vertex-figures cannot have more than $2n!$ flags, since there are at least $n+1$ vertices, so the vertex-figures are either simplices or central extensions of simplices. Proposition 4.7 then implies that $\mathcal{P}$ is itself a central extension of a simplex. $\square$

Proposition 4.9. *Suppose $\mathcal{P}$ is the third smallest non-flat regular n-polytope.*

1. *If $n = 3$, then $\mathcal{P}$ has 60 flags.*
2. *If $n = 4$, then $\mathcal{P}$ has 384 flags.*
3. *If $n \geqslant 5$, then $\mathcal{P}$ is a central extension of a simplex, with $4(n+1)!$ flags.*

Proof. For $n = 3, 4$, and 5, we may verify the claim directly using [4]. For $n \geqslant 6$, the proof is essentially the same as the proof of Proposition 4.8. $\square$

Proposition 4.10. *Suppose $\mathcal{P}$ is the fourth smallest non-flat regular n-polytope, with $n \geqslant 5$. Then $\mathcal{P}$ has at least $(16/3)(n+1)!$ flags.*

Proof. The claim can be verified for $n = 5$ using [4]. Now suppose that $n \geqslant 6$. Up to duality, we may assume that the facets of $\mathcal{P}$ have at least as many flags as the vertex-figures. If $\mathcal{P}$ is a central extension of a simplex with more than $4(n+1)!$ flags, then it has at least $8(n+1)!$ flags. Otherwise, if $\mathcal{P}$ is not a central extension of a simplex, then its facets must have at least $(16/3)n!$ flags, and there are at least $n+1$ facets, so $\mathcal{P}$ has at least $(16/3)(n+1)!$ flags. $\square$

Of course, there is no particular reason to stop at the fourth smallest polytopes — except that we have reached the limits of the data we have on small regular polytopes, which has established the base cases in the previous several results. Table 2 summarizes our results.

Rank	smallest	second smallest	third smallest	fourth smallest
3	24	48	60	64
4	120	240	384	480
5	720	1440	2880	3840
$n \geqslant 6$	$(n+1)!$	$2(n+1)!$	$4(n+1)!$	$\geqslant (16/3)(n+1)!$

Table 2: Number of flags of the smallest non-flat regular polytopes

A solution to the following problem would be a good step toward a fuller understanding of small non-flat regular polytopes.

Problem 3. In each rank, determine the smallest non-flat regular polytope that is not a central extension of a simplex.

5 Small chiral polytopes

The restrictions in the previous section help us describe general lower bounds on the size of chiral polytopes. We will need the following result.

Proposition 5.1. *Chiral polytopes have at least 3 vertices and at least 3 facets.*

Proof. If $\mathcal{P}$ is a polytope with 2 vertices, then every edge is incident on both vertices. Thus, the two vertices are indistinguishable, and there is an automorphism of $\mathcal{P}$ that swaps the vertices while fixing all other faces. This yields two adjacent flags that lie in the same orbit, and so $\mathcal{P}$ is not chiral. The proof of the other claim is essentially the same. $\square$

Now we can provide lower bounds on the size of a chiral polytope, depending on whether the facets and vertex-figures are regular or chiral.

Theorem 5.2. *Let $\mathcal{P}$ be a chiral n-polytope with regular facets and vertex-figures. If $n = 5$, then $\mathcal{P}$ has at least 4004 flags. If $n \geqslant 6$ then $\mathcal{P}$ has at least $(16/3)n \cdot n!$ flags.*

Proof. There are only three chiral 5-polytopes with at most 4000 flags, and all have chiral facets and vertex-figures (see [3]). Since the number of flags of a polytope is always divisible by 4, $\mathcal{P}$ must have at least 4004 flags in this case.

Now suppose that $n \geqslant 6$, and that $\mathcal{P}$ is of type $\{\mathcal{K}, \mathcal{L}\}$, with $\mathcal{K}$ and $\mathcal{L}$ regular. By Theorem 3.1, both $\mathcal{K}$ and $\mathcal{L}$ must be non-flat since $\mathcal{P}$ is chiral. Furthermore, either $\mathcal{K}$ or $\mathcal{L}$ must not be a central extension of a simplex, because otherwise Proposition 4.7 would imply that $\mathcal{P}$ is also a central extension of a simplex, which is regular. Up to duality, we may assume that $\mathcal{K}$ is not a central extension of a simplex, and thus it has at least $(16/3)n!$ flags. Since $\mathcal{L}$ is not flat, it has at least n facets, and so $\mathcal{P}$ itself also has at least n facets. Thus $\mathcal{P}$ has at least $(16/3)n \cdot n!$ flags. □

Theorem 5.3. *Let $\mathcal{P}$ be a chiral n-polytope with chiral facets and regular vertex-figures. If $n = 6$, then $\mathcal{P}$ has more than 18432 flags. If $n \geqslant 7$ then $\mathcal{P}$ has at least $16(n-1)(n-1)!$ flags.*

Proof. For $n = 6$, the result follows from [7, Thm 1.1], which proves that the unique smallest chiral 6-polytope has 18432 flags; it has chiral facets and vertex-figures. Let $\mathcal{K}$ be the facet type of $\mathcal{P}$. Since the facets of the facets of a chiral polytope must be regular, $\mathcal{K}$ has regular facets. Furthermore, since the vertex-figures of $\mathcal{K}$ are also the facets of the regular vertex-figures of $\mathcal{P}$, $\mathcal{K}$ has regular vertex-figures. So $\mathcal{K}$ is a chiral $(n-1)$-polytope with regular facets and vertex-figures. By Proposition 5.1, $\mathcal{P}$ must have at least 3 facets, and combining this with Theorem 5.2 yields the desired result. □

Theorem 5.4. *Let $\mathcal{P}$ be a chiral n-polytope with chiral facets and chiral vertex-figures. If $n = 7$, then $\mathcal{P}$ has more than 55296 flags. If $n \geqslant 8$ then $\mathcal{P}$ has at least $48(n-2)(n-2)!$ flags.*

Proof. The proof is essentially the same as for Theorem 5.3; the facets of $\mathcal{P}$ must be chiral with regular facets and either regular or chiral vertex-figures, and there are at least 3 facets. Applying Theorem 5.2 and (the dual of) Theorem 5.3 for the facets provides the desired result. □

We note that if $n \geqslant 8$, then $48(n-2)(n-2)! < 16(n-1)(n-1)! < (16/3)n \cdot n!$, providing us with the following theorem.

Theorem 5.5. *For $n \geqslant 8$, a chiral n-polytope has at least $48(n-2)(n-2)!$ flags.*

Compare Theorem 5.5 to [2, Thm. 1.1], which states that for $n \geqslant 9$, the smallest nondegenerate regular n-polytope has $2 \cdot 4^{n-1}$ flags. Note that these regular polytopes are all flat, and in fact, they all have flat regular facets and flat regular vertex-figures.

5.1 The smallest chiral polytopes in each rank

The smallest chiral n-polytopes for $n = 3, 4$, and 5 can be found in [5]. In rank 3, the smallest chiral polytope is the torus map $\{4, 4\}_{(1,2)}$, with 40 flags. In rank 4, the smallest chiral polytopes have 240 flags. This includes the universal $\{\{4, 4\}_{(1,2)}, \{4, 3\}\}$ and the universal $\{\{4, 4\}_{(2,1)}, \{4, 4\}_{(1,2)}\}$; the former has chiral facets and regular vertex-figures, and the latter has chiral facets and chiral vertex-figures. The smallest chiral 5-polytope is the universal $\{\{\{3, 4\}, \{4, 4\}_{(2,1)}\}, \{\{4, 4\}_{(1,2)}, \{4, 3\}\}\}$, with 1440 flags. This polytope has chiral facets and vertex-figures.

The smallest chiral 4-polytope with regular facets and vertex-figures is a polytope of type $\{3, 3, 8\}$ with 384 flags. It has automorphism group

$$\langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^3 = \sigma_2^3 = \sigma_3^8 = (\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = \sigma_3^{-1}\sigma_1\sigma_3\sigma_2^{-1}\sigma_1\sigma_3^{-2}\sigma_2 = 1\rangle.$$

There are no chiral 5-polytopes with up to 4000 flags and regular facets or vertex-figures (see [3]). The smallest known chiral 5-polytopes with either regular facets or regular vertex-figures are described in [7]. One is of type $\{3, 3, 4, 6\}$, with regular facets and chiral vertex-figures, and the other is of type $\{3, 4, 6, 3\}$, with chiral facets and regular vertex-figures. Both have 4608 flags. The smallest chiral 6-polytope has the former 5-polytope as facets, the latter as vertex-figures, and has 18432 flags. We summarize our data in Table 3.

Rank	regular facets regular vertex-figures	chiral facets regular vertex-figures	chiral facets chiral vertex-figures
3	40	-	-
4	384	240	240
5	$\geqslant 4004$	$\geqslant 4004, \leqslant 4608$	1440
6	$\geqslant 23040$	> 18432	18432
7	$\geqslant 188160$	$\geqslant 69120$	> 55296
$n \geqslant 8$	$\geqslant (16/3)n \cdot n!$	$\geqslant 16(n-1)(n-1)!$	$\geqslant 48(n-2)(n-2)!$

Table 3: Number of flags of the smallest chiral polytopes

Let $f_{rr}(n)$, $f_{cr}(n)$, and $f_{cc}(n)$ be the minimal number of flags among chiral n-polytopes with regular facets and vertex-figures, with chiral facets and regular vertex-figures, and with chiral facets and vertex-figures, respectively. It is straightforward to prove that $f_{cr}(n) \geqslant 3f_{rr}(n-1)$ and that $f_{cc}(n) \geqslant 3f_{cr}(n-1)$. From the data available, it seems likely that $f_{rr}(n) \geqslant f_{cr}(n) \geqslant f_{cc}(n)$, but it is unclear whether this trend will continue to hold.

Problem 35 in [15] asks to find the size of the smallest chiral n-polytope for each n. It may be useful to split that problem into the following subproblems:

Problem 4. Determine the functions $f_{rr}(n)$, $f_{cr}(n)$, and $f_{cc}(n)$.

Problem 5. Determine whether it is always the case that $f_{rr}(n) \geqslant f_{cr}(n) \geqslant f_{cc}(n)$.

The next smallest step in this direction would be to solve the following problem:

Problem 6. Determine the smallest chiral 5-polytope with regular facets and regular vertex-figures.

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