Perfect fractional matchings in k-out hypergraphs*

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Abstract

Extending the notion of (random) k-out graphs, we consider when the k-out hypergraph is likely to have a perfect fractional matching. In particular, we show that for each r there is a $k = k(r)$ such that the k-out r-uniform hypergraph on n vertices has a perfect fractional matching with high probability (i.e., with probability tending to 1 as $n \to \infty$) and prove an analogous result for r-uniform r-partite hypergraphs. This is based on a new notion of hypergraph expansion and the observation that sufficiently expansive hypergraphs admit perfect fractional matchings. As a further application, we give a short proof of a stopping-time result originally due to Krivelevich.

Keywords: random hypergraphs, perfect fractional matchings, k-out model, hypergraph expansion

1 Introduction

Hypergraphs constitute a far-reaching generalization of graphs and a basic combinatorial construct but are notoriously difficult to work with. A *hypergraph* is a collection $\mathcal{H}$ of subsets (*"edges"*) of a set V of *"vertices."* Such an $\mathcal{H}$ is r*-uniform* (or an r*-graph*) if each edge has cardinality r (so 2-graphs are graphs). A *perfect matching* in a hypergraph is a collection of edges partitioning the vertex set. For any $r > 2$, deciding whether an r-graph has a perfect matching is an NP-complete problem [18]; so instances of the problem tend to be both interesting and difficult. Of particular interest here has been trying to understand conditions under which a *random* hypergraph is likely to have a perfect matching.

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The most natural model of a random r-graph is the "Erdős-Rényi" model, in which each r-set is included in $\mathcal{H}$ with probability p, independent of other choices. One is then interested in the "threshold," roughly, the order of magnitude of $p = p_r(n)$ required to make a perfect matching likely. Here the graph case was settled by Erdős and Rényi [7, 8], but for $r > 2$ the problem—which became known as Shamir's Problem following [6]—remained open until [17]. In each case, the obvious obstruction to containing a perfect matching is existence of an isolated vertex (that is, a vertex contained in no edges), and a natural guess is that this is the *main* obstruction. A literal form of this assertion—the *stopping time* version—says that if we choose random edges *sequentially*, each uniform from those as yet unchosen, then we w.h.p.[1] have a perfect matching as soon as all vertices are covered. This nice behavior does hold for graphs [3], but for hypergraphs remains conjectural (though at least the value it suggests for the threshold is correct).

An interesting point here is that taking p large enough to avoid isolated vertices produces many more edges than other considerations—e.g., wanting a large *expected* number of perfect matchings—suggest. This has been one motivation for the substantial body of work on models of random graphs in which isolated vertices are automatically avoided, notably random *regular* graphs (e.g., [23]) and the k-out model. The generalization of the latter to hypergraphs, which we now introduce, will be our main focus here.

> **The k-out model.** For a ("host") hypergraph $\mathcal{H}$ on V, $\mathcal{H}(k\text{-out})$ is the random subhypergraph $\cup_{v\in V} E_v$, where E_v is chosen uniformly from the k-subsets of $\mathcal{H}_v := \{A \in \mathcal{H} : v \in A\}$ (or—but we won't see this—$E_v = \mathcal{H}_v$ if $|\mathcal{H}_v| < k$), these choices made independently.

The k-out model for $\mathcal{H} = K_{n,n}$ (the complete bipartite graph) was introduced by Walkup [22], who showed that w.h.p. $K_{n,n}(2\text{-out})$ is Hamiltonian, so in particular contains a perfect matching, and Frieze [13] proved the nonbipartite counterpart of the matching result, showing that $K_{2n}(2\text{-out})$ has a perfect matching w.h.p. (Hamiltonicity in the latter case turned out to be more challenging; it was studied in [9, 14, 4] and finally resolved by Bohman and Frieze [2], who proved $K_n(3\text{-out})$ is Hamiltonian w.h.p.). The idea of a general host G was introduced by Frieze and T. Johansson [11]; see also e.g., Ferber *et al.* [10] for (*inter alia*) a nice connection with $G_{n,p}$.

For *hypergraphs* the k-out model seems not to have been studied previously (random regular hypergraphs *have* been considered, e.g., in [5]). Here the two most important examples would seem to be $\mathcal{H} = K_n^{(r)}$ (the complete r-graph on n vertices) and $\mathcal{H} = K_{[n]^r}$ (the complete r-partite r-graph with n vertices in each part). It is natural to expect that for each of these there is some $k = k(r)$ for which $\mathcal{H}(k\text{-out})$ has a perfect matching w.h.p.. Note that, while almost certainly correct, these are likely to be difficult, as either would imply the aforementioned resolution of Shamir's Problem, as well as the following natural guess regarding a beautiful problem of Frieze and Sorkin [12], which is what got us interested in the first place.

[1] As usual we use *with high probability (w.h.p.)* to mean with probability tending to 1 as the relevant parameter—here always n—tends to infinity.

Conjecture 1. *Fix r and let $w(A)$, $A \in K^r_{rn}$, be independent Exp(1) "weights." Then w.h.p. some perfect matching has weight $O(n^{-(r-1)})$.*

(And similarly for $\mathcal{H} = K_{[n]^r}$, which is the version in [12]; the (corresponding) conjecture is not stated there, but, again, is an obvious guess.)

These cautionary notes notwithstanding, we would like to regard the following linear relaxations as small steps toward *actual* perfect matchings in the k-out model. (Relevant definitions are recalled in Section 2.)

Theorem 2. *For each r, there is a k such that w.h.p. $K_n^{(r)}$(k-out) admits a perfect fractional matching and $w \equiv 1/r$ is the only fractional cover of weight n/r.*

Theorem 3. *For each r, there is a k such that w.h.p. $\mathcal{H} = K_{[n]^r}$(k-out) admits a perfect fractional matching and each minimum weight fractional cover of $\mathcal{H}$ is constant on each block of the r-partition.*

Our upper bounds on the k's are quite large (roughly r^r), but in fact we don't even know that they must be larger than 2 (though this sounds optimistic), and we make no attempt to optimize. In the more interesting case of (ordinary) perfect matchings, consideration of the expected number of perfect matchings shows that k does need to be be at least exponential in r.

We will make substantial use of the next observation (or, in the r-partite case, of the analogous Proposition 7, whose statement we postpone), in which the notion of expansion may be of some interest. Recall that an *independent set* in a hypergraph is a set of vertices containing no edges.

Proposition 4. *Suppose $\mathcal{H}$ is an r-graph in which, for all disjoint $X, Y \subseteq V$ with X independent and*

$$|Y| < (r-1)|X|, \tag{1}$$

there is some edge meeting X but not Y. Then $\mathcal{H}$ has a perfect fractional matching. If, moreover we replace "$<$" by "$\leqslant$" in (1), then $w \equiv 1/r$ is the only fractional cover of weight n/r.

It's not hard to see that for $r > 2$ the proof of this can be tweaked to give the stronger conclusion even under the weaker hypothesis. (For $r = 2$ this is clearly false, e.g., if G is a matching.)

Related notions of expansion (respectively stronger than and incomparable to ours) appear in [19] and [15]. An additional application of Proposition 4, given in Section 5, is a short alternate proof of the following result of Krivelevich [19].

Theorem 5. *Let $\{\mathcal{H}_t\}_{t \geqslant 0}$ denote the random r-graph process on V in which each step adds an edge chosen uniformly from the current non-edges, let T denote the first t for which $\mathcal{H}_t$ has no isolated vertices. Then $\mathcal{H}_T$ has a perfect fractional matching w.h.p..*

Outline. Section 2 includes definitions and brief linear programming background. Section 3 treats $K_n^{(r)}$, proving Proposition 4 and Theorem 2, and the corresponding results for $K_{[n]^r}$ are proved in Section 4. Finally, Section 5 returns to $K_n^{(r)}$, using Proposition 4 to give an alternate proof of Theorem 5.

2 Preliminaries

Except where otherwise specified, $\mathcal{H}$ is an r-graph on $V = [n]$. As usual, we use $[t]$ for $\{1, 2, \ldots, t\}$ and $\binom{X}{t}$ for the collection of t-element subsets of X. Throughout we use log for ln and take asymptotics as $n \to \infty$ (with other parameters fixed), pretending (following a common abuse) that all large numbers are integers and assuming n is large enough to support our arguments.

We need to recall a minimal amount of linear programming background (see e.g., [21] for a more serious discussion). For a hypergraph $\mathcal{H}$, a *fractional (vertex) cover* is a map $w : V \to [0, 1]$ such that $\sum_{v \in e} w(v) \geqslant 1$ for all $e \in \mathcal{H}$; the *weight* of a cover w is $|w| = \sum_v w(v)$; and the *fractional cover number*, $\tau^*(\mathcal{H})$, is the smallest such weight. Similarly a *fractional matching* of $\mathcal{H}$ is a $\varphi : \mathcal{H} \to [0, 1]$ such that $\sum_{e \ni v} \varphi(e) \leqslant 1$ for all $v \in V$; the weight of such a φ is defined as for fractional covers; and the *fractional matching number*, $\nu^*(\mathcal{H})$, is the *maximum* weight of a fractional matching.

In this context, LP-duality says that $\nu^*(\mathcal{H}) = \tau^*(\mathcal{H})$ for any hypergraph. For r-graphs the common value is trivially at most n/r (e.g., since $w \equiv 1/r$ is a fractional cover). A fractional matching in an r-graph is *perfect* if it achieves this bound; that is, if $\sum \varphi_e = n/r$ (equivalently $\sum_{e \ni v} \varphi_e = 1 \ \forall v$, which would be the definition of perfection in a nonuniform $\mathcal{H}$).

Finally, given $\mathcal{H}$ we say a nonempty $X \subseteq V$ is λ-*expansive* if for all $Y \subseteq V \setminus X$ of size at most $\lambda|X|$, there is some edge meeting X but not Y.

3 Proofs of Proposition 4 and Theorem 2

Proof of Proposition 4. It is enough to show that if w is a fractional cover with $t_0 := 1/r - \min_v w(v) > 0$, then $|w| \geqslant n/r$, with the inequality strict if we assume the stronger version of (1). We give the argument under this stronger assumption; for the weaker, just replace the few strict inequalities below by nonstrict ones. Given w as above, set, for each $t > 0$,

$$W_t = \{v \in [n] \ : \ w(v) \leqslant \tfrac{1}{r} - t\}, \quad W^t = \{v \in [n] \ : \ w(v) \geqslant \tfrac{1}{r} + t\}.$$

Since w is a fractional cover, each edge meeting W_t must also meet $W^{t/(r-1)}$ (or the weight on the edge would be less than 1); so, since W_t is independent, the hypothesis of Proposition 4 gives $|W^{t/(r-1)}| > (r-1)|W_t|$ for $t \in (0, t_0]$ (the t's for which $W_t \neq \emptyset$).

For $s \in \mathbb{R}$, define $f(s) = |\{v \in [n] \ : \ w(v) \geqslant s\}|$. Then

$$\begin{aligned}\int_0^1 f(s)\, ds &= \int_0^1 \sum_{v \in [n]} \mathbf{1}_{\{w(v) \geqslant s\}}\, ds \\ &= \sum_{v \in [n]} \int_0^1 \mathbf{1}_{\{w(v) \geqslant s\}}\, ds = \sum_{v \in [n]} w(v) = \tau^*(\mathcal{H}).\end{aligned}$$

We also have $|W^t| = f(1/r + t)$ and $|W_t| \geqslant n - f(1/r - t)$, implying

$$f(1/r + t/(r-1)) \geqslant (r-1)(n - f(1/r - t)),$$

with the inequality strict if $t \in (0, t_0]$. Thus,

$$\begin{aligned}
\tau^*(\mathcal{H}) &= \int_0^1 f(s)\,ds = \int_0^{1/r} f(s)\,ds + \int_{1/r}^1 f(s)\,ds \\
&= \int_0^{1/r} f(1/r - t)\,dt + \int_0^{(r-1)^2/r} \frac{f(1/r + t/(r-1))}{r-1}\,dt \\
&\geqslant \int_0^{1/r} \left[f(1/r - t) + \frac{f(1/r + t/(r-1))}{r-1} \right] dt \\
&> \int_0^{1/r} \left[f(1/r - t) + (r-1)\frac{n - f(1/r - t)}{r-1} \right] dt = \frac{n}{r}. \qquad \square
\end{aligned}$$

We should perhaps note that the converse of Proposition 4 is not true in general (failing, e.g., if $r > 2$ and $\mathcal{H}$ is itself a perfect matching). But in the graphic case ($r = 2$) the converse *is* true (and trivial), and the proposition provides an alternate proof of the following characterization, which is [20, Thm. 2.2.4] (and is also contained in [1, Thm. 2.1], e.g.).

Corollary 6. *A graph has a perfect fractional matching iff $|N(I)| \geqslant |I|$ for all independent I.*

(where $N(I)$ is the set of vertices with at least one neighbor in I).

Proof of Theorem 2. Given r, let (without trying to optimize) $k = (2r^2)^r$ and $c = k^{-1/r} = 1/(2r^2)$, and let $\mathcal{H} = K_n^{(r)}(k\text{-out})$. Theorem 2 (with this k) is an immediate consequence of Proposition 4 and the next two routine lemmas. (As usual $\alpha(\mathcal{H})$ is the size of a largest independent set in $\mathcal{H}$.) $\square$

Lemma 3.1. *W.h.p. $\alpha(\mathcal{H}) < cn$.*

Lemma 3.2. *W.h.p. every $X \subseteq V(\mathcal{H})$ with $|X| \leqslant cn$ is $(r-1)$-expansive.*

Proof of Lemma 3.1. The probability that $S \in \binom{[n]}{s}$ is independent in $\mathcal{H}$ is

$$\left[1 - \tfrac{(s-1)_{r-1}}{(n-1)_{r-1}}\right]^{sk} < \exp\left[-sk\left(\tfrac{s-r}{n}\right)^{r-1}\right].$$

(where $(a)_b = a(a-1)\cdots(a-b+1)$), and summing this over S of size cn bounds $\mathbb{P}(\alpha \geqslant cn)$ by

$$2^n \exp\left[-cnk(c - r/n)^{r-1}\right] = \exp\left[n\left(\ln 2 - (1 - o(1))kc^r\right)\right],$$

which tends to 0 as desired. $\square$

Proof of Lemma 3.2. For X, Y disjoint subsets of $[n]$, let $B(X,Y)$ be the event that Y meets all edges meeting X. Then, with $x = |X|$ and $y = |Y|$,

$$\mathbb{P}(B(X,Y)) \leqslant \left[1 - \tfrac{(n-y-1)_{r-1}}{(n-1)_{r-1}}\right]^{kx} \leqslant \left[1 - \left(\tfrac{n-y-r}{n}\right)^{r-1}\right]^{kx} \leqslant \left[\tfrac{r(y+r)}{n}\right]^{kx},$$

the last inequality following from

$$1 - (1-x)^m \leqslant mx \tag{2}$$

(valid for $x \in [0,1]$ and nonnegative integer m). The probability that the conclusion of the lemma fails is thus less than

$$\begin{aligned}\sum \tbinom{n}{rx}\tbinom{rx}{x}\left[\tfrac{r(y+r)}{n}\right]^{kx} &< \sum \left(\tfrac{ne}{rx}\right)^{rx} 2^{rx} \left[\tfrac{r(y+r)}{n}\right]^{kx} \\ &= \sum \left[(2e)^r \left(\tfrac{rx}{n}\right)^{k-r} ((r-1)+r/x)^k\right]^x \\ &< \sum \left[(4er)^r (r(2r-1)x/n)^{k-r}\right]^x \ = o(1),\end{aligned}$$

where the sums are over $1 \leqslant x \leqslant cn$. $\square$

4 Proof of Theorem 3

As in the proof of Theorem 2 we first show that the conclusions of Theorem 3 are implied (deterministically) by sufficiently good expansion and then show that $K_{[n]^r}(k\text{-out})$ w.h.p. expands as desired. We take $V = V_1 \cup \cdots \cup V_r$ to be our r-partition (so $|V_i| = n \ \forall i$) and below always assume $\mathcal{H} \subseteq K_{[n]^r}$.

Proposition 7. *Suppose $\varepsilon \in (0,1/2)$ and $\lambda > 2r^3$ are fixed and $\mathcal{H}$ satisfies: for any $i \in [r]$, $T \subseteq V_i$, $U_j \subseteq V_j$ for $j \neq i$ and $U = \cup_{j\neq i} U_j$, there is an edge meeting T but not U provided either*

(i) $|T| \leqslant \varepsilon n$ and $|U_j| \leqslant \lambda |T| \ \forall j \neq i$, or

(ii) $|T| \geqslant \varepsilon n$ and $|U_j| \leqslant (1-\varepsilon)n \ \forall j \neq i$.

Then $\mathcal{H}$ admits a perfect fractional matching, and every minimum weight fractional cover of $\mathcal{H}$ is constant on each V_i.

Proof. Define a *balanced assignment* to be a $w : V \to \mathbb{R}$ with $\sum_{v\in V_i} w(v) = 0$ and $w(e) \geqslant 0$ for all $e \in \mathcal{H}$.

We claim that (under our hypotheses) the only balanced assignment is the trivial $w \equiv 0$. To get Proposition 7 from this, let f be a minimum weight fractional cover, and let $w_f(v) = f(v) - \sum_{u\in V_i} f(u)/n$, for each i and $v \in V_i$. Then w_f is a balanced assignment: $\sum_{v\in V_i} w_f(v) = 0$ is obvious and nonnegativity holds since $f(e) \geqslant 1$ and, by minimality, $\sum_{v\in V} f(v) \leqslant n$. Thus $w_f \equiv 0$, implying f is as promised.

Suppose then that w is a balanced assignment. For $X \subseteq V$ and $t \geqslant 0$, set $X^t = \{v \in X : w(v) \geqslant t\}$, $X_t = \{v \in X : w(v) < -t\}$, $X^+ = X^0$ and $X^- = X_0$, and define the *value* of X to be $\psi(X) = \sum_{v\in X} |w(v)|$. Let $S = \{i \in [r] : |V_i^-| \leqslant \varepsilon n\}$ and $B = [r] \setminus S$.

Lemma 4.1. *If $X \subseteq V^-$ and $|X| \leqslant \varepsilon n$, then $\psi(X) \leqslant r\psi(V^+)/\lambda$.*

Proof. For any $t > 0$, note that every edge meeting X_t meets $V^{t/(r-1)}$ since otherwise, we could find an edge of negative weight. So since $|X_t| \leqslant |X| \leqslant \varepsilon n$, condition (i) implies $|V^{t/(r-1)}| \geqslant \lambda |X_t|$. Thus,

$$\begin{aligned}\psi(V^+) &= \int_0^\infty |V^u|\, du = \frac{1}{r-1}\int_0^\infty |V^{t/(r-1)}|\, dt \\ &\geqslant \frac{\lambda}{r-1}\int_0^\infty |X_t|\, dt = \frac{\lambda}{r-1}\psi(X).\end{aligned}$$

□

Lemma 4.2. *If $|(V_i)_t| \geqslant \varepsilon n$, then $\max_{j\in S} |V_j^{t/(r-1)}| \geqslant (1-\varepsilon)n$.*

Proof. Since any edge meeting $(V_i)_t$ meets $\cup_{j\neq i} V_j^{t/(r-1)}$ and $|V_j^+| \leqslant (1-\varepsilon)n$ for $j \in B$, there must (see (ii)) be some $j \in S$ with $|V_j^{t/(r-1)}| \geqslant (1-\varepsilon)n$. □

We now claim $\psi(V_i) \leqslant 2r^2\psi(V)/\lambda$ for all i. For $i \in S$, we do a little better: Lemma 4.1 gives $\psi(V_i^-) \leqslant r\psi(V^+)/\lambda$, and balance (of w) then implies $\psi(V_i) = 2\psi(V_i^-) \leqslant r\psi(V)/\lambda$. For $i \in B$ write W for V_i (just to avoid some double subscripts) and set $T = \sup\{t : |W_t| \geqslant \varepsilon n\}$. Then

$$\psi(W^-) = \psi(W_T) + \psi(W^- \setminus W_T) \leqslant \psi(W_T) + T|W^- \setminus W_T|.$$

Since $|W_T| < \varepsilon n$, Lemma 4.1 gives $\psi(W_T) \leqslant r\psi(V^+)/\lambda$. On the other hand, $|W_t| \geqslant \varepsilon n$ for $t \in [0,T)$, with Lemma 4.2, implies that there is a $j \in S$ with $|V_j^{t/(r-1)}| \geqslant (1-\varepsilon)n$ for all such t. Thus

$$\begin{array}{rcl}(1-\varepsilon)T|W^-\setminus W_T| & \leqslant & (1-\varepsilon)nT \;\leqslant\; \int_0^T |V_j^{t/(r-1)}|\, dt \leqslant \int_0^\infty |V_j^{t/(r-1)}|\, dt \\ & = & (r-1)\psi(V_j^+) \;\leqslant\; r^2\psi(V^+)/\lambda.\end{array}$$

So, combining, we have $\psi(W) = 2\psi(W^-) \leqslant 2r^2\psi(V)/\lambda$ (establishing the claim) and

$$\psi(V) = \textstyle\sum_i \psi(V_i) \leqslant 2r^3\psi(V)/\lambda.$$

But since $2r^3 < \lambda$, this forces $\psi(V) = 0$ and so $w \equiv 0$. □

Proof of Theorem 3. Set $\lambda = 4r^3$, $\varepsilon = (2r\lambda)^{-1}$ and $k = 2r\varepsilon^{-r}$ (so k is a little more than r^{4r}). We show that w.h.p. $\mathcal{H} = K_{[n]^r}(k\text{-out})$ is as in Proposition 7. As earlier, let $B(X,Y)$ be the event that every edge meeting X meets Y.

Suppose first that T and U are fixed with $|U_i| = \lambda|T| \leqslant \lambda\varepsilon n$. Then

$$\mathbb{P}(B(T,U)) \leqslant \left[1 - \left(1 - \tfrac{\lambda|T|}{n}\right)^{r-1}\right]^{k|T|} \leqslant \left(\tfrac{r\lambda|T|}{n}\right)^{k|T|}.$$

Summing over choices of T and U bounds the probability that $\mathcal{H}$ violates the assumptions of the proposition for some T and U as in (i) by

$$\begin{aligned} r\sum_{t=1}^{\varepsilon n}\binom{n}{t}\binom{n}{\lambda t}^{r-1}\left(\frac{r\lambda t}{n}\right)^{kt} &\leqslant r\sum_{t=1}^{\varepsilon n}\left(\frac{en}{t}\right)^{t}\left(\frac{en}{\lambda t}\right)^{\lambda t(r-1)}\left(\frac{r\lambda t}{n}\right)^{kt} \\ &\leqslant \sum_{t=1}^{\varepsilon n}\left[(r\lambda t/n)^{k-r\lambda}\lambda(er)^{r\lambda}\right]^{t}=o(1). \end{aligned}$$

Now say T and U are fixed with $|T|=\varepsilon n$ and $|U_i|=(1-\varepsilon)n$. Then

$$\mathbb{P}(B(T,U))\leqslant(1-\varepsilon^{r-1})^{k|T|}\leqslant\exp\left[-k|T|\varepsilon^{r-1}\right]\leqslant\exp\left[-kn\varepsilon^{r}\right].$$

So summing over possibilities for (T,U) bounds the probability of a violation with T and U as in (ii) by

$$r2^{nr}\exp\left[-kn\varepsilon^{r}\right]\leqslant\exp\left[n(r-k\varepsilon^{r})\right]=o(1).$$

□

5 Proof of Theorem 5

We now turn to our proof of Theorem 5, for which we work with the following standard device for handling the process $\{\mathcal{H}_t\}$.

Let ξ_S, $S\in\binom{[n]}{r}$, be independent random variables, each uniform from $[0,1]$, and for $\lambda\in[0,1]$, let $G(\lambda)$ be the r-graph on $[n]$ with edge set $\mathcal{E}(\lambda)=\{S\ :\ \xi_S\leqslant\lambda\}$. Members of $\mathcal{E}(\lambda)$ will be called *λ-edges*. Note that with probability one, $G(0)$ is empty, $G(1)$ is complete, and the ξ_S's are distinct.

Provided the ξ_S's are distinct, this defines the discrete process $\{\mathcal{H}_t\}$ in the natural way, namely by adding edges S in the order in which their associated ξ_S's appear in $[0,1]$. We will work with the following quantities, where $\gamma=\varepsilon\log n$ for some small fixed (positive) ε and g is a suitably slow $\omega(1)$.

- $\Lambda=\min\{\lambda : G(\lambda)\text{ has no isolated vertices}\}$;
- $W_\lambda=\{v\in[n]\ :\ d_{G(\lambda)}(v)\leqslant\gamma\}$;
- $\sigma=\frac{\log n-g(n)}{\binom{n-1}{r-1}}$ and $\beta=\frac{\log n+g(n)}{\binom{n-1}{r-1}}$;
- $N=\{v:\exists e\in\mathcal{E}(\beta),\ v\in e,\ e\cap W_\sigma\neq\emptyset\}$

(so N is W_σ together with its $\mathcal{E}(\beta)$-neighbors).

Preview. With the above framework, our assignment is to show that $G(\Lambda)$ has a perfect matching w.h.p.. Perhaps the nicest part of this—and the point of coupling the different $G(\lambda)$'s—is that, so long as $\Lambda\in[\sigma,\beta]$, which we will show holds w.h.p., the desired assertion on $G(\Lambda)$ follows *deterministically* from a few properties ((b)-(d) of Lemma 5.1) involving $G(\sigma)$, $G(\beta)$ *or both*; so by showing that the latter properties hold w.h.p. we avoid the need for a union bound to cover possibilities for Λ. Production of the fractional matching is then similar to (though somewhat simpler than) what happens in [19]: the relatively few vertices of W_Λ (and some others) are covered by an (ordinary) matching, and the hypergraph induced by what's left has the expansion needed for Proposition 4.

Lemma 5.1. *With the above setup (for fixed r) and $Z = n(\log n)^{-1/r}$, w.h.p.*

(a) $\Lambda \in [\sigma, \beta]$*;*

(b) $\alpha(G(\sigma)) < Z$*;*

(c) no β-edge meets W_σ more than once and no $u \notin W_\sigma$ lies in more than one β-edge meeting $N \setminus \{u\}$;

(d) each $X \subseteq V \setminus W_\sigma$ of size at most Z is r-expansive in $G(\sigma)$.

Proof. For (a), note that the expected number of isolated vertices in $G(\lambda)$ is $h(\lambda) := n(1-\lambda)^{\binom{n-1}{r-1}}$. The upper bound (i.e. $\Lambda < \beta$ w.h.p.) then follows from $h(\beta) = o(1)$, and the lower bound is given by Chebyshev's Inequality (applied to the number of isolated vertices).

For (b), we have

$$\begin{aligned} \mathbb{P}(\alpha(G(\beta)) \geqslant Z) &< \tbinom{n}{Z}(1-\beta)^{\binom{Z}{r}} < (en/Z)^Z \exp\left[-\beta\tbinom{Z}{r}\right] \\ &= \exp\left[Z\log(en/Z) - (1-o(1))(n/r)\log n(Z/n)^r\right] \\ &= \exp\left[Z\log(en/Z) - \Omega(n)\right] = o(1). \end{aligned}$$

The proofs of (c) and (d) are similarly routine but take a little longer. Aiming for (c), set $p = \mathbb{P}(\zeta \leqslant \gamma)$, where ζ is binomial with parameters $\binom{n-2}{r-1}$ and σ. Since $\mu := \mathbb{E}\zeta \sim \log n$, a standard large deviation estimate (e.g., [16, Thm. 2.1]) gives

$$p < \exp[-\mu\varphi(-(\mu-\gamma)/\mu)] < n^{-1+\delta},$$

where $\varphi(x) = (x+1)\log(x+1) - x$ for $x \geqslant -1$ and $\delta \approx \varepsilon\log(1/\varepsilon)$.

Failure of the first assertion in (c) implies existence of $S \in K_n^{(r)}$ and (distinct) $u, v \in S$ with $S \in G(\beta)$ and $u, v \in W_\sigma$. The probability that this occurs for a given S, u, v is less than βp^2 (the p^2 bounding the probability that each of u, v lies in at most γ edges not containing the other), so the probability that the assertion fails is less than

$$\tbinom{n}{r} r^2 \beta p^2 \sim nr(\log n)p^2 = o(1).$$

If the second part of (c) fails, then we must be able to find a $u \notin W_\sigma$ as well as one of the following configurations, in which $x, y \in W_\sigma$, $S_i \in G(\beta)$, and $a, b \in [n]$ (and vertices and edges within a configuration are distinct):

(i) x, S_1, S_2 with $x, u \in S_1 \cap S_2$;

(ii) x, y, S_1, S_2 with $x, u \in S_1$, $y, u \in S_2$;

(iii) x, a, S_1, S_2, S_3 with $x, u \in S_1$, $x, a \in S_2$, $u, a \in S_3$;

(iv) x, y, a, S_1, S_2, S_3 with $x, u \in S_1$, $y, a \in S_2$, $u, a \in S_3$;

(v) x, a, S_1, S_2, S_3 with $x, a \in S_1,\ u, a \in S_2 \cap S_3$;

(vi) $x, a, b, S_1, S_2, S_3, S_4$ with $x, a \in S_1,\ x, b \in S_2,\ u, a \in S_3,\ u, b \in S_4$;

(vii) $x, y, a, b, S_1, S_2, S_3, S_4$ with $x, a \in S_1,\ y, b \in S_2,\ u, a \in S_3,\ u, b \in S_4$;

(viii) x, a, b, S_1, S_2, S_3 with $x, a, b \in S_1, u, a \in S_2, u, b \in S_3$.

Thus, with $M = \binom{n-2}{r-2}$, summing probabilities for these possibilities bounds the probability of violating the second part of (c) by

$$n^2pM^2\beta^2 + n^3p^2M^2\beta^2 + n^3pM^3\beta^3 + n^4p^2M^3\beta^3 + n^3pM^3\beta^3 \\ + n^4pM^4\beta^4 + n^5p^2M^4\beta^4 + n^4pM^2\tbinom{n-3}{r-3}\beta^3 = o(1).$$

For (d) it is enough to bound (by $o(1)$) the probability that for some (nonempty) $X \subseteq V$ of size $x \leqslant Z$ and $Y \subseteq V \setminus X$ of size rx,

$$\textit{there are at least } \gamma x/r\ \sigma\textit{-edges meeting both } X \textit{ and } Y. \tag{3}$$

For given X, Y the expected number of such edges is less than

$$x \cdot rx\tbinom{n-2}{r-2}\sigma < xr^2\tfrac{Z\log n}{n-1} =: bx.$$

(The first inequality is a significant giveaway for small x, but we have lots of room.) So, again using [16, Thm. 2.1], we find that the probability of (3) is less than

$$\exp[-(\gamma x/r)\log(\gamma/(erb)] < \exp[-\Omega(\gamma x \log\log n)],$$

while the number of possibilities for (X, Y) is less than

$$\tbinom{n}{x}\tbinom{n}{rx} < \exp[(r+1)x(1+\log(n/x))] = \exp[O(x\log n)],$$

and the desired $o(1)$ bound follows. □

Proof of Theorem 5. By Lemma 5.1 it is enough to show that if (a)-(d) of the lemma hold then $G(\Lambda)$ has a perfect fractional matching; so we assume we have these conditions and proceed (working in $G(\Lambda)$).

According to (c) (and the definition of Λ), $G(\Lambda)$ admits a *matching*, M, covering W_σ (each edge of which contains exactly one vertex of W_σ). Let W be the set of vertices covered by M (so W consists of W_σ plus some subset of $N \setminus W_\sigma$), and $H = G(\Lambda) - W$ (as usual meaning that the edges of H are the edges of $G(\Lambda)$ that miss W). It is enough to show that H has a perfect fractional matching, which will follow from Proposition 4 if we show

$$\text{each independent set } X \text{ of } H \text{ is } (r-1)\text{-expansive.} \tag{4}$$

Proof. Since such an X is also independent in $G(\sigma)$, (b) gives $|X| \leqslant Z$, and (d) then says X is r-expansive in $G(\sigma)$, *a fortiori* in $G(\Lambda)$. On the other hand, since $X \cap W_\sigma = \emptyset$, (c) guarantees that the β-edges (so also the Λ-edges) meeting X and *not* contained in $V(H)$ can be covered by some $U \subseteq W$ of size at most $|X|$ (namely, (c) says each $x \in X$ lies in at most one such edge). It follows that the Λ-edges meeting X that *do* belong to H cannot be covered by $(r-1)|X|$ vertices of $V(H) \setminus X$. □

□

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