Asymptotic behavior of Odd-Even partitions

Min-Joo Jang*

Mathematical Institute
University of Cologne
50931 Cologne, Germany

min-joo.jang@uni-koeln.de

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Abstract

Andrews studied a function which appears in Ramanujan's identities. In Ramanujan's "Lost" Notebook, there are several formulas involving this function, but they are not as simple as the identities with other similar shape of functions. Nonetheless, Andrews found out that this function possesses combinatorial information, odd-even partition. In this paper, we provide the asymptotic formula for this combinatorial object. We also study its companion odd-even overpartitions.

Keywords: Odd-Even partitions; Overpartitions; Asymptotics; Wright's circle method

1 Introduction and Statement of results

Andrews [1] considered a certain family of functions and noticed a mysterious phenomenon. More precisely, Andrews looked into q-series identities involving hypergeometric functions, for example in particular ([1], [2, eq. (4.10) and (4.12)] and [3, p. 19 and 104])

$$1+\sum_{n=1}^{\infty}\frac{q^n}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=1}^{\infty}\frac{1}{(1-q^n)},$$
$$1+\sum_{n=1}^{\infty}\frac{q^{\frac{n(n+1)}{2}}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=1}^{\infty}(1+q^n),$$
$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{5n-4})(1-q^{5n-1})},$$

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$$1+\sum_{n=1}^{\infty}\frac{q^n}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}=\prod_{n=1}^{\infty}\frac{1}{(1-q^{2n-1})},$$

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}=\prod_{n=1}^{\infty}\left(1+q^{2n-1}\right),$$

$$1+\sum_{n=1}^{\infty}\frac{q^{\frac{n(n+1)}{2}}}{(1-q^2)(1-q^4)\cdots(1-q^{2n})}=?, \tag{1}$$

$$1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1-q^4)(1-q^8)\cdots(1-q^{4n})}=\prod_{n=1}^{\infty}\frac{1}{(1+q^{2n})(1-q^{5n-4})(1-q^{5n-1})}, \tag{2}$$

$$1+\sum_{n=1}^{\infty}\frac{q^{n(n+2)}}{(1-q^4)(1-q^8)\cdots(1-q^{4n})}=\prod_{n=1}^{\infty}\frac{1}{(1+q^{2n})(1-q^{5n-3})(1-q^{5n-2})}. \tag{3}$$

While the others can be nicely written in terms of infinite product (so that it turns out that they are modular forms up to q powers), Andrews did not find any such shape of identities for (1). Moreover, Zagier [14, Table 1] figured out that (1) is not modular. Nonetheless, Andrews [1] provided a combinatorial interpretation for this function, namely *odd-even partitions.*

Recall that a *partition* of positive integer n is a nowhere increasing sequence of positive integers whose sum is n. Define a partition function $OE(n)$ by the number of partitions of n in which the parts alternate in parity starting with the smallest part odd. In other words, $OE(n)$ counts the number of *odd-even partitions of n.* For instance, there are no odd-even partitions of 2 and the odd-even partitions of 3 are 3 and 2+1. Therefore $OE(2) = 0$ and $OE(3) = 2$. By Andrews' proof a generating function (in Eulerian form) for the odd-even partitions is given by

$$\mathscr{O}(q) := 1+\sum_{n=1}^{\infty}OE(n)q^n=\sum_{m=0}^{\infty}\frac{q^{\frac{m(m+1)}{2}}}{(q^2;q^2)_m}, \tag{4}$$

which is (1). Here the q-Pochhammer symbol or q-shifted factorial is defined as $(a)_n := (a;q)_n := \prod_{j=1}^{n}(1-aq^{j-1})$ for $n \in \mathbb{N}_0 \cup \{\infty\}$.

In this paper we investigate the asymptotic behavior of $OE(n)$. In order to study the asymptotic behavior of the coefficients of a series, one can either use the Circle Method [6, 11, 13] or apply Ingham's Tauberian Theorem [7]. Since $\mathscr{O}(q)$ has a pole at every root of unity and it is not easy to find the bounds for $\mathscr{O}(q)$ at every root of unity, it is difficult to use the Circle Method in our case. Moreover, as $OE(n)$ is not monotonically increasing, we cannot directly apply Ingham's Tauberian Theorem to our case either (see Section 2 for more details). Thus, we need to slightly modify our function so that we can apply Ingham's Tauberian Theorem.

Theorem 1. *We have*

$$OE(n) \sim \frac{1}{2\sqrt{5}n^{\frac{3}{4}}}e^{\pi\sqrt{\frac{n}{5}}}$$

as $n \to \infty$.

We also investigate the asymptotics of *odd-even overpartitions*, studied by Lovejoy [9]. Recall that an *overpartition* of positive interger n is a partition of n in which the first occurrence (equivalently, the final occurrence) of a number may be overlined. An *odd-even overpartition* is an overpartition with the smallest part odd and such that the difference between successive parts is odd if the smaller is nonoverlined and even otherwise. For example, there are no odd-even overpartitions of 2, the odd-even overpartitions of 3 are $\overline{3}$, 3, $\overline{2}+1$, and $2+1$, and the odd-even partitions of 4 are $\overline{3}+\overline{1}$ and $3+\overline{1}$. Notice that if all parts are non-overlined, then we have the odd-even partitions. We denote $\overline{OE}(n)$ by the number of odd-even overpartitions of n and define $\overline{OE}(0) := 1$. The generating function is given in [9]

$$\overline{\mathscr{O}}(q) := \sum_{n=0}^{\infty} \overline{OE}(n) q^n = \sum_{m=0}^{\infty} \frac{(-1)_m q^{\frac{m(m+1)}{2}}}{(q^2;q^2)_m} = (-q)_\infty f(q),$$

where

$$f(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q)_n^2} \tag{5}$$

is one of Ramanujan's third order mock theta functions. These functions appeared in Ramanujan's deathbed letter to Hardy and are now known as the holomorphic parts of weight 1/2 *harmonic Maass forms* (see [15]). We remark that the generating function for the odd-even overpartitions is a *mixed mock modular form*, i.e., the product of a modular form and a mock theta function. From this fact, we can apply Wright's Circle Method [13] to obtain the asymptotic formula for $\overline{OE}(n)$.

Theorem 2. *We have*

$$\overline{OE}(n) \sim \frac{1}{3^{\frac{5}{4}} n^{\frac{3}{4}}} e^{\pi\sqrt{\frac{n}{3}}}$$

as $n \to \infty$.

This paper is organized as follows. In Section 2 we study some basic properties of odd-even partitions and introduce an auxiliary theorem which play important roles to prove Theorem 1. The proof is given in Section 3. We conclude the paper with the proof of Theorem 2 in Section 4.

2 Preliminaries

2.1 Basic properties of odd-even partitions

First we look into the first few values of the odd-even partition function $OE(n)$ given in Table 1. From these values, we see that $OE(n)$ is not monotonically increasing. Nevertheless, $OE(n) \leqslant OE(n+2)$ holds for every n due to the fact that we can always make an odd-even partition of $n+2$ from the one of n by adding 2 to the largest part. Thus, $OE(n)$ is monotonically increasing for even (odd resp.) n. This suggests that the appropriate

n	relevant partitions of n	$OE(n)$
1	1	1
2	—	0
3	3, 1+2	2
4	—	0
5	5, 1+4	2
6	1+2+3	1
7	7, 1+6, 3+4	3
8	1+2+5	1
⋮	⋮	⋮

Table 1: Values of $OE(n)$

approach to understand the asymptotic behavior of $OE(n)$ is to split the power series of $OE(n)$ into two parts, one with even n and the other with odd n, as follows:

$$\mathscr{O}(q) = \sum_{n=0}^{\infty} OE(n)q^n = \sum_{n=0}^{\infty} OE(2n)q^{2n} + \sum_{n=0}^{\infty} OE(2n+1)q^{2n+1} =: \mathscr{O}_e(q) + \mathscr{O}_o(q).$$

Here, for convenience we define $OE(0) := 1$. We further split the q hypergeometric series in (4) accordingly by considering the parity of powers of q for each summand. Since the q-Pochhammer symbol $(q^2;q^2)_m$ in the denominator always produces even powers of q, the parity of powers of q depends only on $m(m+1)/2$. Note that $m(m+1)/2$ is even iff $m \equiv 0,3 \pmod 4$ and odd iff $m \equiv 1,2 \pmod 4$. Hence

$$\mathscr{O}_e(q) = \sum_{\substack{m\geqslant 0 \\ m\equiv 0,3 \pmod 4}} \frac{q^{\frac{m(m+1)}{2}}}{(q^2;q^2)_m}, \qquad \mathscr{O}_o(q) = \sum_{\substack{m\geqslant 0 \\ m\equiv 1,2 \pmod 4}} \frac{q^{\frac{m(m+1)}{2}}}{(q^2;q^2)_m}.$$

2.2 Ingham's Tauberian Theorem

From the asymptotic behavior of a power series, Ingham's Tauberian Theorem [7] gives an asymptotic formula for its coefficients.

Theorem (Ingham [7])**.** *Let $f(q) = \sum_{n\geqslant 0} a(n)q^n$ be a power series with weakly increasing nonnegative coefficients and radius of convergence equal to 1. If there are constants $A > 0$, $\lambda, \alpha \in \mathbb{R}$ such that*

$$f\left(e^{-\varepsilon}\right) \sim \lambda \varepsilon^{\alpha} e^{\frac{A}{\varepsilon}}$$

as $\varepsilon \to 0^+$, then

$$a(n) \sim \frac{\lambda}{2\sqrt{\pi}} \frac{A^{\frac{\alpha}{2}+\frac{1}{4}}}{n^{\frac{\alpha}{2}+\frac{3}{4}}} e^{2\sqrt{An}}$$

as $n \to \infty$.

3 Proof of Theorem 1

3.1 Asymptotics for the generating functions

In this section we estimate the functions $\mathscr{O}_e(q)$ and $\mathscr{O}_o(q)$. Throughout the section we set $q = e^{-t}$. In order to get the asymptotic formulas for these functions, we exploit the second proof of [14, Proposition 5]. The idea of the proof is based on the asymptotics of the individual terms in the series. We first study the asymptotic behavior of the summand and then sum up the asymptotics. We denote the mth term in the series (4) by

$$f_m = f_m(q) := \frac{q^{\frac{m(m+1)}{2}}}{(q^2;q^2)_m}.$$

The sequence $(f_m)_{m\in\mathbb{N}}$ is unimodal, meaning that f_m increases until f_m reaches a maximum value and then decreases. More precisely, for $0 < |q| < 1$ the ratio

$$\frac{f_m}{f_{m-1}} = \frac{q^m}{1-q^{2m}} \tag{6}$$

goes to ∞ as $m \to 0$, decreases as m grows, and tends to 0 as $m \to \infty$. To determine when f_m takes the maximum value, we check when the ratio (6) becomes 1. This ratio is equal to 1 exactly for q^{2m} the unique root of the equation $Q^{\frac{1}{2}} + Q = 1$ in the interval $(0,1)$, namely $Q := \frac{3-\sqrt{5}}{2}$. In other words, f_m approaches the maximum value when q^{2m} is close to Q and m near $\mathrm{Log}(Q)/(2\,\mathrm{Log}(q))$. We further note that

$$\frac{\mathrm{Log}(Q)}{2\,\mathrm{Log}(q)} \to \infty, \quad q^{2m} \to Q \qquad \text{as} \quad q \to 1^-.$$

Thus, the main contribution occurs when the terms are of the form $q^{2m} = Qq^{-2\nu}$ (or $q^m = Q^{\frac{1}{2}}q^{-\nu}$) with $\nu \in \nu_0 + \mathbb{Z}$ satisfying $\nu = o(m)$ and ν_0 denotes the fractional part of $\mathrm{Log}(Q)/(2\,\mathrm{Log}(q))$. In this setting, we evaluate the size of f_m. For this, we use the asymptotic expansion from Zagier [14, Page 53]. Here the *dilogarithm function* $\mathrm{Li}_2(z)$ is defined for $|z| < 1$ by

$$\mathrm{Li}_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$

Lemma 3. *Let $A, B \in \mathbb{R}$ and $A > 0$. For the unique root $R \in (0,1)$ of the equation $R + R^A = 1$ and $q = e^{-t}$ with $q^n = Rq^{-\nu}$, $\nu = o(n)$ as $n \to \infty$, we have*

$$\begin{aligned}
&\mathrm{Log}\left(\frac{q^{\frac{1}{2}An^2+Bn}}{(q)_n}\right)\\
&\quad= \left(\frac{\pi^2}{6} - \mathrm{Li}_2(R) - \frac{1}{2}\mathrm{Log}(R)\,\mathrm{Log}(1-R)\right)t^{-1} - \frac{1}{2}\mathrm{Log}\left(\frac{2\pi}{t}\right) + \mathrm{Log}\left(\frac{R^B}{\sqrt{1-R}}\right)\\
&\qquad - \left(\frac{A+R-AR}{2(1-R)}\nu^2 - \left(B + \frac{R}{2(1-R)}\right)\nu + \frac{1+R}{24(1-R)}\right)t + O\left(t^2\right),
\end{aligned}$$

as $t \to 0$.

Remark 4. In fact, Zagier obtained the asymptotic expansion with arbitrary many main terms. Since we only use the first few main terms in this paper, we do not need to consider the complete expansion.

We set $q \mapsto q^2$, $A \mapsto 1/2$, and $B \mapsto 1/4$ in Lemma 3. Thus, R becomes Q and we have, recalling that $Q^{\frac{1}{2}} + Q = 1$ and $Q = \frac{3-\sqrt{5}}{2}$,

$$\operatorname{Log}\left(\frac{q^{\frac{m(m+1)}{2}}}{(q^2;q^2)_m}\right) = \left(\frac{\pi^2}{6} - \operatorname{Li}_2(Q) - \left(\frac{1}{2}\operatorname{Log}(Q)\right)^2\right)\frac{1}{2t} - \frac{1}{2}\operatorname{Log}\left(\frac{\pi}{t}\right) - \frac{\sqrt{5}}{2}\left(\nu^2 - \nu + \frac{1}{6}\right)t + O\left(t^2\right). \quad (7)$$

Furthermore, we use the special value of the dilogarithm function from [14, Section I.1]

$$\operatorname{Li}_2(Q) = \frac{\pi^2}{15} - \left(\operatorname{Log}\left(\frac{1+\sqrt{5}}{2}\right)\right)^2 \quad (8)$$

and note that

$$\left(\frac{1}{2}\operatorname{Log}(Q)\right)^2 = (\operatorname{Log}(1-Q))^2 = \left(\operatorname{Log}\left((1-Q)^{-1}\right)\right)^2 = \left(\operatorname{Log}\left(\frac{1+\sqrt{5}}{2}\right)\right)^2. \quad (9)$$

Combining (7), (8), and (9) gives

$$\begin{aligned}\operatorname{Log}\left(\frac{q^{\frac{m(m+1)}{2}}}{(q^2;q^2)_m}\right) &= \frac{\pi^2}{20t} - \frac{1}{2}\operatorname{Log}\left(\frac{\pi}{t}\right) - \frac{\sqrt{5}}{2}\left(\nu^2 - \nu + \frac{1}{6}\right)t + O\left(t^2\right) \\ &= \operatorname{Log}(\varphi(\nu)) + O\left(t^2\right),\end{aligned} \quad (10)$$

where

$$\varphi(\nu) := \sqrt{\frac{t}{\pi}}\exp\left[\frac{\pi^2}{20t} - \frac{\sqrt{5}}{2}\left(\nu^2 - \nu + \frac{1}{6}\right)t\right].$$

We additionally define for $j \in \{0,1,2,3\}$

$$\mathcal{S}_j := \sum_{m \equiv j \pmod 4} \frac{q^{\frac{m(m+1)}{2}}}{(q^2;q^2)_m},$$

so that we can write

$$\mathcal{O}_e(q) = \mathcal{S}_0 + \mathcal{S}_3, \qquad \mathcal{O}_o(q) = \mathcal{S}_1 + \mathcal{S}_2. \quad (11)$$

Theorem 5. *We have*

$$\mathcal{O}_e\left(e^{-t}\right) \sim \mathcal{O}_o\left(e^{-t}\right) \sim \frac{1}{\sqrt{2\sqrt{5}}}e^{\frac{\pi^2}{20t}}$$

as $t \to 0^+$.

Proof. Using (10), we can also rewrite $\mathcal{S}_j$ in terms of $\varphi(\nu)$ as

$$\mathcal{S}_j = \left(1 + O\left(t^2\right)\right) \sum_{\nu \equiv \nu_0 + j \pmod 4} \varphi(\nu). \tag{12}$$

To estimate $\mathcal{S}_j$, we begin by rewriting the sum in ν on the right-hand side of (12) as

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \varphi(4n+\nu_0+j) &= \sum_{n\in\frac{1}{2}+\mathbb{Z}} \varphi(4n+\alpha) \\
&= \sqrt{\frac{t}{\pi}} e^{\frac{\pi^2}{20t}} \sum_{n\in\frac{1}{2}+\mathbb{Z}} e^{-\frac{\sqrt{5}}{2}\left((4n+\alpha)^2-(4n+\alpha)+\frac{1}{6}\right)t} \\
&= \sqrt{\frac{t}{\pi}} e^{\frac{\pi^2}{20t}-\frac{\sqrt{5}}{2}\left(\alpha^2-\alpha+\frac{1}{6}\right)t} \vartheta\left(\frac{\sqrt{5}\,(2\alpha-1)\,ti}{\pi} - \frac{1}{2}; \frac{8\sqrt{5}ti}{\pi}\right),
\end{aligned} \tag{13}$$

where $\alpha := 2 + \nu_0 + j$ and the Jacobi Theta function is given for $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ by

$$\vartheta\left(z;\tau\right) := \sum_{n\in\frac{1}{2}+\mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n\left(z+\frac{1}{2}\right)}.$$

The modular inversion formula for the Jacobi theta function [15, Proposition 1.3 (7)] implies that for $a, b \in \mathbb{C}$ with $\mathrm{Re}(a) > 0$

$$\begin{aligned}
\vartheta\left(\frac{bti}{\pi} - \frac{1}{2}; \frac{ati}{\pi}\right) &= i\sqrt{\frac{\pi}{at}} e^{-\frac{\pi^2}{at}\left(\frac{bti}{\pi}-\frac{1}{2}\right)^2} \vartheta\left(\frac{b}{a} + \frac{\pi i}{2at}; \frac{\pi i}{at}\right) \\
&= \sqrt{\frac{\pi}{at}} \sum_{n\in\mathbb{Z}} (-1)^n e^{-\frac{\pi^2}{at}\left(n-\frac{bti}{\pi}\right)^2}.
\end{aligned}$$

Plugging in $a \mapsto 8\sqrt{5}$ and $b \mapsto \sqrt{5}(2\alpha - 1)$ and simplifying the summation yields that

$$\begin{aligned}
\vartheta\left(\frac{\sqrt{5}\,(2\alpha-1)\,ti}{\pi} - \frac{1}{2}; \frac{8\sqrt{5}ti}{\pi}\right) &= \sqrt{\frac{\pi}{8\sqrt{5}t}} \sum_{n\in\mathbb{Z}} (-1)^n e^{-\frac{\pi^2}{8\sqrt{5}t}\left(n-\frac{\sqrt{5}(2\alpha-1)ti}{\pi}\right)^2} \\
&= \sqrt{\frac{\pi}{8\sqrt{5}t}} e^{\frac{\sqrt{5}(2\alpha-1)^2 t}{32}} \left(1 + O\left(\sum_{n\in\mathbb{Z}\setminus\{0\}} e^{-\frac{\pi^2 n^2}{8\sqrt{5}t}}\right)\right) \\
&= \sqrt{\frac{\pi}{8\sqrt{5}t}} \left(1 + O\left(t\right)\right).
\end{aligned} \tag{14}$$

The last equality comes directly from the fact that as $t \to 0^+$

$$e^{\frac{\sqrt{5}(2\alpha-1)^2 t}{32}} = 1 + O\left(t\right),$$

and

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} e^{-\frac{\pi^2 n^2}{8\sqrt{5}t}} \ll e^{-\frac{\pi^2}{8\sqrt{5}t}}.$$

From (12), (13), and (14), we obtain for any $j \in \{0,1,2,3\}$

$$\mathcal{S}_j \sim \frac{1}{2\sqrt{2\sqrt{5}}} e^{\frac{\pi^2}{20t}-\frac{\sqrt{5}}{2}\left(\alpha^2-\alpha+\frac{1}{6}\right)t} \sim \frac{1}{2\sqrt{2\sqrt{5}}} e^{\frac{\pi^2}{20t}}$$

as $t \to 0^+$. Recalling (11), we have the desired result.

□

Moreover, since $\mathscr{O}(q) = \mathscr{O}_e(q) + \mathscr{O}_o(q)$, we have following Corollary.

Corollary 6. *We have*

$$\mathscr{O}\left(e^{-t}\right) \sim \sqrt{\frac{2}{\sqrt{5}}} e^{\frac{\pi^2}{20t}}$$

as $t \to 0^+$.

Remark 7. One can directly estimate the series $\mathscr{O}(q)$ by using the Constant Term Method, inserting an additional variable to identify the series as the constant term of the product of more familiar number-theoretic functions in a new variable. (See [14, First proof of Proposition 5] for more details.)

Remark 8. McIntosh [10] derived the complete asymptotic expansion of the more general q-series

$$\sum_{n=0}^{\infty} \frac{a^n q^{bn^2+cn}}{(q)_n}$$

in full detail using elementary methods (Euler-Maclaurin sum formula). The asymptotic of $\mathscr{O}(q)$ is the case $c = 1/4$ and $q \to q^2$ (gence $t \to 2t$) in the following formula [10, p. 134]

$$\begin{aligned}
\log \sum_{n=0}^{\infty} \frac{q^{n^2/4+cn}}{(q)_n} = \frac{\pi^2}{10t} &+ 2c \log\left(\frac{\sqrt{5}-1}{2}\right) - \frac{1}{2}\log\left(\frac{5-\sqrt{5}}{4}\right) \\
&+ \left(\frac{4c-1}{40} + \frac{c(2c-1)}{10}\sqrt{5}\right) t \\
&- c(2c-1)\left(\frac{1}{25} + \frac{4c-1}{150}\sqrt{5}\right) t^2 \\
&+ c(2c-1)\left(\frac{4c-1}{250} + \frac{2c^2-c+3}{750}\sqrt{5}\right) t^3 \\
&- c(2c-1)\left(\frac{2c^2-c+13}{3750} - \frac{(4c-1)\left(12c^2-6c-31\right)}{45000}\sqrt{5}\right) t^4 \\
&+ O\left(t^5\right),
\end{aligned}$$

where $q = e^{-t}$ and $t \to 0^+$.

3.2 Applying Ingham's Tauberian Theorem

Now we are ready to apply Ingham's Tauberian Theorem to the functions $\mathscr{O}_e\left(e^{-t}\right)$ and $\mathscr{O}_o\left(e^{-t}\right)$. We first deal with the even case. Setting $a(n) = OE(2n)$ and replacing q by q^2 in Theorem 2.2 determines the constants

$$\lambda = \frac{1}{\sqrt{2\sqrt{5}}}, \qquad \alpha = 0, \qquad A = \frac{\pi^2}{10}.$$

We remark that since $OE(n)$ does not satisfy weakly increasing property with $n = 0$, we only consider when $n \geqslant 1$. Thus, we have

$$OE(2n) \sim \frac{1}{2\sqrt{5}(2n)^{\frac{3}{4}}} e^{2\pi\sqrt{\frac{n}{10}}}.$$

By letting $n \mapsto n/2$, we obtain the desired asymptotic formula for $OE(n)$ with even n, namely

$$OE(n) \sim \frac{1}{2\sqrt{5}n^{\frac{3}{4}}} e^{\pi\sqrt{\frac{n}{5}}}. \tag{15}$$

For odd n, we rewrite the series as

$$\mathscr{O}_o(q) = \sum_{n=0}^{\infty} OE(2n+1)q^{2n+1} = q\sum_{n=0}^{\infty} OE(2n+1)q^{2n}.$$

Since by Theorem 5

$$\mathscr{O}_o\left(e^{-t}\right) = e^{-t}\sum_{n=0}^{\infty} OE(2n+1)e^{-2tn} \sim \frac{1}{\sqrt{2\sqrt{5}}} e^{\frac{\pi^2}{20t}},$$

we have

$$\sum_{n=0}^{\infty} OE(2n+1)e^{-2tn} \sim \frac{1}{\sqrt{2\sqrt{5}}} e^{\frac{\pi^2}{20t}}.$$

Similar to the case of even n, setting $a(n) = OE(2n+1)$ and replacing q by q^2 yields

$$OE(2n+1) \sim \frac{1}{2\sqrt{5}(2n)^{\frac{3}{4}}} e^{2\pi\sqrt{\frac{n}{10}}}.$$

As before we let $n \mapsto n/2$ and thus we have for even n

$$OE(n+1) \sim \frac{1}{2\sqrt{5}n^{\frac{3}{4}}} e^{\pi\sqrt{\frac{n}{5}}}. \tag{16}$$

Finally from (15) and (16) we get the desired asymptotic formula for $OE(n)$, for every n,

$$OE(n) \sim \frac{1}{2\sqrt{5}n^{\frac{3}{4}}} e^{\pi\sqrt{\frac{n}{5}}}$$

as $n \to \infty$.

4 Proof of Theorem 2

We follow the same method of the proof of Theorem 4 in [5]. The strategy is to estimate the generating function near and away from a dominant pole, and then apply Wright's Circle Method. Although the method of proof is not new, because we are dealing with a different function, the result does not follow directly from the statement of Theorem 4 in [5], and thus we include its proof here. However, it is basically the same proof.

4.1 Asymptotics of $\overline{\mathscr{O}}(q)$

Using the Watson's identity for Ramanujan's third order mock theta function $f(q)$ [12]

$$f(q)=\frac{2}{(q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1+q^n},$$

we rewrite $\overline{\mathscr{O}}(q)$ as

$$\overline{\mathscr{O}}(q)=\frac{2(-q)_\infty}{(q)_\infty}\sum_{n\in\mathbb{Z}}\frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1+q^n}. \tag{17}$$

From this expression we can see that $\overline{\mathscr{O}}(q)$ has a dominant pole at $q=1$.

Theorem 9. *Let $M>0$ be fixed.*

(i) *For $|x|\leqslant My$, as $y\to 0^+$*

$$\overline{\mathscr{O}}(q)=\frac{2\sqrt{2}}{3}e^{\frac{\pi i}{24\tau}}+O\left(ye^{\frac{\pi}{24}\operatorname{Im}\left(\frac{-1}{\tau}\right)}\right).$$

(ii) *For $My<|x|\leqslant 1/2$, as $y\to 0^+$*

$$\overline{\mathscr{O}}(q)\ll\frac{1}{y\sqrt{2}}\exp\left[\frac{1}{y}\left(\frac{\pi}{8}-\frac{1}{\pi}\left(1-\frac{1}{\sqrt{1+M^2}}\right)\right)\right].$$

Remark 10. One can find $M>\sqrt{\left(\frac{12}{12-\pi^2}\right)^2-1}=5.543\ldots$, so that the bound in the part (ii) is indeed an error term.

Proof. (i) To estimate the function $\overline{\mathscr{O}}(q)$ near $q=1$, we first examine $f(q)$. By Taylor's theorem, we have

$$f(q)=f(1)+O\left(|\tau|\right),$$

and from (5) we see that

$$f(1)=\sum_{n=0}^{\infty}\frac{1}{4^n}=\frac{4}{3}.$$

Thus, we have for $|x|\leqslant My$

$$f(q)=\frac{4}{3}+O\left(y\right) \tag{18}$$

as $y \to 0^+$.

Now we turn to the infinite product $(-q)_\infty$ in front of $f(q)$. Recall that, from the modular inversion formula for Dedekind's eta-function ([8, p. 121, Proposition 14]),

$$(q;q)_\infty = \frac{1}{\sqrt{-i\tau}} e^{-\frac{\pi i\tau}{12} - \frac{\pi i}{12\tau}} \left(1 + O\left(e^{-\frac{2\pi i}{\tau}}\right)\right). \tag{19}$$

Therefore, we find that

$$(-q)_\infty = \frac{(q^2;q^2)_\infty}{(q)_\infty} = \frac{1}{\sqrt{2}} e^{\frac{\pi i}{24\tau}} + O\left(y e^{\frac{\pi}{24}\mathrm{Im}\left(\frac{-1}{\tau}\right)}\right). \tag{20}$$

Combining (18) and (20) gives the proof of the part (i).

(ii) In the case of $\overline{\mathscr{O}}(q)$ away from $q=1$, we consider the expression in (17). Note that

$$\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1+q^n} = \frac{1}{2} + 2\sum_{n\geqslant 1} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1+q^n}$$

and that, for $My < |x| \leqslant 1/2$,

$$\left|\sum_{n\geqslant 1} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1+q^n}\right| \leqslant \frac{1}{1-|q|} \sum_{n\geqslant 1} |q|^{\frac{n(3n+1)}{2}} \ll \frac{1}{y}\cdot y^{-\frac{1}{2}} = y^{-\frac{3}{2}}.$$

This implies

$$\left|\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{\frac{n(3n+1)}{2}}}{1+q^n}\right| \ll y^{-\frac{3}{2}}. \tag{21}$$

Now it remains to bound the infinity product

$$\frac{(-q)_\infty}{(q)_\infty} = \frac{(q^2;q^2)_\infty}{(q)_\infty^2}.$$

We write this as

$$\begin{aligned}
\mathrm{Log}\left(\frac{(q^2;q^2)_\infty}{(q)_\infty^2}\right) &= \sum_{n\geqslant 1}\left(\mathrm{Log}\left(1-q^{2n}\right) - 2\,\mathrm{Log}\left(1-q^n\right)\right) \\
&= \sum_{n\geqslant 1}\sum_{m\geqslant 1} \frac{2q^{nm}}{m} - \sum_{n\geqslant 1}\sum_{m\geqslant 1} \frac{q^{2nm}}{m} \\
&= \sum_{m\geqslant 1}\left(\frac{2q^m}{m\left(1-q^m\right)} - \frac{q^{2m}}{m\left(1-q^{2m}\right)}\right) \\
&= \sum_{m\geqslant 1} \frac{2q^{2m-1}}{(2m-1)\left(1-q^{2m-1}\right)}.
\end{aligned}$$

Thus,

$$\begin{aligned}\left|\operatorname{Log}\left(\frac{(q^2;q^2)_\infty}{(q)_\infty^2}\right)\right| &\leqslant \sum_{m\geqslant 1}\frac{2|q|^{2m-1}}{(2m-1)\left|1-q^{2m-1}\right|}\\ &\leqslant \sum_{m\geqslant 1}\frac{2|q|^{2m-1}}{(2m-1)\left(1-|q|^{2m-1}\right)}+\frac{2|q|}{|1-q|}-\frac{2|q|}{1-|q|}\\ &= \operatorname{Log}\left(\frac{(|q|^2;|q|^2)_\infty}{(|q|)_\infty^2}\right)-2|q|\left(\frac{1}{1-|q|}-\frac{1}{|1-q|}\right).\end{aligned}$$

From (19), we have

$$\frac{(|q|^2;|q|^2)_\infty}{(|q|)_\infty^2}=\sqrt{\frac{y}{2}}e^{\frac{\pi}{8y}}\left(1+O\left(e^{-\frac{\pi}{y}}\right)\right).$$

To evaluate the remaining term, we note that for $My<|x|\leqslant \frac{1}{2}$, $\cos(\pi My)>\cos(\pi x)$. Therefore,

$$|1-q|^2=1-2e^{-2\pi y}\cos(2\pi x)+e^{-4\pi y}>1-2e^{-2\pi y}\cos(2\pi My)+e^{-4\pi y}.$$

By the Taylor expansion around $y=0$, we conclude that

$$|1-q|>2\pi y\sqrt{1+M^2}+O\left(y^2\right).$$

Since $1-|q|=2\pi y+O\left(y^2\right)$, we arrive at

$$\left|\frac{(q^2;q^2)_\infty}{(q)_\infty^2}\right|\ll\sqrt{\frac{y}{2}}\exp\left[\frac{1}{y}\left(\frac{\pi}{8}-\frac{1}{\pi}\left(1-\frac{1}{\sqrt{1+M^2}}\right)\right)\right]. \tag{22}$$

Plugging (21) and (22) into (17) yields the part (ii). □

Corollary 11. *For $My<|x|\leqslant 1/2$ with $M>\sqrt{\left(\frac{12}{12-\pi^2}\right)^2-1}$, there exists $\epsilon>0$ such that as $y\to 0^+$*

$$\overline{\mathscr{O}}(q)\ll\frac{1}{y\sqrt{2}}e^{\frac{\pi}{24}\left(\operatorname{Im}\left(\frac{-1}{\tau}\right)-\epsilon\right)}.$$

4.2 Wright's Circle Method

In this section we complete the proof of Theorem 2 by applying Wright's Circle Method. By Cauchy's Theorem, we see for $y=\frac{1}{4\sqrt{3n}}$ that

$$\begin{aligned}\overline{OE}(n)=\frac{1}{2\pi i}\int_{\mathcal{C}}\frac{\overline{\mathscr{O}}(q)}{q^{n+1}}dq&=\int_{-\frac{1}{2}}^{\frac{1}{2}}\overline{\mathscr{O}}\left(e^{2\pi ix-\frac{\pi}{2\sqrt{3n}}}\right)e^{-2\pi inx+\frac{\pi\sqrt{n}}{2\sqrt{3}}}dx\\ &=\int_{|x|\leqslant My}\overline{\mathscr{O}}\left(e^{2\pi ix-\frac{\pi}{2\sqrt{3n}}}\right)e^{-2\pi inx+\frac{\pi\sqrt{n}}{2\sqrt{3}}}dx\\ &\quad+\int_{My<|x|\leqslant\frac{1}{2}}\overline{\mathscr{O}}\left(e^{2\pi ix-\frac{\pi}{2\sqrt{3n}}}\right)e^{-2\pi inx+\frac{\pi\sqrt{n}}{2\sqrt{3}}}dx\qquad=:\mathcal{I}_1+\mathcal{I}_2,\end{aligned}$$

where $\mathcal{C} = \{|q| = e^{-\frac{\pi}{2\sqrt{3n}}}\}$. In fact, the integral $\mathcal{I}_1$ contributes the main term as the integral $\mathcal{I}_2$ is an error term.

In order to evaluate $\mathcal{I}_1$, we introduce a function $P_s(u)$, defined by Wright [13], for fixed $M > 0$ and $u \in \mathbb{R}^+$

$$P_s(u) := \frac{1}{2\pi i}\int_{1-Mi}^{1+Mi} v^s e^{u\left(v+\frac{1}{v}\right)} dv.$$

This function is rewritten in terms of the I-Bessel function up to an error term.

Lemma 12 ([13]). *As $n \to \infty$*

$$P_s(u) = I_{-s-1}(2u) + O\left(e^u\right),$$

where I_ℓ denotes the usual the I-Bessel function of order ℓ.

Using Theorem 9 (i), we write the integral $\mathcal{I}_1$ as

$$\mathcal{I}_1 = \int_{|x|\leqslant \frac{M}{4\sqrt{3n}}} \left(\frac{2\sqrt{2}}{3}e^{\frac{\pi i}{24\tau}} + O\left(n^{-\frac{1}{2}}e^{\frac{\pi\sqrt{n}}{2\sqrt{3}}}\right)\right) e^{-2\pi i n x + \frac{\pi\sqrt{n}}{2\sqrt{3}}} dx.$$

By making the change of variables $v = 1 - i4\sqrt{3n}x$, we arrive at

$$\begin{aligned}
\mathcal{I}_1 &= \int_{1-Mi}^{1+Mi} \frac{-i}{4\sqrt{3n}} \left(\frac{2\sqrt{2}}{3}e^{\frac{\pi\sqrt{n}}{2\sqrt{3}v}} + O\left(n^{-\frac{1}{2}}e^{\frac{\pi\sqrt{n}}{2\sqrt{3}}}\right)\right) e^{\frac{\pi\sqrt{n}v}{2\sqrt{3}}} dv \\
&= \frac{\pi\sqrt{2}}{3\sqrt{3n}} P_0\left(\frac{\pi\sqrt{n}}{2\sqrt{3}}\right) + O\left(n^{-\frac{3}{2}}e^{\frac{\pi\sqrt{n}}{\sqrt{3}}}\right) \\
&= \frac{\pi\sqrt{2}}{3\sqrt{3n}} I_{-1}\left(\frac{\pi\sqrt{n}}{\sqrt{3}}\right) + O\left(n^{-\frac{3}{2}}e^{\frac{\pi\sqrt{n}}{\sqrt{3}}}\right) \\
&= \frac{1}{3^{\frac{5}{4}}n^{\frac{3}{4}}} e^{\frac{\pi\sqrt{n}}{\sqrt{3}}} + O\left(n^{-\frac{3}{2}}e^{\frac{\pi\sqrt{n}}{\sqrt{3}}}\right), \qquad (23)
\end{aligned}$$

where we use the asymptotic formula for the I-Bessel function [4, 4.12.7]

$$I_\ell(x) = \frac{e^x}{\sqrt{2\pi x}} + O\left(\frac{e^x}{x^{\frac{3}{2}}}\right).$$

Now we turn to the integral $\mathcal{I}_2$. From the Corollarly 11, we have for $My < |x| \leqslant 1/2$

$$\mathcal{I}_2 \ll \int_{My<|x|\leqslant\frac{1}{2}} 2\sqrt{6n} e^{\frac{1}{y}\left(\frac{\pi}{24}-\epsilon\right)} e^{\frac{\pi\sqrt{n}}{2\sqrt{3}}} dx \ll n^{\frac{1}{2}} e^{\frac{\pi\sqrt{n}}{\sqrt{3}}(1-\epsilon)},$$

which together with (23) completes the proof.

5 Concluding remarks

The referee has kindly pointed out that in Section 3.1 by the unimodality of the sequence $(f_m)_{m\in\mathbb{N}}$ all of the $\mathcal{S}_j$ have the same asymptotic expansion since any differences in the $\mathcal{S}_j$ are exponentially small compared to other terms in their asymptotic expansions. Therefore the asymptotic expansion of each $\mathcal{S}_j$ is equal to the asymptotic expansion of the full q-series multiplied by $1/4$. In this case it is not necessary to use theta functions for the proof. We refer the interested reader to [10] for details.

The referee has also suggested a interesting discussion. In [10] McIntosh obtained

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)/4}}{(q)_n}=\sqrt{\frac{2}{\sqrt{5}}}\exp\left\{\frac{\pi^2}{10}t^{-1}-\frac{\sqrt{5}}{80}t+\frac{1}{200}t^2-\frac{23\sqrt{5}}{48000}t^3+\frac{103}{240000}t^4+O\left(t^5\right)\right\}$$

and

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1/2)}}{(q)_n}=\sqrt{\frac{1}{\sqrt{5}}}\exp\left\{\frac{\pi^2}{15}t^{-1}+\left(\frac{1}{48}+\frac{\sqrt{5}}{80}\right)t\right.$$
$$\left.+\frac{1}{200}t^2+\frac{23\sqrt{5}}{48000}t^3+\frac{103}{240000}t^4+O\left(t^5\right)\right\},$$

where $q=e^{-t}$ and $t\to 0^+$. These expansions appear to agree up to sign from the t^2 term onward. If we replace q by q^4 in the first series, then in some sense it is in the middle of the identities (2) and (3). Watson [12] discussed the second series, denoted by $G_{1/2}(q)$, in terms of Ramanujan's concept of 'closed' and 'unclosed' asymptotic expansion. This appears to be somewhat related to the concept of 'modular' and 'nonmodular'.

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