Unimodal Permutations and Almost-Increasing Cycles

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Abstract

In this paper, we establish a natural bijection between the almost-increasing cyclic permutations of length n and unimodal permutations of length $n-1$. This map is used to give a new characterization, in terms of pattern avoidance, of almost-increasing cycles. Additionally, we use this bijection to enumerate several statistics on almost-increasing cycles. Such statistics include descents, inversions, peaks and excedances, as well as the newly defined statistic called low non-inversions. Furthermore, we refine the enumeration of unimodal permutations by descents, inversions and inverse valleys. We conclude this paper with a theorem that characterizes the standard cycle notation of almost-increasing permutations.

Keywords: permutation statistics, cyclic permutations, unimodal permutations, almost-increasing permutations

1 Introduction

So-called *almost-increasing* permutations were the subject of prior research, and in Section 2.1 below we define them in a precise sense. These permutations were introduced by Knuth [20] in the context of sorting algorithms and were enumerated using generating functions. Elizalde [15] showed that these permutations have a characterization in terms of pattern avoidance and presented a bijective proof for the enumeration for almost-increasing permutations. This bijection was constructed between the set of almost-increasing permutations and the set of colored Motzkin paths (a generalization of Dyck paths). Elizalde further used this bijection to refine the enumeration with respect to several permutation statistics, including number of cycles, number of excedances, number of fixed points and number of inversions. Our focus in this paper, is on almost-increasing cyclic permutations, i.e., those almost-increasing permutations comprised of a single cycle. Recently, cyclic permutations received a great deal of attention in connection with pattern avoidance and permutation statistics; we discuss such connections in Section 2.3.

In this paper, we establish a natural bijection between the almost-increasing cyclic permutations of length n and unimodal permutations of length $n-1$. This map is used to give a new characterization of almost-increasing cycles in terms of pattern avoidance for the standard cycle notation. Furthermore, this bijection allows us to enumerate several statistics on almost-increasing cycles, including descents, peaks, excedances, inversions, and the newly defined statistic called low non-inversions. Table 1 summarizes the results obtained In this paper, on the refinement of the enumeration for almost-increasing cycles with respect to these statistics; the proofs of such results can be found in Sections 4 and 5. Let $p_{\leqslant n}(k)$ denote the number of partitions of the integer k into parts of size at most n. We note that the number of almost-increasing cycles with k excedances can also be recovered from Theorem 6.1 in [15].

Unimodal Permutations on $[n-1]$	**Almost-Increasing Cycles on $[n]$**	**Enumeration**	**Results**
$k-1$ inverse valleys	k descents	$\binom{n-1}{2k-1}$	Theorem 7 Theorem 13
	k peaks	$\binom{n-1}{2k}$	Corollary 15
k inversions	k low non-inversions	$p_{\leqslant n-2}(k)$	Theorem 10 Theorem 16
$k-1$ ascents	k excedances	$\binom{n-2}{k-1}$	Theorem 17
	k inversions	$\begin{cases} 2^{n-2}, & \text{if } k=n-1 \\ 0, & \text{otherwise} \end{cases}$	Theorem 18

Table 1: Statistics on unimodal permutations and almost-increasing cycles

Pattern avoidance in terms of a permutation's one-line notation is well-studied; there are numerous articles that explored statistics on permutations avoiding a prescribed pattern or a set of patterns, including descents in [6, 5, 21], excedances in [8, 17], inversions in [12, 22, 9] and peaks in [7]. Additionally, permutation statistics on permutations with a prescribed cycle type were studied, as discussed in Section 2.3. There are also some results regarding occurrences of patterns in the cycle notation of a permutation, see for example [10], among others. However, analyzing how patterns avoided in a permutation's one-line can determine patterns avoided in the permutation's standard cycle notation is relatively unexplored, and little is known about this cycle structure. We conclude this paper with a discussion of the relationship between a permutation's one-line notation and its standard cycle notation. These results, seen in Theorem 19, completely characterize almost-increasing permutations written in standard cycle notation.

2 Background

Let $\mathcal{S}_n$ be the set of permutations on $[n] = \{1, 2, \ldots, n\}$. A permutation $\pi \in \mathcal{S}_n$ can be written in its one-line notation as $\pi = \pi_1\pi_2 \ldots \pi_n$, where $\pi_i = \pi(i)$; we make use of both the notations π_i and $\pi(i)$ throughout the remainder of this paper. Alternatively, a permutation $\pi \in \mathcal{S}_n$ can be written in cycle notation as a product of disjoint cycles of the form $(i, \pi(i), \pi^2(i), \ldots, \pi^{k-1}(i))$, where $\pi^k(i) = i$. A permutation is in *standard cycle notation* if each cycle is written with its smallest element first and the cycles appear in increasing order with respect to their smallest element. For instance, the standard cycle notation of the permutation $\pi = 4361527 \in \mathcal{S}_7$ is $(14)(236)(7)$. A permutation composed of a single n-cycle is *cyclic*, and we let $\mathcal{C}_n$ denote the set of cyclic permutations (or *cycles* for short).

2.1 Pattern avoidance

For $m < n$, a permutation $\pi \in \mathcal{S}_n$ *contains* the pattern $\tau \in \mathcal{S}_m$ if there are indices $i_1 < i_2 < \cdots < i_m$ so that the subsequence $\pi_{i_1}\pi_{i_2} \ldots \pi_{i_m}$ appears in the same relative order as $\tau = \tau_1\tau_2 \ldots \tau_m$. If π does not contain τ, then we say π *avoids* τ. For example, the permutation 4361527 contains the pattern 213 because the subsequence 437 appears in the same relative order as the pattern 213. This permutation 4361527 avoids the pattern 1234 since there is no subsequence of length four that is increasing. A permutation $\pi \in \mathcal{S}_n$ is *unimodal* if there is some $i \in [n]$ such that $\pi_1 < \pi_2 < \cdots < \pi_i$ and $\pi_n < \pi_{n-1} < \cdots < \pi_i$, i.e., the permutation π is increasing then decreasing. It is well-known that these permutations are characterized as avoiding the patterns 312 and 213, and are enumerated by 2^{n-1}. Let $\mathcal{U}_n$ denote the set of unimodal permutations on $[n]$.

A permutation $\pi \in \mathcal{S}_n$ is *almost-increasing* if for every element $i \in [n]$, there is at most one $j \leqslant i$ such that $\pi_j > i$. Elizalde [15] characterized these permutations as avoiding the patterns 3412, 3421, 4321 and 4312. The set of almost-increasing permutations in $\mathcal{S}_n$ is denoted by $\mathcal{A}_n$, and the set of almost-increasing cyclic permutations in $\mathcal{C}_n$ is denoted by $\mathcal{A}_n^{\mathcal{C}}$.

2.2 Statistics on permutations

Let $\pi \in \mathcal{S}_n$. The permutation π has a *descent* at position i if $\pi_i > \pi_{i+1}$. The *descent set* of π is

$$\mathrm{Des}(\pi) = \{i \in [n-1] : \pi_i > \pi_{i+1}\},$$

and the *descent number*, denoted by $\mathrm{des}(\pi)$, is $|\mathrm{Des}(\pi)|$. An *inversion* of π is a pair (i, j) such that $i < j$ and $\pi_i > \pi_j$, and a *non-inversion* of π is a pair (i, j) such that $i < j$ and $\pi_i < \pi_j$. There are nine inversions of the permutation $134569782 \in \mathcal{S}_9$, namely $(2, 9)$, $(3, 9)$, $(4, 9)$, $(5, 9)$, $(6, 7)$, $(6, 8)$, $(6, 9)$, $(7, 9)$, and $(8, 9)$. An *excedance* of a permutation π is a position $i \in [n]$ such that $\pi_i > i$; the excedances of the permutation 134569782 are 2, 3, 4, 5 and 6, and the values of these excedances are 3, 4, 5, 6 and 9, repsectively. On the other hand, a *non-excedance* of π is a position $i \in [n]$ such that $\pi_i \leqslant i$. The permutation π has a *peak* at position i if $\pi_{i-1} < \pi_i$ and $\pi_{i+1} < \pi_i$. A permutation π has a *valley* at position i if $\pi_i < \pi_{i-1}$ and $\pi_i < \pi_{i+1}$, and *inverse valley* at position i if π_i is a valley of the inverse of π. The permutation 134569782 has peaks at positions 5 and 7, a valley at position 7, and an inverse valley at position 2. A *consecutive run* of a permutation is a maximal increasing contiguous subsequence $\pi_i\pi_{i+1}\dots\pi_{i+k}$ of consecutive integers. For example, the consecutive runs of 134569782 are 1, 3456, 9, 78 and 2. If $\pi_i\pi_{i+1}\dots\pi_{i+k}$ is a consecutive run, then the position i is the *start* of the run and the position $i+k$ is the *end* of the run.

We conclude this subsection by establishing a new statistic on permutations; define a *low non-inversion* of a permutation $\pi \in \mathcal{S}_n$ to be a pair (i, j) such that $\pi_i < i$, $i < j$ and $\pi_i < \pi_j$. The number of low non-inversions is denoted by

$$\operatorname{lni}(\pi) = \sum_{i:\pi_i<i} \{j \in [n] : i < j,\ \pi_i < \pi_j\}.$$

The low non-inversions are exactly the non-inversions of π for which the smaller index is associated to a non-excedance. With the aid of the pictorial presentation of the permutation $31624785 \in \mathcal{S}_8$ seen in Figure 3(b), we assert that π has 13 low non-inversions, namely the pairs (i, j) for which $i \in \{2, 4, 5\}$ and $j \in \{i+1, i+2, \dots, 8\}$.

2.3 Results on cyclic permutations

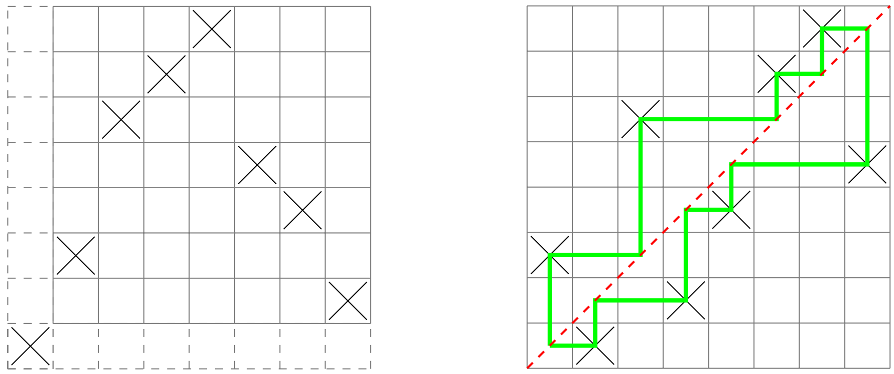
Many interesting results were discovered regarding permutation statistics of cyclic permutations. Using a clever bijection and quasi-symmetric functions, Gessel and Reutenauer [19] enumerated permutations by descent set and cycle type. Elizalde [14] established a bijection between permutations on $[n-1]$ with descent set D and cycles on $[n]$ with descent set D or $D \cup \{n-1\}$. Adin and Roichman [1] found a connection of cyclic permutations with a given descent set to characters of the symmetric group; these were further explored in [2]. In addition to descents, other permutation statistics were considered. For example, Diaconis, Fulman and Holmes [11] enumerated permutations by the number of peaks and cycle type.

In contrast, relatively little is known about pattern-avoiding cycles. The cycle structure of alternating permutations (which can be characterized as avoiding the consecutive patterns 321 and 123) was explored by Stanley [23], using the ideas of Gessel and Reutenauer. Cyclic permutations avoiding certain vector grid classes (which can also be characterized as avoiding a finite set of patterns) were enumerated in [3, 4, 18, 24, 25]. However, even "small" cases like the enumeration of 321-avoiding cycles remain unanswered.

3 The bijection φ

Let st denote the standardization map that sends a sequence of n distinct positive integers to a permutation in $\mathcal{S}_n$ in the same relative order as the given sequence. For example, $\operatorname{st}(24197) = 23154$. Define the map $\Phi : \mathcal{C}_n \to \mathcal{S}_{n-1}$ to send a permutation $\pi \in \mathcal{C}_n$ to the image $\Phi(\pi) = \operatorname{st}(\pi(1), \pi^2(1), \dots, \pi^{n-1}(1))$. Essentially, one writes the permutation in standard cycle notation, removes the parentheses and leading 1, and standardizes the resulting permutation. For instance, the standard cycle notation of the permutation $\pi = 365827914$ is (135267948), which maps to $\Phi(\pi) = \mathrm{st}(35267948) = 24156837$. The map Φ is clearly invertible, and thus is a bijection. Let $\varphi : \mathcal{A}_n^{\mathcal{C}} \to \mathcal{S}_{n-1}$ denote the map Φ restricted to $\mathcal{A}_n^{\mathcal{C}}$. If $\sigma \in \mathcal{U}_{n-1}$, then for notational convenience we denote the permutation $1 \oplus \sigma$ by π and denote the image of σ under φ^{-1} by $\hat{\pi}$.

Example 1. Consider the permutation $\sigma = 2567431 \in \mathcal{U}_7$; with the notation described above, $\pi = 13678542$ and $\hat{\pi} = 31624785 \in \mathcal{A}_8^{\mathcal{C}}$. Thus the permutation $\hat{\pi} = 31624785$ is mapped to the permutation to $\sigma = 2567431$ under the map φ. Figure 1 presents pictorial representations of the permutations π and $\hat{\pi}$.



(a) $\pi = 13678542 \in \mathcal{U}_8$ (b) $\hat{\pi} = 31624785 \in \mathcal{A}_8^{\mathcal{C}}$

Figure 1: Pictorial representations of the permutations π and $\hat{\pi}$. The figure on the right illustrates the cycle structure of $\hat{\pi}$, called the cycle diagram of the permutation (see [15]). The figure on the left illustrates π; the permutation within the solid grid represents σ.

Remark 2. The map $\Phi^{-1} : \mathcal{S}_{n-1} \to \mathcal{C}_n$ is very similar to the map found in [3, 13], called $\theta : \mathcal{S}_n \to \mathcal{C}_n$. This map θ sends any permutation $\pi = \pi_1\pi_2\ldots\pi_n$ to the permutation $\hat{\pi} = (\pi_1\pi_2\ldots\pi_n)$. In this form, θ is an n to 1 map, and thus is not bijective. Although the map Φ^{-1} is closer to convention, for notational convenience we use Φ, which when restricted to $\mathcal{A}_n^{\mathcal{C}}$ is φ.

Below we show that the image of $\mathcal{A}_n^{\mathcal{C}}$ under φ is exactly $\mathcal{U}_{n-1}$, and thus φ is a bijection between these two sets.

Theorem 3. *For $n \geqslant 2$, $\varphi(\mathcal{A}_n^{\mathcal{C}}) = \mathcal{U}_{n-1}$.*

Proof. We first establish that $\varphi(\mathcal{A}_n^{\mathcal{C}}) \subseteq \mathcal{U}_{n-1}$. For each $\hat{\pi} \in \mathcal{A}_n^{\mathcal{C}}$, it is clear that $\hat{\pi}(1) > 1$ and $\hat{\pi}(n) < n$. Thus it suffices to show that for all $i \in [n]$, if $i < \hat{\pi}(i) < n$, then $\hat{\pi}(i) < \hat{\pi}^2(i)$ and if $i > \hat{\pi}(i) > 1$, then $\hat{\pi}(i) > \hat{\pi}^2(i)$. Set $j = \hat{\pi}(i)$, and towards a contradiction assume that $i < j < n$ and $j > \hat{\pi}(j)$. By definition of an almost-increasing function, there is exactly one $k \leqslant j-1$ such that $\hat{\pi}(k) > j-1$, namely $\hat{\pi}(i) = j$. Our assumption $j > \hat{\pi}(j)$ implies there is no element $k \leqslant j$ for which $\hat{\pi}(k) > j$. Therefore,

$[j]$ maps to itself under the permutation $\hat{\pi}$. Since $j < n$, it follows that $\hat{\pi}$ is not cyclic, a contradiction. Now, towards another contradiction, assume that $i > j > n$ and $j < \hat{\pi}(j)$. Because $\hat{\pi}$ is almost-increasing there is exactly one $k \leqslant j$ such that $\hat{\pi}(k) > j$, namely j itself. This, along with the fact that $\hat{\pi}^{-1}(j) = i > j$, imply there is no element $k \leqslant j-1$ with $\hat{\pi}(k) > j-1$. Therefore, $[j-1]$ maps to itself under $\hat{\pi}$, implying the permutation is not cyclic, a contradiction. Thus $\varphi(\mathcal{A}_n^{\mathcal{C}}) \subseteq \mathcal{U}_{n-1}$.

It remains to show that all of $\mathcal{U}_{n-1}$ is obtained in the image of φ. For each $\sigma \in \mathcal{U}_{n-1}$, $\hat{\pi} = \varphi^{-1}(\sigma)$ is a cyclic permutation, and thus each element of $\hat{\pi}$ is of the form $\hat{\pi}^j(1)$ with $0 \leqslant j \leqslant n-1$. If $\hat{\pi}^k(1) = n$, then $\hat{\pi}^j(1) < \hat{\pi}^{j+1}(1)$ when $0 \leqslant j < k$ and $\hat{\pi}^j(1) > \hat{\pi}^{j+1}(1)$ when $k \leqslant j < n$. The elements $\hat{\pi}^j(1)$ with $0 \leqslant j < k$ are the excedances of $\hat{\pi}$ and the cooresponding values of these exceedance comprise an left-to-right increasing subsequence of $\hat{\pi}$; the elements $\hat{\pi}^j(1)$ with $k \leqslant j < n$ are the non-excedances of $\hat{\pi}$ and these values comprise a right-to-left decreasing subsequence of $\hat{\pi}$. Thus given any $i \in [n]$ with $\hat{\pi}_i > i$, the value of any previous excedance is less than or equal to i. It follows that for any $i \in [n]$, there is at most one element less than or equal to i whose image is larger than i, and so $\hat{\pi}$ is almost-increasing. Hence φ is a bijection between $\mathcal{A}_n^{\mathcal{C}}$ and $\mathcal{U}_{n-1}$, as desired. □

Remark 4. The proof of Theorem 3 relies on the following geometric intuition. For any almost-increasing cycle, tracing the cycle structure clockwise yields a path that crosses the diagonal once. This crossing corresponds to the peak of the unimodal permutation obtained via φ. Notice that on the right in Figure 1, we can observe this phenomenon illustrated by the green line.

The bijection φ given in Theorem 3 provides the following enumeration for almost-increasing cyclic permutations.

Corollary 5. *For $n \geqslant 2$, the number of almost-increasing cycles on $[n]$ is equal to 2^{n-2}.*

This corollary also follows from Theorem 6.1 in [15], which gives a generating function for the number of almost-increasing permutations on $[n]$ with k cycles, i fixed points and j excedances.

We conclude this section with a proposition that gives a new, simpler characterization of almost-increasing cyclic permutations in terms of pattern avoidance. It is well-known that 321-avoiding permutations are composed of two increasing, interleaved sequences. As a result the forthcoming proposition follows from the last paragraph of the proof of Theorem 3.

Proposition 6. *For $n \geqslant 1$, $\mathcal{C}_n(3412, 3421, 4312, 4321) = \mathcal{C}_n(321, 3412)$.*

We continue by proving some statistics on unimodal permutations.

4 Statistics on unimodal permutations

In this section, we enumerate unimodal permutations according to the number of inverse valleys, number of inversions and number of descents.

Theorem 7. *The number of unimodal permutations on* $[n]$ *with* k *inverse valleys is equal to* $\binom{n}{2k+1}$.

Proof. We establish a bijection between unimodal permutations with k inverse valleys and subsets of $[n]$ of size $2k+1$. Suppose σ is a unimodal permutation with k inverse valleys, and let $\sigma = \sigma_I n \sigma_D$, where σ_I is the increasing leg of σ and σ_D is the decreasing leg of σ, neither of which include the peak n. Notice that if there is an $2 \leqslant i \leqslant n$ so that $\sigma^{-1}(i-1) > \sigma^{-1}(i) < \sigma^{-1}(i+1)$, then i appears before $i+1$ in σ, and thus i must appear in σ_I. Since $i-1$ also appears after i in σ, $i-1$ must appear in σ_D. In fact, whenever an element $\sigma_j = i$ appears in σ_I, with $i-1$ appearing in σ_D, the position j is a position of an inverse valley. Notice that if $\sigma_1 = 1$, then there are $k+1$ consecutive runs of the segment σ_I and if $\sigma_1 \neq 1$, there are k consecutive runs of σ_I. Similarly, there are k consecutive (decreasing) runs of σ_D.

Construct a set S_σ by appending the following elements to S_σ:

1. $\ell := \sigma^{-1}(n)$;
2. the first k elements $j < \ell$ such that $\sigma_j < \sigma_{j+1} - 1$ or $\sigma_{j+1} = n$ (by the previous paragraph there are either k or $k+1$ such elements); and
3. the k elements $j > \ell$ such that $\sigma_j < \sigma_{j-1} - 1$.

We claim this set S uniquely determines the permutation σ. Specifically, order S in increasing order and take the $(k+1)$st element of S, say ℓ, to be the position of the peak. Construct the permutation σ as follows. Start with n at position i, decrease by one each time, and place a new label in each position starting from the position of n and working away, switching sides exactly when a position in S is to be encountered. If $\ell+1$ is an element of S, start with $n-1$ on the left of n, and place it to the right otherwise. This clearly returns the permutation σ. □

Below we give two examples of the bijection given in the proof of Theorem 7.

Example 8. The permutation $\sigma = 345789621 \in \mathcal{U}_9$ has two inverse valleys at positions 1 and 4 because σ^{-1} has valleys at positions 3 and 7. Since $\sigma^{-1}(9) = 6$, we have $6 \in S$. The positions 3 and 5 are the ends of the first two consecutive runs of 34578; as a result, we have $3, 5 \in S$. Finally, $\sigma_7 < \sigma_6 - 1$ and $\sigma_8 < \sigma_7 - 1$ so that $7, 8 \in S$. Therefore $S = \{3, 5, 6, 7, 8\}$ is the set associated to the permutation $\sigma = 345789621$.

We can recover σ by noting that the 3rd element of the set, namely 6, is the position of the peak and thus $\sigma_6 = 9$. Since $7 \in S$, the value $n-1 = 8$ will appear to the left of 9 and thus $\sigma_5 = 8$, since $4 \notin S$, we keep going and label $\sigma_4 = 7$. However, $3 \in S$ and so we switch to the other side of the peak and label $\sigma_7 = 6$. Since $8 \in S$, we switch to the other side and label $\sigma_1\sigma_2\sigma_3 = 345$. Since we have reached the end, we switch back to the other side and label $\sigma_8\sigma_9 = 21$.

Example 9. The permutation $145679832 \in \mathcal{U}_9$ has one inverse valley at position 2 and is associated to the set $S = \{1, 6, 8\}$.

Theorem 10. *The number of unimodal permutations on* $[n]$ *with* k *inversions is equal to the number of partitions of* k *into distinct parts of size at most* $n-1$.

Proof. We establish a bijection between unimodal permutations on $[n]$ with k inversions and the number of partitions of k into distinct parts of size at most $n-1$. Suppose $\sigma \in \mathcal{U}_n$ has k inversions. Define $D = \{i \in [n] : i > \sigma^{-1}(n)\}$, that is, D is the set of consecutive integers $\{j, j+1, j+2, \ldots, n\}$ where $\sigma_{j-1} = n$. Since the permutation σ is unimodal, D is the set of indices associated to the decreasing leg of σ. Notice that if (m, i) is an inversion, that is, $m < i$ with $\sigma_m > \sigma_i$, then $i \in D$. Moreover, for every $i \in D$, there is some inversion of this form; for example, $(m, i) = (\sigma^{-1}(n), i)$ is such and inversion.

Let $\text{inv}_\sigma(i)$ be the number of inversions of σ that are of the form (m, i) for some m. Consider the partition λ of the the number of inversions k with parts $\text{inv}_\sigma(i)$ for $i \in D$. Notice that for two elements of $i_1, i_2 \in D$, if $i_1 < i_2$, then $\text{inv}_\sigma(i_1) < \text{inv}_\sigma(i_2)$ and the inequality is strict because $\sigma_{i_1} > \sigma_{i_2}$. Therefore the elements of λ are increasing and thus distinct. As a result the partition obtained is a partition of k into distinct elements of size at most $n-1$.

We can see that this map is invertible as follows. Let λ be a partition of k into distinct parts of size at most n listed in increasing order. The number of parts of λ determines the length of the decreasing leg of σ. The elements of the decreasing leg, which uniquely determines the permutation σ, are given by $\sigma_j = n - \lambda_1, \sigma_{j+1} = n - \lambda_2, \ldots, \sigma_n = n - \lambda_{n-j+1}$. □

We continue by giving an example of the bijection detailed in the proof of Theorem 10.

Example 11. Consider the permutation $\sigma = 345789621 \in \mathcal{U}_9$, which has 18 inversions. Since $\sigma^{-1}(9) = 6$, $D = \{7, 8, 9\}$. Notice that $\text{inv}_\sigma(7) = 3, \text{inv}_\sigma(8) = 7$ and $\text{inv}_\sigma(9) = 8$. Therefore this permutation is associated to the partition of 18 into distinct parts of size at most 8: $\lambda = 3 + 7 + 8$. Reversing this process, we see that the decreasing leg of σ is $\sigma_D = \sigma_7, \sigma_8, \sigma_9 = (9-3), (9-7), (9-8) = 6, 2, 1$.

The next proposition is immediate because all descents of a permutation in $\mathcal{U}_n$ occur in the decreasing leg of the permutation, and thus the unimodal permutations with k descents are exactly those with $\sigma_{n-k} = n$.

Proposition 12. *The number of unimodal permutations on* $[n]$ *with* k *descents is equal to* $\binom{n-1}{k}$.

In the next section, we consider statistics on almost-increasing cyclic permutations.

5 Statistics on almost-increasing cycles

In this section, we establish a refinement of the bijection φ, given in Section 3, in order to enumerate almost-increasing cycles with respect to certain statistics.

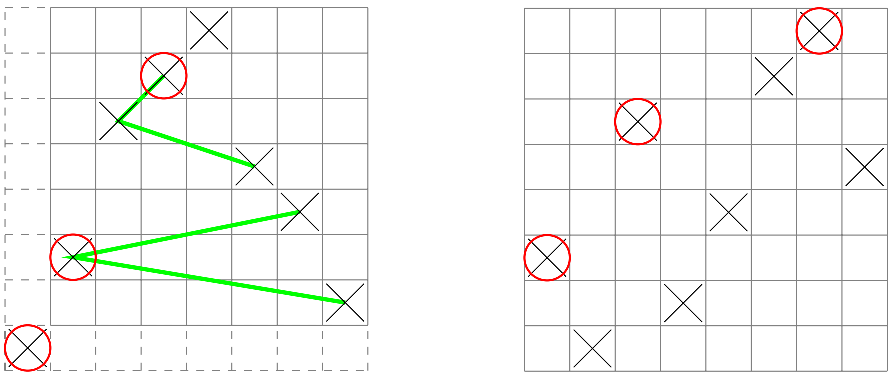
Theorem 13. *For* $n \geqslant 2$, *the number of almost-increasing cyclic permutations on* $[n]$ *with* k *descents is equal to* $\binom{n-1}{2k-1}$.

Proof. We will show that if $\sigma \in \mathcal{U}_{n-1}$ has $k-1$ inverse valleys, then $\varphi^{-1}(\sigma)$ is an almost-increasing cycle on $[n]$ with k descents. Let $\sigma \in \mathcal{U}_{n-1}$ with $k-1$ inverse valleys, and suppose $\pi = 1 \oplus \sigma$. Notice that σ and π have the same number of inverse valleys because appending 1 to σ does not introduce a new inverse valley. Let $\pi = \pi_I n \pi_D$, where π_I is the increasing leg of π and π_D is the decreasing leg of π, neither of which include the peak n. The positions of the valleys in π^{-1} are exactly the values at the start of consecutive runs in π_I, with the exception of $\pi_1 = 1$ (as detailed in the proof of Theorem 7).

If i is in π_I and $i+1$ is in π_D, then $\hat{\pi}_i > i$ and $\hat{\pi}_{i+1} < i+1$. Note that even if $\pi_n = i+1$, since $\pi_1 = 1$, we would have $\hat{\pi}_{i+1} = 1 < i+1$. Thus there is a descent at position i in $\hat{\pi}$. The last entry of π_I is also the position of a descent since $\hat{\pi}$ would take the value n at the position. Observe that since $\hat{\pi}$ is cyclic, n does not appear in the last position. Hence the values at the ends of the consecutive runs of π_I give the positions of the descents in $\hat{\pi}$. Since the number of consecutive runs of π is both equal to the number of descents and to one less than the number of inverse valleys of σ, the result follows. $\square$

Below we exemplify the proof of Theorem 13.

Example 14. For $n = 8$, consider the almost-increasing cycle associated to the permutation $\sigma = 2567431$ with 2 inverse valleys, $\varphi^{-1}(\sigma) = \hat{\pi} = 31624785$. Notice that the permutation $\pi = 1 \oplus \sigma = 13678542$ also has 2 inverse valleys. The positions of the valleys in π^{-1}, namely 3 and 6, are the start of the consecutive runs in the increasing leg 1367 of π (with the exception of the first entry). The values at the ends of these consecutive runs, namely 3 and 7, along with the the value at the end of the first increasing run, namely 1, give the descents in the permutation $\hat{\pi} = 31624785$.



(a) $\pi = 13678542 \in \mathcal{U}_8$ (b) $\hat{\pi} = 31624785 \in \mathcal{A}_8^{\mathcal{C}}$

Figure 2: Pictorial representations of the permutations π and $\hat{\pi}$. The figure on the left illustrates π together with its inverse valleys highlighted in green. These valleys correspond to the start of the consecutive runs with the exception of the first entry. The red circles correspond to the ends of the consecutive increasing runs, and the values at the end of these runs are the positions of the descents of $\hat{\pi}$ depicted on the right.

The forthcoming corollary follows from the observation that every descent of $\hat{\pi}$ is associated to a peak of $\hat{\pi}$, except possibly the first descent.

Corollary 15. *For $n \geqslant 2$, the number of almost-increasing cyclic permutations on $[n]$ with k peaks is equal to $\binom{n-1}{2k}$.*

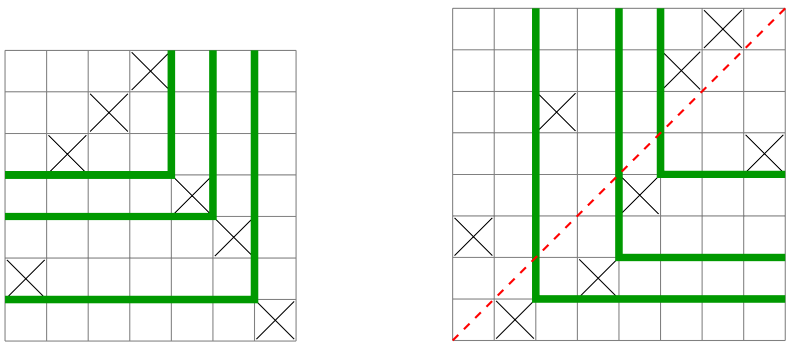
In the next theorem, we make use of the newly defined statistic low non-inversions; see Section 2.2 to recall its definition.

Theorem 16. *For $n \geqslant 2$, the number of almost-increasing cyclic permutations $\hat{\pi}$ on $[n]$ so that $\mathrm{lni}(\hat{\pi}) = k$ is equal to the number of partitions of k into parts of size at most $n-2$.*

Proof. For the permutation $\sigma \in \mathcal{U}_{n-1}$ and the corresponding permutation $\hat{\pi} = \varphi^{-1}(\sigma) \in \mathcal{A}_n^{\mathcal{C}}$, we establish a bijection between the inversions of σ and the low non-inversions of $\hat{\pi}$ as follows. Let (i_1, i_2) be an inversion of σ, that is, suppose $i_1 < i_2$ and $\sigma_{i_2} < \sigma_{i_1}$. We claim that there is a corresponding low non-inversion (j_1, j_2) of $\hat{\pi}$, where $j_1 = \sigma_{i_2} + 1$ and $j_2 = \sigma_{i_1} + 1$.

If $\pi = 1 \oplus \sigma$, then $\sigma_{i_2} + 1 = \pi_{i_2+1}$ and $\sigma_{i_1} + 1 = \pi_{i_1+1}$. Hence (i_1+1, i_2+1) is an inversion of π. Furthermore, by appending a 1 to σ, no new inversions are introduced to π. Since (i_1+1, i_2+1) is an inversion of π, we must have that $j_1 = \pi_{i_2+1}$ is in the decreasing leg of π. Thus $\hat{\pi}_{j_2} < j_2$, and since $\sigma_{i_2} < \sigma_{i_1}$, we have $j_1 < j_2$. Finally, $\pi_{i_2+1} < \pi_{i_1+1}$ implies that $j_2 = \pi_{i_1+1}$ either appears in the increasing leg of π or earlier in the decreasing leg of π than $j_1 = \pi_{i_2+1}$. Either way, we have that $\hat{\pi}_{j_1} < \hat{\pi}_{j_2}$. Thus (j_1, j_2) is indeed a low non-inversion. Moreover, all low non-inversions are associated to a corresponding inversion of σ. □

Reconsider the permutation $\hat{\pi} = 31624785 \in \mathcal{A}_8^{\mathcal{C}}$, which maps to $\sigma = 2567431 \in \mathcal{U}_7$ under the bijection φ. Consider Figure 3 that gives pictorial representations of σ and $\hat{\pi}$. Observe that the low non-inversions involving 4 are $(4,5)$, $(4,6)$, $(4,7)$ and $(4,8)$. These low non-inversions correspond to the inversions of σ involving 6, namely $(2,6)$, $(3,6)$, $(4,6)$ and $(5,6)$.



(a) $\sigma = 2567431 \in \mathcal{U}_7$ (b) $\hat{\pi} = 31624785 \in \mathcal{A}_8^{\mathcal{C}}$

Figure 3: Pictorial representations of the permutations σ and $\hat{\pi}$. The inversions of σ of the form (i,j) can be seen in (a). There are three inversions of the form $(i,5)$, four inversions of the form $(i,6)$, and six inversions of the form $(i,7)$. The corresponding low non-inversions of $\hat{\pi}$ can be seen in (b). In this case there are three low non-inversions of the form $(5,j)$, four inversions of the form $(4,j)$, and six inversions of the form $(2,j)$.

We can also use the bijection φ to recover the following result obtained originally by Elizalde in [15].

Theorem 17. *For $n \geqslant 2$, the number of almost-increasing cyclic permutations on $[n]$ with k excedances is equal to $\binom{n-2}{k-1}$.*

This theorem follows since excedences of the almost-increasing permutation corresponded exactly to the ascents of the corresponding unimodal permutation. Indeed there is an ascent at k of $\pi = 1 \oplus \sigma$ if and only if there is an excedence at the position $i = \pi_k$ of $\hat{\pi}$. From this observation, we can also obtain the following theorem.

Theorem 18. *For $n \geqslant 2$, almost-increasing cyclic permutations on $[n]$ have $n-1$ inversions.*

Proof. Let π be a unimodal permutation on $[n]$ and let $\hat{\pi}$ be the corresponding almost-increasing cyclic permutation. For each inversion (i, j) of $\hat{\pi}$, i must be an excedance of $\hat{\pi}$. For each excedance i of $\hat{\pi}$, i is associated so some ascent (as described in the discussion of Theorem 17). If excedance i of $\hat{\pi}$ is associated to an ascent at k of π, there are $\pi_{k+1} - \pi_k$ inversions of $\hat{\pi}$ of the form (i, j). This gives $n-1$ total inversions of $\hat{\pi}$. □

6 Other cycle types

We conclude with a theorem generalizing the characterization given in Theorem 3 to all cycle types.

Theorem 19. *When written in standard form, an almost-increasing permutation is completely characterized as satisfying the conditions:*

- *each cycle is unimodal, and*
- *for any two cycles, excluding fixed points, all elements in one cycle are each less than all elements in the other.*

The proof utilizes very similar ideas to the proof of Theorem 3 for almost-increasing cycles. Consider the following examples of Theorem 19. The permutations $\pi_1 = 32418657$ and $\pi_2 = 32185467$ are almost-increasing permutations in $\mathcal{A}_8$; their standard forms $\pi_1 = (134)(587)(6)$ and $\pi_2 = (13)(2)(4876)(5)$ satisfy the both conditions of Theorem 19.

This theorem allows us to enumerate almost-increasing permutations by their cycle type.

Theorem 20. *Almost-increasing permutations with cycle type $\lambda = (1^{k_1}, 2^{k_2}, \ldots, n^{k_n})$ are enumerated by the formula*

$$\binom{n}{k_1} \frac{(|\lambda| - k_1)!}{k_2! k_3! \cdots k_n!} \cdot 2^{n-2|\lambda|+k_1}.$$

Proof. The k_1 fixed points can be any k_1 numbers in $[n]$. The remaining cycles are ordered by the multinomial coefficient $\frac{(|\lambda|-k_1)!}{k_2!k_3!\cdots k_n!}$. The values these use are determined and it remains to account for the number of unimodal cycles, which is 2^{i-2} for a cycle of size i. Therefore, we have

$$2^{0\cdot k_2}\cdot 2^{1\cdot k_3}\cdot 2^{2\cdot k_4}\cdots 2^{(n-1)k_n}.$$

Using the fact that $k_1+2k_2+3k_3+\cdots+nk_n=n$, we obtain the desired formula. □

This formula agrees with, and generalizes, the generating function the number of cycles (denoted by $\mathrm{cyc}(\pi)$) of an almost-increasing permutation π, given by Elizalde in [15]:

$$F(t,x)=\sum_{c,n\geqslant 0}|\{\pi\in\mathcal{A}_n:\mathrm{cyc}(\pi)=c\}|t^c x^n=\frac{1}{1-tx-\frac{tx^2}{1-(2+t)x}}.$$

7 Open Questions

In this paper, we saw that cyclic permutations whose one-line notations avoid the set of patterns $\{3412,3421,4312,4321\}$ are exactly those whose standard cycle notations avoid the set of patterns $\{213,312\}$. We are interested in finding more occurrences of this phenomenon. That is, for which other pattern-avoiding permutations can the standard cycle notation also be characterized in terms of pattern avoidance. More generally, one can ask whether is there is any nice description of the standard cycle notation of certain pattern-avoiding permutations.

This question is closely related to the work in [3], where it is found that cyclic permutations in certain vector grid classes (and thus avoid a finite set of patterns) are exactly those whose cycle notation comprises the relative order of the iterates of a periodic point of a dynamical system under iteration. For example, cyclic permutations avoiding the set of permutations $\{213,312\}$, when written in cycle notation gives the relative order of iterates of periodic points of the well-known tent map (also seen in [25]).

Moreover, this work furthers the study of the enumeration of pattern-avoiding permutations by cycle type. While much is known about pattern-avoiding involutions, and fixed points of pattern-avoiding permutations, little is known about other cycle types of pattern-avoiding permutations. In addition to almost-increasing permutations in this manuscript, unimodal permutations were enumerated by cycle type in [18, 24], and permutations avoiding $\{231,312\}$, $\{231,321\}$, and $\{132,321\}$ were enumerated by cycle type in Chapter 5 of [16]. Enumerating permutations avoiding a single pattern of length 3 by cycle type, and even enumerating the cyclic permutations in these cases, remains an open question.

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