A Conjecture of Norine and Thomas for Abelian Cayley Graphs*

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Abstract

A graph Γ_1 is a matching minor of Γ if some even subdivision of Γ_1 is isomorphic to a subgraph Γ_2 of Γ, and by deleting the vertices of Γ_2 from Γ the left subgraph has a perfect matching. Motivated by the study of Pfaffian graphs (the numbers of perfect matchings of these graphs can be computed in polynomial time), we characterized Abelian Cayley graphs which do not contain a $K_{3,3}$ matching minor. Furthermore, the Pfaffian property of Cayley graphs on Abelian groups is completely characterized. This result confirms that the conjecture posed by Norine and Thomas in 2008 for Abelian Cayley graphs is true.

Keywords: perfect matchings; Pfaffian graphs; Cayley graphs; Abelian groups

1. Introduction

All graphs in this paper are finite and simple. Let $\Gamma = (V, E)$ be a graph. A *perfect matching* of Γ is a set of independent edges of Γ covering all vertices of Γ. A subgraph Γ' of Γ is *nice* if $\Gamma - \Gamma'$ has a perfect matching, where $\Gamma - \Gamma'$ denotes the subgraph of Γ obtained from Γ by deleting the vertices of Γ'. The length of a path is the number of its edges. We say that a graph Γ' is a subdivision of a graph Γ if Γ' is obtained from Γ by replacing the edges of Γ by internally disjoint paths of length at least one. If each replacing path is of odd length, then Γ' is called an *even subdivision* of Γ. A graph Γ' is a *matching minor* of Γ if some even subdivision of Γ' is isomorphic to a nice subgraph of Γ. For $F \subseteq E(\Gamma)$, we denote by $\Gamma - F$ the graph obtained from Γ by removing the edges in F. Let $\overrightarrow{\Gamma}$ be an orientation of Γ. An even cycle C is called *oddly oriented* if traversing

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C we encounter an odd number of edges of $\overrightarrow{\Gamma}$ oriented in the direction of the traversal. The orientation $\overrightarrow{\Gamma}$ is a *Pfaffian orientation* if every nice even cycle in Γ is oddly oriented relative to $\overrightarrow{\Gamma}$. A graph is said to be *Pfaffian* if it admits a Pfaffian orientation.

It is known that the number of perfect matchings of a Pfaffian graph can be computed in polynomial time [13]. Enumeration problem for perfect matchings in general graphs (even in bipartite graphs) is ♯P-complete. So determining whether a graph is Pfaffian is interesting. Obviously, if a graph is Pfaffian, then every nice subgraph of it is Pfaffian. And if some nice subgraph of a graph Γ is not Pfaffian, then Γ is not Pfaffian. Kasteleyn [4, 5] showed that every planar graph is Pfaffian and described a polynomial-time algorithm for finding a Pfaffian orientation of a planar graph. Denote by $K_{s,t}$ the complete bipartite graph with two partitions of s vertices and t vertices. Little [9] gave a sufficient condition containing a $K_{3,3}$ matching minor for a non-Pfaffian graph. And it is also necessary for bipartite graphs. McCuaig [14] and, independently, Robertson et. al. [17] provided a polynomial-time algorithm for determining whether a bipartite graph is Pfaffian.

A graph is *1-extendible* if every edge has a perfect matching containing it. A graph is *near-bipartite* if it is 1-extendible, not bipartite but it has two edges whose removal yields a 1-extendible bipartite graph. Fischer and Little [3] characterised the near-bipartite graph G is not Pfaffian if and only if a $K_{3,3}$, cubeplex or twinplex can be obtained from a matching minor of G by a sequence of odd cycle contractions. Miranda and Lucchesi [15] discovered a polynomial-time algorithm to decide whether a near-bipartite graph is Pfaffian. A 1-extendible graph Γ is *solid* if, for any two disjoint odd cycles C_1 and C_2 of Γ, the subgraph $\Gamma - V(C_1 \cup C_2)$ has no perfect matching. de Carvalho, Lucchesi and Murty [1] characterized the Pfaffian property of solid 1-extendible graphs, which are a kind of generalization of the bipartite graphs. They introduced the notion of a minor, name S-minor, which is stronger than the notion of a matching minor, and proved that every non-Pfaffian solid 1-extendible graphs contains the $K_{3,3}$ as an S-minor. Norine and Thomas [16] exhibited an infinite family of minimal non-Pfaffian graphs with respect to the matching minor. They also stated the following conjecture.

Conjecture 1. There exists a finite collection of valid rules such that every non-Pfaffian graph can be reduced to $K_{3,3}$ by repeated reductions using those rules.

Little and Rendl [10] discussed various operations on graphs and examined their effect on the Pfaffian property. The Cartesian product of two graphs Γ and Γ', denoted by $\Gamma\Box\Gamma'$, is the graph with vertex set $V(\Gamma\Box\Gamma') = \{(u,v)|u \in V(\Gamma), v \in V(\Gamma')\}$ and two vertex (u_1, v_1) and (u_2, v_2) in $\Gamma\Box\Gamma'$ are adjacent if and only if $v_1 = v_2$ and $u_1u_2 \in E(\Gamma)$, or $u_1 = u_2$ and $v_1v_2 \in E(\Gamma')$. Yan and Zhang [20, 21] considered the Pfaffian property of the Cartesian product of trees and a path with four vertices. Lin and Zhang [6] studied the Pfaffian property of Cartesian product of non-bipartite unicycle graphs and a path with four vertices. Denote by P_k and C_k the path and the cycle with k vertices respectively. Lu and Zhang [11] characterized the Pfaffian property of the Cartesian products $\Gamma\Box P_{2n}$ and $\Gamma\Box C_{2n}$ for any graph Γ.

Given a group G with identity element 1 and a subset S of G such that $1 \notin S$ and $x \in S$ implies $x^{-1} \in S$, the Cayley graph of G with respect to S, denoted by $\Gamma(G, S)$, has the elements of G as its vertices and edges joining x and yx for all $x \in G$ and $y \in S$. For $S \subseteq G$, denote by $\langle S \rangle$ the subgroup of G generated by S. It is known that every Cayley graph is vertex-transitive. When the group is cyclic, or $G \cong Z_n$, the Cayley graph is called a circulant, denoted by $Z_n(S)$. The group G is said to be Abelian if $ab = ba$ for all $a, b \in G$. Obviously, circulants are special Abelian Cayley graphs. We say the edge joining x and sx in $\Gamma(G, S)$ is s-edge. Stong [18] showed that for every generating set a Cayley graph over an Abelian group of even order has a perfect matching. In this paper, the Pfaffian property of Cayley graph on finite Abelian groups is completely characterized, that is the following theorem.

Theorem 2. *Let $\Gamma = \Gamma(S)$ be a connected Cayley graph on an Abelian group of even order. Γ is Pfaffian if and only if Γ contains no $K_{3,3}$ as a matching minor.*

The proof of Theorem 2 is given in Section 3.

2. Structures of $\Gamma(G, S)$ with $|S| \leqslant 4$

Suppose G is a finite Abelian group with $|G| = n$. For convenience, write $\Gamma(S)$ for the Abelian Cayley graph $\Gamma(G, S)$ with respect to the subset S of G. If $|S|$ is odd, then S contains an odd number of elements of order 2 since $S = S^{-1}$. For the element of order 2, we have the following proposition.

Proposition 3. *Suppose $|S| = 2$. Then $\Gamma(S)$ is a union of disjoint cycles. And, except the elements in S, there exist at most two elements of order 2 in every cycle of $\Gamma(S)$, specially, there exists at most one element of order 2 in the cycle containing 1.*

Proof. Since $|S| = 2$, Γ is 2-regular, Hence it is a union of cycles. We may assume the following two cases: $S = \{a, a^{-1}\}$ where $a \neq a^{-1}$, or $S = \{a, b\}$ where $a^2 = b^2 = 1$. And assume that there is an elements of order 2, denoted by c ($c \notin S$), in some cycle of $\Gamma(G, S)$. If $S = \{a, a^{-1}\}$, then every element of order 2 in the cycle has the form $a^s c$, where s is an integer. Thus $1 = (a^s c)^2 = a^{2s} c^2 = a^{2s}$. Hence, $s = 0$ or $s = |a|/2$. So at most two elements of order 2, that is $a^0 c = c$ and $(a^{|a|/2} c)$, in this cycle. If $S = \{a, b\}$, then this cycle is of length four. Except the elements in S, only two elements left. If 1 lies in the same cycle, then every element of order 2 in the cycle has the form a^s. Only one element of order 2, since $a^0 = 1$. $\square$

Obviously, a Cayley graph $\Gamma(S)$ is connected when $\langle S \rangle = G$. The following proposition is about the subset of S and the subgraph of $\Gamma(S)$.

Proposition 4. *[7]Suppose G is a finite Abelian group which is generated by S. And let G' be the subgroup of G which is generated by $S' = S \backslash \{s, s^{-1}\}$, where $s \in S$. If $G' \cap \langle s, s^{-1} \rangle = \{1\}$, then $\Gamma(S) = \Gamma(G', S') \square \Gamma(\langle s, s^{-1} \rangle, \{s, s^{-1}\})$.*

Specially, if $s \in S, s \notin S \backslash \{s\}$ and $s^2 = 1$, then $\Gamma(S) = \Gamma(\langle S \backslash \{s\} \rangle, S \backslash \{s\}) \square P_2$, by Proposition 4.

Proposition 5. *Suppose the Abelian Cayley graph $\Gamma(S)$ is connected with $|S| = 3$. Then $\Gamma(S)$ is isomorphic to $Z_n(1, n-1, n/2)$ or $C_{n/2}\Box P_2$.*

Proof. By $|S| = 3$, S contains at least one element of order 2. Suppose $S = \{a, b, c\}$, where $a^2 = 1$. By Proposition 1, $\Gamma(\langle b, c\rangle)$ is a cycle or a union of disjoint cycles. And the a-edges consist of a perfect matching of $\Gamma(S)$. Since Γ is connected, $\Gamma(\langle b, c\rangle)$ has at most two cycles. If it is a cycle, then $\Gamma(S)$ is isomorphic to $Z_n(1, n-1, n/2)$. If it is a union of two disjoint cycles, then $\Gamma(S)$ is isomorphic to $C_{n/2}\Box P_2$. So the result follows. □

Proposition 6. *If n is even, then $\Gamma = Z_{2n}(1, 2n-1, n)$ contains no even division of $K_{2,3}$ as a subgraph after contracting at most one odd cycle.*

Proof. Label the vertices of Γ as $\{v_1, v_2, \ldots, v_{2n}\}$ such that $E(\Gamma) = \{v_iv_{i+1} : i = 1, 2, \ldots, 2n-1\} \cup \{v_{2n}v_1\} \cup \{v_iv_{2n+1-i} : i = 1, 2, \ldots, n\}$. If Γ contains an even division of $K_{2,3}$ as a subgraph, denoted by H_1. Since Γ is vertex transitive, without loss of generality, suppose $v_1, v_t \in V(H_1)$ and the degrees of v_1 and v_t are 3 in H_1. Then v_1v_2, v_1v_{2n} and v_1v_{n+1} belong to three different paths in H_1. Note that $\Gamma - \{v_1v_{n+1}, v_nv_{2n}\}$ is a bipartite graph with the two color classes $\{v_i | i \text{ is even}\}$ and $\{v_i | i \text{ is odd}\}$. If t is odd, then all the paths from v_2 (or v_{2n}) to v_t in $\Gamma - \{v_1v_{n+1}, v_nv_{2n}\}$ are of odd length. And all the paths from v_{n+1} to v_t in $\Gamma - \{v_1v_{n+1}, v_nv_{2n}\}$ are of even length since n is even. So there do not exist three different paths of even length from v_1 to v_t in Γ. A contradiction. Similar contradiction occurs when t is even. So Γ contains no even subdivision of $K_{2,3}$ as a subgraph.

Let $E_1 = \{v_iv_{i+1} : i = 1, 2, \ldots, 2n-1\} \cup \{v_{2n}v_1\}$. For any odd cycle C of Γ, C contains $|E_1|/2$ edges in E_1. Let Γ' be obtained from Γ by contracting C to a vertex v. Then v is adjacent to all the other vertices of Γ', and $\Gamma' - v$ consists of disjoint paths. So Γ' contain no even subdivision of $K_{2,3}$. □

Thomassen [19] defined a kind of quadrilateral tilling on the torus, named $Q_{m,s,r}$(also called r-pseudo-cartesian products in [2, 7, 8]), where $s \geqslant 1$ and $0 \leqslant r < m$. If $s > 1$, write the vertices of the two cycles on the boundary of $C_m\Box P_s$ as $x_1, x_2, \ldots, x_m$ and $y_1, y_2, \ldots, y_m$, respectively, where x_i corresponds to $y_i (i = 1, 2, \ldots, m)$. The graph $Q_{m,s,r}$ is the graph obtained from $C_m\Box P_s$ by adding edges x_iy_{i+r} ($i = 1, 2, \ldots, m$, r is an integer and $i + r$ is modulo m). If $s = 1$, the graph $Q_{m,s,r}$ can be obtained from an m-cycle $w_1w_2 \ldots w_mw_1$ by adding edges w_iw_{i+r}, $i = 1, 2, \ldots, m$, where $i + r$ is modulo m. It is easy to see that $Q_{m,1,r}$ is isomorphic to $Z_m(1, m-1, r, -r)$ and $Q_{m,s,0}$ is isomorphic to $C_m\Box C_s$. A 4-regular circular is isomorphic to some $Q_{m,s,r}$ [12].

Proposition 7. *Let $\Gamma = \Gamma(S)$ be a connected Cayley graph on an Abelian group with $|S| = 4$.*

(1) *If S contains no any element of order 2, then for $a, b \in S, a \neq b^{-1}$, there exists $0 \leqslant r < |a|$ such that $a^r = b^{n/|a|}$, and Γ is isomorphic to $Q_{|a|,n/|a|,r}$.*

(2) *If S contains some element of order 2, then Γ is isomorphic to $Q_{n/4,4,0}$, $Z_{n/2}(1, n/2-1, n/4)\Box P_2$ or $Q_{n/2,2,n/4}$.*

Proof.

(1) Suppose $S = \{a, a^{-1}, b, b^{-1}\}$. If $\langle a \rangle = G$, then a-edges consist of a hamiltonian cycle. Therefore, Γ is isomorphic to $Z_n(1, n-1, t, -t)$, where $a^t = b$. If $|a| < |G|$, then a-edges consist of a union of disjoint cycles. And those cycles are connected by b-edges in Γ. Therefore, Γ is isomorphic to $Q_{|a|,s,r}$, where $s = n/|a|$, $r(< |a|)$ is an integer satisfied $b^s = a^r$.

(2) Suppose $S = \{a, b, c, d\}$, where $c^2 = 1 = d^2$. By Proposition 3, $\Gamma(\{a, b\})$ is not a cycle, hence $\langle a, b \rangle \neq G$ and $\langle a, b \rangle$ does not contain both c and d. If $c, d \notin \langle a, b \rangle$ and $c \notin \langle a, b, d \rangle$, then $\langle a, b \rangle \cap \langle d \rangle = \{1\}$ and $\langle a, b, d \rangle \cap \langle c \rangle = \{1\}$. By Proposition 4, Γ is isomorphic to some $Q_{n/4,4,0}$ where $n/4 = |\langle a, b \rangle|$. If $c, d \notin \langle a, b \rangle$ and $d \in \langle a, b, c \rangle$, then $\langle a, b \rangle \cap \langle c \rangle = \{1\}$ and $\langle a, b \rangle \cap \langle d \rangle = \{1\}$. Therefore $\Gamma\langle a, b, c \rangle$ and $\Gamma\langle a, b, d \rangle$ are isomorphic to $C_{n/2} \square P_2$ where $n/2 = |\langle a, b \rangle|$, by Proposition 4. So Γ is isomorphic to some $Q_{n/2,2,n/4}$. We now suppose that one of c and d in $\langle a, b \rangle$. Assume that $c \in \langle a, b \rangle$ and $d \notin \langle a, b \rangle$, then $\langle a, b, c \rangle \cap \langle d \rangle = \{1\}$ and each component of $\Gamma(a, b, c)$ is isomorphic $Z_{n/2}(1, n/2 - 1, n/4)$ where $n/2 = |\langle a, b \rangle|$. Therefore Γ is isomorphic to $Z_{n/2}(1, n/2 - 1, n/4) \square P_2$ by Proposition 4. So the result follows. □

It is easy to see that $K_{3,3}$ contains an even subdivision of $K_{2,3}$ as a subgraph. Hence, Lemma 12 and Theorem 16 in [12] imply the following proposition.

Proposition 8. *Suppose $\Gamma = \Gamma(S)$ be a circulant graph with $|S| = 4$. Then Γ contains an even subdivision of $K_{2,3}$ as a subgraph, after contracting at most one odd cycle.*

Proposition 9. *Let $\Gamma = \Gamma(S)$ be a connected Cayley graph on an Abelian group with $|S| = 4$. Then Γ contains an even subdivision of $K_{2,3}$ as a subgraph, after contracting at most one odd cycle.*

Proof. By Proposition 5, Γ is isomorphic to $Q_{m,s,r}$ or $Z_{n/2}(1, n/2-1, n/4) \square P_2$. If Γ is isomorphic to $Z_{n/2}(1, n/2-1, n/4) \square P_2$, then $n/2$ is even and the spanning subgraph $C_{n/2} \square P_2$ of $Z_{n/2}(1, n/2-1, n/4) \square P_2$ contains an even subdivision of $K_{2,3}$. Therefore, $Z_{n/2}(1, n/2-1, n/4) \square P_2$ contains an even subdivision of $K_{2,3}$. In the following, we need to show that $Q_{m,s,r}$ contains an even subdivision of $K_{2,3}$ as a subgraph, after contracting at most one odd cycle, to complete the proof.

If $s = 1$, then Γ is a circulant. By Proposition 6, Γ contains an even subdivision of $K_{2,3}$, after contracting at most one odd cycle. If $s \geqslant 3$, it is easy to see that a subgraph $P_3 \square P_3$ of Γ contains an even subdivision of $K_{2,3}$ as a spanning subgraph. So we consider the case when $s = 2$. The spanning subgraph $C_m \square P_2$ of Γ contains an even subdivision of $K_{2,3}$ when m is even. When m is odd, label the vertices of $Q_{m,2,r}$ as $\{v_i, u_i : i = 1, 2, \ldots, m\}$ such that

$$E(Q_{m,2,r}) = \bigcup_{i=1}^{m} \{v_i u_i, v_i u_{i+r}, v_i v_{i+1}, u_i u_{i+1}\},$$

where the subscripts are modulo m. If r is odd, let

$$P_1 = v_1 v_m u_m u_{m-1} u_{m-2} u_{m-3} \ldots u_{3+r} u_{2+r}.$$

Then P_1 contains an odd number of vertices. Therefore, the three paths $v_1v_2u_{2+r}$, $v_1u_{r+1}u_{2+r}$ and P_1 consist of an even subdivision of $K_{2,3}$. If r is even, let

$$P_2 = v_1u_{1+r}u_ru_{r-1}\dots u_3u_2.$$

Then P_2 contains an odd number of vertices. Therefore, the three paths $v_1v_2u_2, v_1u_1u_2$ and P_2 consist of an even subdivision of $K_{2,3}$. Thus, the result holds. □

3. The $K_{3,3}$ matching minor of $\Gamma(S)$

For proving our main results, we need some known results.

Theorem 10. *[11] Let Γ be a connected graph.*

(1) $\Gamma\Box P_2$ *is Pfaffian if and only if* Γ *contains no subgraph which is, after the contraction of at most one cycle of odd length, an even subdivision of* $K_{2,3}$*;*

(2) $\Gamma\Box P_4$ *is Pfaffian if and only if* Γ *contains neither an even subdivision of Q-graph nor two edge-disjoint odd cycles as its subgraph, where Q-graph is the graph obtained from a 4-cycle and a vertex not in the 4-cycle by joining the vertex to one vertex of the 4-cycle;*

(3) $\Gamma\Box P_{2n}(n \geqslant 3)$ *is Pfaffian if and only if* Γ *contains no Y-tree as its subgraph, where Y-tree is a graph obtained from* $K_{1,3}$ *by replacing an edge by a path of length two.*

If a graph Γ contains an even subdivision of $K_{2,3}$ (after possibly contracting an odd cycle), then $\Gamma\Box P_2$ contains $K_{3,3}$ as a matching minor (see the proof of Theorem 10 in [11]). For the Pfaffian property of $Q_{m,s,r}$, we have the following result.

Theorem 11. *[22] The graph* $Q_{m,s,r}$ *of even order is Pfaffian if and only if it is not a bipartite graph, if and only if it has no matching minor isomorphic to* $K_{3,3}$.

Theorem 12. *[12] Let* $\Gamma = Z_n(\pm a_1, \pm a_2, \dots, \pm a_m)$ *be a connected circulant graph of even order. Then* Γ *is Pfaffian if and only if* $m = 1$ *or,* $m = 2$ *and* $a_1 + a_2$ *is odd, if and only if it has no matching minor isomorphic to* $K_{3,3}$.

For $s > 1$ and an even number m, label the vertices of $Z_m(1, m-1, m/2)\Box P_s$ as v_i^j, $i = 1, 2, \dots, m, j = 1, 2, \dots, s$, such that

$$E(Z_m(1,m-1,m/2)\Box P_s) = \bigcup_{j=1}^{s}\left((\bigcup_{i=1}^{m}\{v_i^jv_{i+1}^j\}) \bigcup (\bigcup_{i=1}^{m/2}\{v_i^jv_{i+m/2}^j\}) \right) \bigcup \left(\bigcup_{j=1}^{s-1}\bigcup_{i=1}^{m}\{v_i^jv_i^{j+1}\} \right)$$

where the subscripts are modulo m. The graph $Q'_{m,s,r}$ is the graph obtained from $Z_m(1, m-1, m/2)\Box P_s$ by adding edges $v_i^1v_{i+r}^s$ ($i = 1, 2, \dots, m$, r is an integer, $0 \leqslant r < m$ and $i + r$ is modulo m). Obviously, $Q'_{m,s,0}$ is isomorphic to $Z_m(1, m-1, m/2)\Box C_s$.

Proposition 13. *The graph $Q'_{m,s,r}$ contains $K_{3,3}$ as a matching minor.*

Proof. If $m \equiv 2 \pmod 4$, then $Z_m(1, m-1, m/2)$contains $K_{3,3}$ as a matching minor by Theorem 12, so does $Q'_{m,s,r}$ for $Z_m(1, m-1, m/2)$ is a nice subgraph of $Q'_{m,s,r}$. In the following, we consider the case when $m \equiv 0 \pmod 4$. Label the vertices of $Q'_{m,s,r}$ as $\{u^i_j : i = 1, 2, \ldots, s; j = 1, 2, \ldots, m\}$ such that

$$E(Q'_{m,s,r}) = \bigcup_{i=1}^{s}\left(\bigcup_{j=1}^{m}\{u^i_j u^i_{j+1}\}\bigcup(\bigcup_{j=1}^{m/2}\{u^i_j u^i_{j+m/2}\})\right)\bigcup\left(\bigcup_{i=1}^{s-1}\bigcup_{j=1}^{m}\{u^i_j u^{i+1}_j\}\right)\bigcup\left(\bigcup_{j=1}^{m}(v^1_i v^s_{i+r})\right)$$

where the subscripts are modulo m. If $m \geqslant 8$ and $s \geqslant 3$, then $Q'_{m,s,r}$ contains $Z_8(1, 7, 4)\Box P_3$ as a nice subgraph. The subgraph spanned by $\{u^1_1, u^1_2, u^2_1, u^2_2, u^1_6, u^1_5\}$, together with the paths $u^2_2 u^3_2 u^3_6 u^3_5 u^3_4 u^2_4 u^1_4 u^1_5$ and $u^2_1 u^2_5 u^2_6 u^1_6$, consists of an even division of $K_{3,3}$, and after its removal, the left graph has a perfect matching $\{u^1_3 u^1_7, u^2_3 u^2_7, u^3_3 u^3_7, u^1_8 u^2_8, u^3_8 u^3_1\}$. That is $Z_8(1, 7, 4)\Box P_3$ contains $K_{3,3}$ as a matching minor (see Figure 1). If $s = 2$ and r is even, then $Q'_{m,s,r}$ contains $Q_{m,2,r}$ as a spanning subgraph, and $Q_{m,2,r}$ is a bipartite graph when m and r are even and therefore contains $K_{3,3}$ as a matching minor by Theorem 11. If $s = 2$ and r is odd, let H be the subgraph of $Q'_{m,2,r}$ which is generated by vertices $\{v^1_1, v^1_2, v^2_{1+r}, v^2_{2+r}, v^2_{1+r+m/2}, v^2_{2+r+m/2}\}$ together with the two paths $v^1_2 v^1_3 \ldots v^1_{r+3} v^2_{r+3} v^2_{r+4} \ldots v^2_{r+1+m/2}$ and $v^1_1 v^2_1 v^2_m v^2_{m-1} \ldots v^2_{r+2+m/2}$. Then H is an even subdivision of $K_{3,3}$. And $Q'_{m,2,r} - H$ has a perfect matching

$$\{v^1_{r+4} v^1_{r+5}, v^1_{r+6} v^1_{r+7}, \ldots, v^1_{m-1} v^1_m, v^2_2 v^2_3, v^2_4 v^2_5, \ldots, v^2_{r-1} v^2_r\}.$$

So $Q'_{m,2,r}$ contains $K_{3,3}$ as a matching minor.

If $m = 4$ and $s = 3$, then $Q'_{4,3,r}$ contains $K_{3,3}$ as a matching minor for $r = 1, 2$ (see Figure 2, the colored edges consist of an even division of $K_{3,3}$). After removing the edges $\bigcup_{j=4}^{s}\left((\bigcup_{i=1}^{m}\{v^j_i v^j_{i+1}\})\bigcup(\bigcup_{i=1}^{m/2}\{v^j_i v^j_{i+m/2}\})\right)$ from $Q'_{m,s,r}$ when $s > 3, r \neq 0$ and s is odd, an even division of $Q'_{4,3,r}$ can be gotten. After removing the edges $\bigcup_{j=3}^{s}\left((\bigcup_{i=1}^{m}\{v^j_i v^j_{i+1}\})\bigcup(\bigcup_{i=1}^{m/2}\{v^j_i v^j_{i+m/2}\})\right)$ from $Q'_{m,s,r}$ when $s > 3, r \neq 0$ and s is even, an even division of $Q'_{4,2,r}$ can be gotten. Similarly, $Q'_{4,s,0}$ contains an even division of $Q'_{4,3,0}$ when s is odd, or $Q'_{4,4,0}$ when s is even. Both $Q'_{4,3,0}$ and $Q'_{4,4,0}$ contain $K_{3,3}$ matching minor (see Figure 3). So the result follows. □

The following result which is a characterization of the Pfaffian property of Abelian Cayley graphs, extending the result of circulants [12].

Theorem 14. *Let $\Gamma(G, S)$ be a connected Cayley graph on the Abelian group G with even order. Denote by S_2 the set of elements with order 2 in S. Then $\Gamma(G, S)$ is not Pfaffian if and only if $|S| \geqslant 5$, or one of the following statements holds:*

(1) $|S| = 3$, $|G| \equiv 2 \pmod 4$ *and S contains an element of order $|G|$;*

(2) $|S| = 4$, $|S_2| = 0$, *all elements in S are even order and for two elements $a, b \in S(a \neq b^{-1})$, there exists an integer $0 \leqslant r < |a|$ such that $|G|/|a| + r$ is even and $b^{|G|/|a|} = a^r$;*

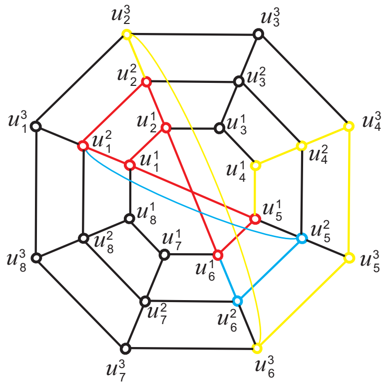
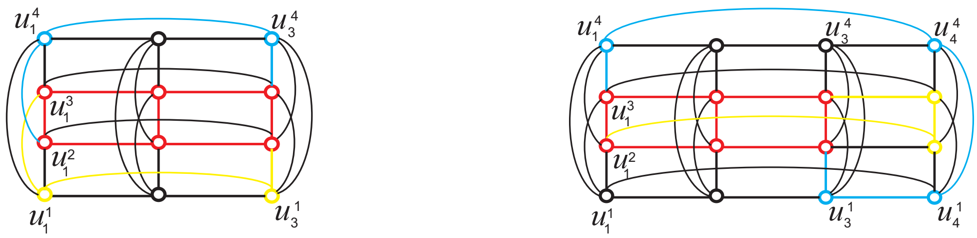


Figure 1: a $K_{3,3}$ matching minor of $Z_8(1,7,4)\Box P_3$.

(a) (b)

Figure 2: (a) A $K_{3,3}$ matching minor of $Q'_{4,3,1}$; (b) A $K_{3,3}$ matching minor of $Q'_{4,3,2}$.



(a) (b)

Figure 3: (a) A $K_{3,3}$ matching minor of $Q'_{4,3,0}$; (b) A $K_{3,3}$ matching minor of $Q'_{4,4,0}$.

(3) $|S| = 4$, $|S_2| \geqslant 2$, *$n/4$ is even when there exist $a, b \in S_2$ such that $a, b \notin \langle S \setminus \{a, b\}\rangle$, otherwise $n/4$ is odd.*

Firstly, we prove the following Lemma 15–Lemma 19.

Lemma 15. *Let $\Gamma = \Gamma(S)$ be a connected Abelian Cayley graph of n vertices, where $|S| = 3$. Then Γ is not Pfaffian if and only if $n \equiv 2 \pmod 4$ and S contains an element of order n.*

Proof. By Proposition 5, Γ is isomorphic to $C_{n/2}\Box P_2$ or $Z_n(1, n-1, n/2)$. If Γ is isomorphic to $C_{n/2}\Box P_2$, then there has no any element in S with order n and Γ is Pfaffian since it is a planar graph. If Γ is isomorphic to $Z_n(1, n-1, n/2)$, then S has an element with order n. By Theorem 12, Γ is Pfaffian if and only if $n/2$ is even. Thus for $a \in S$ with $|a| = n$, Γ is not Pfaffian if and only if $|a| \equiv 2 \pmod 4$. So the result follows. □

Lemma 16. *Let $\Gamma = \Gamma(G, S)$ be a connected Cayley graph on an Abelian group of even order with $|S| = 4$ and $|G| = n$. Denote by $S_2(\subset S)$ the set of elements with order 2. Then $\Gamma(S)$ is not Pfaffian if and only if one of the following statements holds:*

(1) $|S_2| = 0$, *all elements in S are even order and for two elements $a, b \in S(a \neq b^{-1})$, there exists an integer $0 \leqslant r < |a|$ such that $n/|a| + r$ is even and $b^{n/|a|} = a^r$;*

(2) $|S_2| \geqslant 2$, *$n/4$ is even if there exist $a, b \in S_2$ such that $a, b \notin \langle S \setminus \{a, b\}\rangle$, else $n/4$ is odd.*

Proof.

(1) For $|S_2| = 0$, Γ is isomorphic to some $Q_{|a|,s,r}$ where $a, b \in S(a \neq b^{-1})$, $s = n/|a|$, $0 \leqslant r < |a|$ satisfied $b^s = a^z$, by Proposition 7 1). By Theorem 11, $Q_{m,s,r}$ of even order is Pfaffian if and only if it is not a bipartite graph. And it can be checked that $Q_{m,s,r}$ of even order is bipartite if and only if both m and $s + r$ are even. Therefore Γ is not Pfaffian if and only if Γ satisfied 1).

(2) By Proposition 7 2), if there exist $a, b \in S_2$ such that $a, b \notin \langle S \setminus \{a, b\}\rangle$, then Γ is isomorphic to $Q_{n/4,4,0}$ or $Q_{n/2,2,n/4}$, otherwise Γ is isomorphic to $Z_{n/2}(1, n/2 - 1, n/4)\Box P_2$. If Γ is isomorphic to $Z_{n/2}(1, n/2-1, n/4)\Box P_2$, then $Z_{n/2}(1, n/2-1, n/4)$ contains no even division of $K_{2,3}$ as a subgraph after contracting at most one odd cycle when $n/4$ is even, by Proposition 6, and $Z_{n/2}(1, n/2-1, n/4)$ contains an even division of $K_{3,3}$ as a subgraph when $n/4$ is odd by Theorem 16 in [13]. So $Z_{n/2}(1, n/2-1, n/4)\Box P_2$ is not Pfaffian if and only if $n/4$ is odd by Theorem 10. Note $Q_{n/4,4,0}$ or $Q_{n/2,2,n/4}$ is bipartite if and only if $n/4$ is even. If Γ is isomorphic to $Q_{n/4,4,0}$ or $Q_{n/2,2,n/4}$, then Γ is not Pfaffian if and only if $n/4$ is even. Thus the result follows. □

Lemma 17. *Let $\Gamma = \Gamma(S)$ be a connected Cayley graph on an Abelian group with $|S| = 5$. Then Γ contains $K_{3,3}$ as a matching minor.*

Proof. If there exist an element of order 2 in S, denoted by a, such that $a \notin \langle S' \rangle$ where $S' = S \backslash \{a\}$, then $\Gamma = \Gamma(\langle S' \rangle, S') \Box P_2$ by Proposition 4. Each component of $\Gamma(\langle S' \rangle, S')$ is a 4-regular Abelian Cayley graph. And it contains an even subdivision of $K_{2,3}$, after contracting at most one odd cycle by Proposition 9. So Γ contains $K_{3,3}$ as a matching minor. So we consider that G can be induced by the set gotten by removing an arbitrary element of order 2 from S in the following.

Case 1. There is only one element of order 2 in S.

Suppose $S = \{a, b, b^{-1}, c, c^{-1}\}$ where $a^2 = 1$, and $a \in \langle S \backslash \{a\} \rangle$. Then at least one element in $\{b, c\}$ is of even order since $|G|$ is even. Without loss of generality, suppose b is of even order. If $\Gamma(\{a, b, b^{-1}\})$ is connected, then it is isomorphic to $Z_n(1, n-1, n/2)$ when $|b| = |G| = n$, or $C_{|b|} \Box P_2$ when $|b| < |G|$. In the former case, Γ is a 5-regular circulant since $\langle b \rangle = G$, therefore contains $K_{3,3}$ as a matching minor by Theorem 12. In the latter case, c-edges connect the two disjoint cycles of $\Gamma(\{b, b^{-1}\})$ since $a \in \langle S \backslash \{a\} \rangle$. Denote by C_1 and C_2 the two cycles of $\Gamma(\{b, b^{-1}\})$ such that $1 \in C_1$. Then $c, c^{-1}, a \in C_2$. There exist two vertices in $\{c, c^{-1}, a\}$, without loss of generality, suppose a and c, such that the distance between a and c is even in C_2. By removing all the c^{-1}-edges from Γ, a $Q_{|b|,2,r'}$ can be gotten, where r' is even, which is bipartite. Therefore, Γ contains $K_{3,3}$ as a matching minor by Theorem 11.

If $\Gamma(\{a, b, b^{-1}\})$ is not connected, then c-edges connect those components, and every component is isomorphic to $Z_{|b|}(1, |b|-1, |b|/2)$ or $C_{|b|} \Box P_2$. In the former case, Γ is isomorphic to $Q'_{|b|,n/|b|,r}$, which contains $K_{3,3}$ matching minor by Proposition 13. In the latter case, Γ contains $C_{|b|} \Box P_2 \Box P_2$ as a nice subgraph since $|\langle a, b, b^{-1} \rangle|$ is even. Note that $C_{|b|} \Box P_2$ contains an even subdivision of $K_{2,3}$ since $|b|$ is even. Then $C_{|b|} \Box P_2 \Box P_2$ contains $K_{3,3}$ as a matching minor. So does Γ.

Case 2. S contains at least three elements of order 2.

Suppose $a, b, c \in S$ and $a^2 = b^2 = c^2 = 1$. By Lemma 16 2), $\Gamma(S \backslash a)$ is isomorphic to $Q_{n/4,4,0}$, $Q_{n/2,2,n/4}$ or $Z_{n/2}(1, n-1, n/4) \Box P_2$. If $\Gamma(S \backslash a)$ is isomorphic to $Q_{n/2,2,n/4}$ or $Z_{n/2}(1, n-1, n/4) \Box P_2$, suppose C_1 and C_2 are the two cycles of $\Gamma(S \setminus \{a, b, c\})$. By Proposition 3, C_1 and C_2 contain two different elements in $\{1, a, b, c\}$, respectively. Then Γ is isomorphic to $Q'_{n/2,2,n/4}$, which contains $K_{3,3}$ as a matching minor by Proposition 13. If $\Gamma(S \setminus a)$ is isomorphic to $Q_{n/4,4,0}$, then it contains $K_{3,3}$ as a matching minor when $n/4$ is even by Theorem 11. So we consider when $n/4$ is odd. The elements a and 1 can not lie in the same cycle of $\Gamma(S \setminus \{a, b, c\})$ since $n/4$ is odd. So $a \in \langle b, c \rangle$ since $a \in S \setminus \{a\}$. Therefore each component of $\Gamma(\{a, b, c\})$ is isomorphic to K_4. And Γ is isomorphic to $Q'_{n/4,4,0}$, which contains $K_{3,3}$ as a matching minor by Proposition 13. So the result follows. □

Lemma 18. *Let $\Gamma = \Gamma(S)$ be a connected Cayley graph on an Abelian group of even order and $|S| = 6$. Then Γ contains $K_{3,3}$ as a matching minor.*

Proof. If S contains an element of order 2, denoted by a, then every component of $\Gamma(\langle S \backslash \{a\} \rangle, S \backslash \{a\})$ is 5-regular, therefore contains $K_{3,3}$ as a matching minor by Lemma 17. So does Γ. Suppose no element in S is of order 2 and $S = \{a, a^{-1}, b, b^{-1}, c, c^{-1}\}$. If one of $\Gamma(S \backslash \{a, a^{-1}\})$, $\Gamma(S \backslash \{b, b^{-1}\})$ and $\Gamma(S \backslash \{c, c^{-1}\})$ is not connected, without loss of generality, suppose $\Gamma(S \backslash \{a, a^{-1}\})$ is not connected and has k components. If the graph

$\Gamma(\langle S\backslash\{a,a^{-1}\}\rangle, S\backslash\{a,a^{-1}\})$ contains an odd number of vertices, then k is even, since $|G|$ is even. Therefore Γ contains $\Gamma(\langle S\backslash\{a,a^{-1}\}\rangle, S\backslash\{a,a^{-1}\})\Box P_2$ as a nice subgraph. If $\Gamma(\langle S\backslash\{a,a^{-1}\}\rangle, S\backslash\{a,a^{-1}\})$ contains an even number of vertices, then Γ also contains $\Gamma(\langle S\backslash\{a,a^{-1}\}\rangle, S\backslash\{a,a^{-1}\})\Box P_2$ as a nice subgraph, since $\Gamma(\langle S\backslash\{a,a^{-1}\}\rangle, S\backslash\{a,a^{-1}\})$ has a perfect matching. Note that $\Gamma(\langle S\backslash\{a,a^{-1}\}\rangle, S\backslash\{a,a^{-1}\})$ is isomorphic to some $Q_{m,s,r}$. It contains an even subdivision of $K_{2,3}$ after contracting at most one odd cycle by Proposition 9. So Γ contains $K_{3,3}$ as a matching minor by Theorem 10.

Suppose all of $\Gamma(S\backslash\{a,a^{-1}\})$, $\Gamma(S\backslash\{b,b^{-1}\})$ and $\Gamma(S\backslash\{c,c^{-1}\})$ are connected. Since $|G|$ is even, at least one element in $\{a,b,c\}$ is of even order. Without loss of generality, suppose a does. Then $\Gamma(\{a,a^{-1}\})$ consists of a union of disjoint even cycles, and those cycles connected by b-edges and c-edges. Label the cycles as $C_1, C_2, C_3, \ldots, C_{|G|/|a|}$ such that C_i and $C_{i+1} (i = 1, 2, \ldots, |G|/|a|)$ are connected by b-edges, where $i+1$ is modulo $|G|/|a|$. Note that $\Gamma(S\backslash\{c,c^{-1}\})$ is isomorphic to some $Q_{|a|,|G|/|a|,r}$. If $\Gamma(S\backslash\{c,c^{-1}\})$ is bipartite, then it contains $K_{3,3}$ as a matching minor by Theorem 11. So we suppose $\Gamma(S\backslash\{c,c^{-1}\})$ is not bipartite. After the removal of all the b-edges between C_i and C_{i+1} for any i, the left graph is bipartite. Denote the b-edges between C_1 and C_2 by E_b. Similarly, if C_j and C_k are jointed by c-edges, then after removing all the c-edges between C_j and C_k from $\Gamma(S\backslash\{b,b^{-1}\})$, the left graph is a bipartite graph. Suppose C_1 and C_t are jointed by c-edges and denote the c-edges between C_1 and C_t by E_c.

If the two ends of every edge in E_c are in different colors of $\Gamma(S\backslash\{c,c^{-1}\}) - E_b$, then by adding edges E_b to the subgraph of $\Gamma(S\backslash\{c,c^{-1}\})$ spanned by $V(C_t) \cup V(C_{t+1}) \cup \ldots \cup V(C_{|G|/|a|}) \cup V(C_1)$, we get a bipartite subgraph of Γ. It is isomorphic to some $Q_{|a|,|G|/|a|-t+2,r_1}$, which contains $K_{3,3}$ as a matching minor by Theorem 11. And it is nice subgraph of Γ since every cycle C_i, $i = 2, 3, \ldots, t-1$, is even, therefore Γ contains $K_{3,3}$ as a matching minor. So the two ends of every edge in E_c are in the same colors of $\Gamma(S\backslash\{c,c^{-1}\}) - E_b$. Then by adding edges E_b to the subgraph of $\Gamma(S\backslash\{c,c^{-1}\})$ spanned by $V(C_1) \cup V(C_2) \cup \ldots \cup V(C_t)$, a bipartite nice subgraph of Γ can be gotten. It is isomorphic to some $Q_{|a|,t,r_2}$, which contains $K_{3,3}$ as a matching minor by Theorem 11. Thus the proof is completed. □

Lemma 19. *Let $\Gamma = \Gamma(S)$ be a connected Cayley graph on an Abelian group of even order with $|S| = k > 6$. Then Γ contains $K_{3,3}$ as a matching minor.*

Proof. If k is odd, then S contains at least one element of order 2, denoted by a. Set $S' \subset S$ such that $|S'| = 5$, $a \in S'$ and $S' = S'^{-1}$. Each component of $\Gamma(S')$ is a nice subgraph of $\Gamma(S)$. And each component of $\Gamma(S')$ contains $K_{3,3}$ as a matching minor by Lemma 17. So does $\Gamma(S)$.

If k is even, then S contains at least one element of even order, denoted by b, since Γ has an even number of vertices. Set $S'' \subset S$ such that $|S''| = 6, S'' = S''^{-1}$ and $b \in S_2$. So $|\langle S''\rangle|$ is even. Then each component of $\Gamma(S'')$ is a nice subgraph of $\Gamma(S)$. And each component of $\Gamma(S'')$ contains $K_{3,3}$ as a matching minor by Lemma 18. So does $\Gamma(S)$. Thus the result is true. □

Proof of Theorem 14. It is easy to see that the result can be obtained immediately from Lemma 15–Lemma 19.

Proof of Theorem 2. If $|S| = 3$, then Γ is isomorphic to $C_{n/2}\Box P_2$ or $Z_n(1, n-1, n/2)$ by Proposition 5. $C_{n/2}\Box P_2$ is a planar graph, therefore Pfaffian. And $Z_n(1, n-1, n/2)$ is Pfaffian if and only if it contains no $K_{3,3}$ as a matching minor by Theorem 12. If $|S| = 4$ and $|S_2| = 0$, then Γ is isomorphic to some $Q_{|a|,s,r}$ where $a, b \in S(a \neq b^{-1})$, $s = n/|a|$, $0 \leqslant r < |a|$ satisfied $b^s = a^r$, by Proposition 7 1). By Theorem 11, Γ is Pfaffian if and only if it contains no $K_{3,3}$ as a matching minor. If $|S| = 4$ and $|S_2| \geqslant 2$, then Γ is isomorphic to $Q_{n/4,4,0}$, $Q_{n/2,2,n/4}$ or $Z_{n/2}(1, n/2-1, n/4)\Box P_2$. By the proof of Lemma 16 (2) and Theorem 11, Γ is Pfaffian if and only if it contains no $K_{3,3}$ as a matching minor. Therefore the result follows by Lemma 17–Lemma 19.

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