Bijection between oriented maps and weighted non-oriented maps

Agnieszka Czyżewska-Jankowska

Piałwska 20/16
50-538 Wrocław
Poland

agnieszka.czyzewska@gmail.com

Piotr Śniady*

Institute of Mathematics
Polish Academy of Sciences
ul. Śniadeckich 8
00-956 Warszawa
Poland

psniady@impan.pl

Abstract

We consider bicolored maps, i.e. graphs which are drawn on surfaces, and construct a bijection between (i) *oriented* maps with *arbitary* face structure, and (ii) (weighted) *non-oriented* maps with *exactly one face*. Above, each non-oriented map is counted with a multiplicity which is based on the concept of *the orientability generating series* and *the measure of orientability of a map*. This bijection has the remarkable property of preserving the underlying bicolored graph. Our bijection shows equivalence between two explicit formulas for the top-degree of Jack characters, i.e. (suitably normalized) coefficients in the expansion of Jack symmetric functions in the basis of power-sum symmetric functions.

Keywords: oriented maps, non-oriented maps, topological aspects of graph theory, Jack polynomials, Jack characters

0   Prologue

In order to motivate the Reader and to give her some flavor of the results to expect, we shall present now some selected highlights before getting involved in somewhat lengthy definitions. We also deliberately postpone the bibliographic details.

0.1 Maps and orientability

Roughly speaking, a *map* $M = (G, S)$ is a bicolored graph G which is drawn on a surface S. We require that the connected components of $S \setminus G$, called *faces*, are all homeomorphic to open discs. The set of vertices of G is decomposed into two disjoint sets: the set of white vertices and the set of black vertices. Each edge connects two vertices of opposite colors; multiple edges are allowed. We do not allow isolated vertices. We allow the surface S to be disconnected.
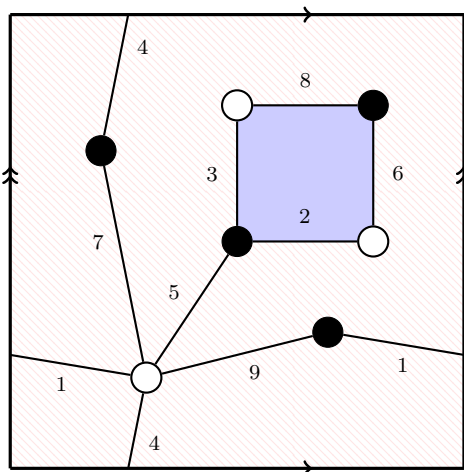

Figure 1: Example of an *oriented* map drawn on the torus. The left side of the square should be glued to the right side, as well as bottom to top, as indicated by the arrows. The statement that it is an *oriented* map means that there is a prescribed consistent choice around each vertex of what *clockwise* means.

The maps which we consider in the current paper come in the following two flavors: *oriented maps* (which are drawn on an orientable surface S which comes with some prescribed choice of the orientation, see Figure 1) and *non-oriented maps* (which are drawn on an arbitrary surface S without any additional structures, see Figure 2). These two flavors are quite distinct, nevertheless some algebraic-combinatorial conjectures which concern *Jack polynomials* suggest that there is some hypothetical natural one-parameter interpolation between them. To be more specific: it has been conjectured that there exists some *measure of non-orientability* of a given non-oriented map M which is a hypothetical polynomial weight$_M(\gamma)$ in the deformation parameter γ with the property that if a summation over *non-oriented* maps is performed, and each map is counted with appropriate multiplicity weight$_M(\gamma)$, the resulting sum becomes a natural interpolation between some generating series of oriented and non-oriented maps.

In the current paper we consider a concrete formula for such a candidate weight, a candidate which will be denoted by mon$_M(\gamma)$, which is an acronym that stands for **m**easure **of n**on-*orientability*.
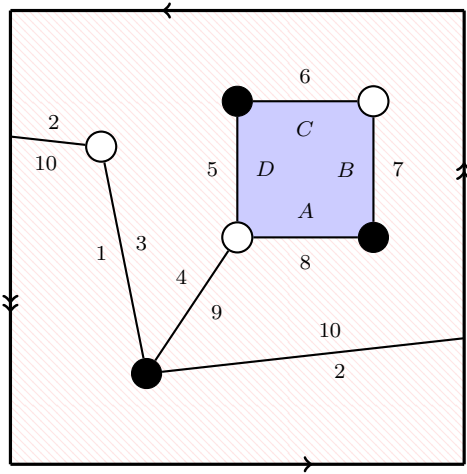
Figure 2: Example of a *non-oriented map* drawn on the projective plane. The left side of the square should be glued with a twist to the right side, as well as bottom to top, as indicated by the arrows. The two faces of the map are indicated by the colors.
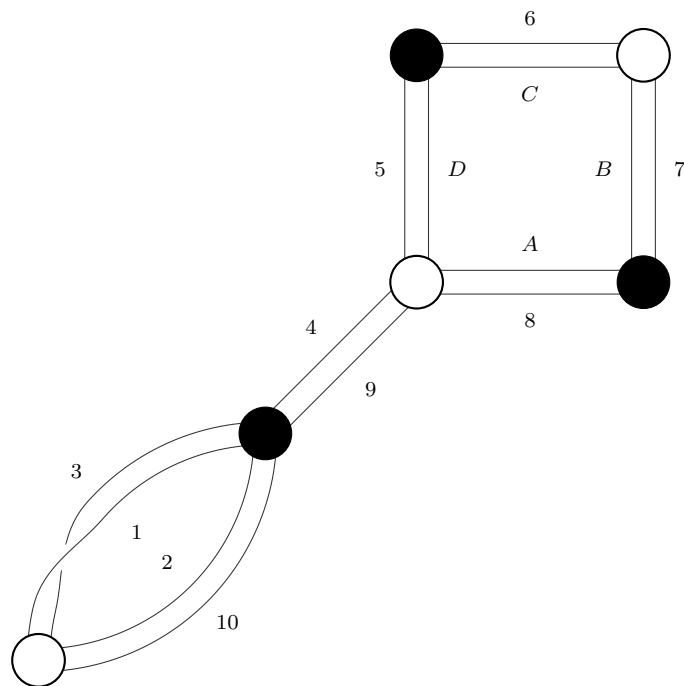

Figure 3: Alternative graphical representation of the map from Figure 2 as a ribbon graph. The edges of the map are represented as thin ribbons attached to the vertices.

0.2 Top-degree of the measure of non-orientability

In order to state the main results of the paper we will not need the definition of the full polynomial $\mathrm{mon}_M(\gamma) \in \mathbb{Q}[\gamma]$ and we shall restrict ourselves to *the leading coefficient* of this polynomial which will be denoted by $\mathrm{mon}_M^{\mathrm{top}}$. We shall present now its definition.

Any non-oriented map M can be equivalently viewed as a *ribbon graph*, see Figure 3. With this viewpoint each vertex becomes a disc, each edge becomes a thin ribbon connecting the discs, and each face of the map becomes a connected component of the boundary of the union of the discs and the ribbons.

From a ribbon graph corresponding to M we shall remove all of its edges, one after another, in a uniformly random order. We use the convention that whenever after an edge removal some vertex becomes isolated, we remove this vertex as well.

The following can be viewed as a *working definition* of the quantity $\mathrm{mon}_M^{\mathrm{top}}$.

Definition 0.1. Assume that M is a non-oriented map. The quantity $\mathrm{mon}_M^{\mathrm{top}}$ is defined to be the probability of *the event that in the above process of uniformly random edge removal, at each step the number of faces of the ribbon graph is equal to the number of its connected components.*

The link between the above quantity $\mathrm{mon}_M^{\mathrm{top}}$ and the polynomial mon_M will be provided later in Proposition 5.4.

Example 0.2. We consider the map M shown in Figure 4; the corresponding ribbon graph is shown in Figure 5.

Consider the case when the ribbon marked $\{3,6\}$ is removed first; then the remaining two ribbons form an annulus which has two faces and only one connected component, thus the event considered in Definition 0.1 does not hold.

Consider now the remaining two cases when either the ribbon $\{1,5\}$ or the ribbon $\{2,4\}$ is removed first; then the remaining two ribbons form the Möbius strip which has one face and one connected component. After yet another ribbon removal the unique remaining ribbon forms a ribbon graph which again consists of one face and one connected component. It follows that the event considered in Definition 0.1 holds.

In this way we found that the probability considered in Definition 0.1 is equal to $\frac{2}{3}$ and thus $\mathrm{mon}_M^{\mathrm{top}} = \frac{2}{3}$.

0.3 Rooted maps

The notion of a *rooted map* takes a different form when we speak about *oriented maps* than in the case of *non-oriented maps*.

More specifically, by a *rooted oriented map* we mean an oriented map in which one *edge* is decorated. For example, one can take the map from Figure 1 and remove the labels of all edges, except for the edge labeled by the symbol 1, and declare that this *edge* is decorated.

By a *rooted non-oriented map* we mean a non-oriented map in which one of the *edge-sides* is decorated (each edge consists of two edge-sides). For example, one can take the
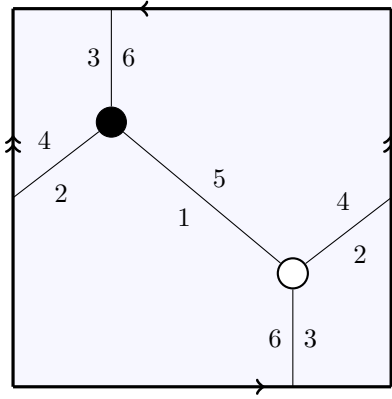
Figure 4: Example of a non-oriented map drawn on the Klein bottle: the left-hand side of the square should be glued to the right-hand side (without a twist) and the top side should be glued to the bottom side (with a twist), as indicated by the arrows. This map has one face.
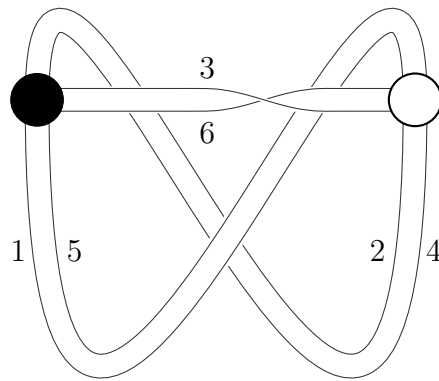
Figure 5: The map from Figure 4 drawn as a *ribbon graph*.

map from Figure 2 and remove the labels of all edge-sides, except for the edge-side labeled by 1, and declare that this *edge-side* is decorated.

0.4   The first main result

The following is one of the main results of the current paper (for the other one see Theorem 4.3). It states that a summation over *oriented maps* with *arbitrary face structure* is equivalent to a weighted summation over *non-oriented maps* with *exactly one face*.

Theorem 0.3 (The first main result)**.** *For all integers $n \geqslant 1$ the following formal linear combinations of* bicolored graphs *are equal:*

$$\sum_{M_1=(G_1,S_1)} G_1 = \sum_{M_2=(G_2,S_2)} \mathrm{mon}_{M_2}^{\mathrm{top}} G_2, \qquad (0.1)$$

where the sum on the left-hand side runs over oriented, unlabeled, rooted, connected maps M_1 with n edges *and the sum on the right-hand side runs over* non-oriented, unlabeled, rooted maps M_2 with n edges and one face.

It should be stressed that on the left-hand side we impose no restrictions on the face-type of the map M_1, in particular the number of the faces is arbitrary.

Equation (0.1) is an equality between formal linear combinations of bicolored graphs. Equivalently, it can be viewed as the following statement: for each bicolored graph G, the number of ways in which G can be drawn on

(A1) some oriented surface S_1 in such a way that (G, S_1) becomes an *oriented map*

is equal to the (weighted) number of ways in which G can be drawn on

(A2) some non-oriented surface S_2 in such a way that (G, S_2) becomes a *non-oriented map with exactly one face.*

Our proof of Theorem 0.3 will be bijective. For a fixed bicolored graph G we will find a bijection between:

(B1) the set of *non-oriented (but orientable) maps* (G, S_1), together with a choice of an *arbitrary linear order* on the set of edges of G, and

(B2) the set of *non-oriented maps* (G, S_2), together with a choice of a *linear order on the set of edges of G with the property that if the edges of a ribbon graph corresponding to the map (G, S_2) are removed according to this linear order, then the condition from Definition 0.1 holds true*, i.e. at each step the number of faces of the ribbon graph is equal to the number of its connected components.

Roughly speaking, our bijection consists of a number of *twists* of the ribbons, see Figure 6. The proof of Theorem 0.3 is postponed to Section 3.

One possible motivation for Theorem 0.3 is purely aesthetical. However, there is also another important motivation related to the study of *Jack polynomials* and *Jack characters* which will be explored in Section 4 and, in particular, in Sections 4.9 and 4.10.

0.5 Overview

The structure of this paper is twofold. The first part (Sections 1–3) is self-contained and devoted to the proof of the first main result, Theorem 0.3. The second part (Sections 4–5) presents the the motivations, the wider context, the bibliographic details and the applications to Jack characters, in particular to the second main result, Theorem 4.3.
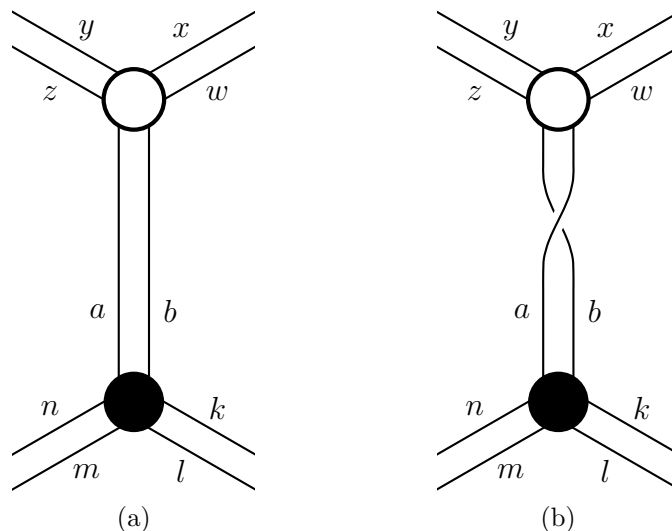
Figure 6: Twist of a ribbon in a ribbon graph. The details of the notation will be explained in Section 1.8. (a) A part of a map. The pair partition describing the structure of black vertices is $\mathcal{B} = \big\{\{b,k\},\{l,m\},\{n,a\},\dots\big\}$, the pair partition describing the white vertices is $\mathcal{W} = \big\{\{b,w\},\{x,y\},\{z,a\},\dots\big\}$ and the pair-partition describing the edges is $\mathcal{E} = \big\{\{a,b\},\{k,l\},\{m,n\},\{w,x\},\{y,z\},\dots\big\}$. (b) The outcome of a twist of the edge $\{a,b\}$. Only the partition describing the structure of white vertices has changed and is equal to $\mathcal{W}' = \big\{\{a,w\},\{x,y\},\{z,b\},\dots\big\}$.

1   Non-oriented maps

The notations presented in this chapter are based on the work of Dołęga, Féray and Śniady [9].

1.1   Non-oriented maps, informal viewpoint

If we draw an edge of a non-oriented map $M = (G, S)$ with a fat pen (or, alternatively, if we regard an edge of a corresponding ribbon graph) its boundary consists of two *edge-sides*. The maps which we consider in the current paper have labeled edge-sides, see Figure 2; in other words each edge carries two labels, one on each of its sides. We also assume that each label is unique.

If we cut the surface S along the edges of the graph G, the map becomes a collection of bicolored polygons, each polygon corresponding to one face of the map, see Figure 7. The original map M can be recovered from this collection of polygons by gluing together pairs of the edges of the polygons; for each pair of glued edges the white (respectively, black) endpoints of the two edges should be glued together.

Thus a non-oriented map can be described alternatively as a collection of bicolored polygons with labeled edges (such as the ones from Figure 7) together with the pair-partition $\mathcal{E}$ telling which pairs of edges should be glued together.
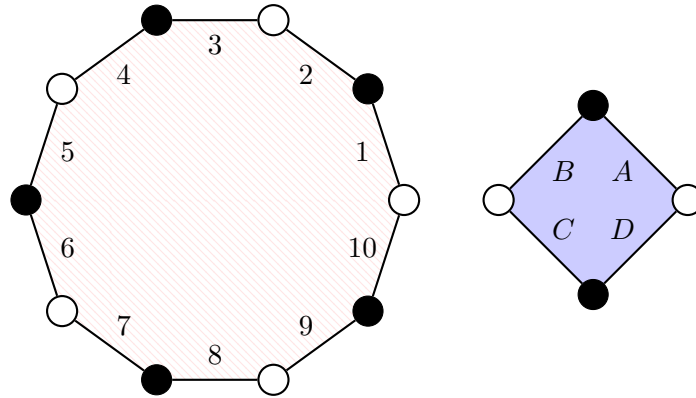
Figure 7: The map from Figure 2 is obtained from the above two polygons by gluing the following pairs of edges: $\{1,3\}$, $\{2,10\}$, $\{4,9\}$, $\{5,D\}$, $\{6,C\}$, $\{7,B\}$, $\{8,A\}$. The colors of the polygons correspond to the colors of the faces of the map from Figure 2.

1.2 Non-oriented maps, more formal viewpoint

1.2.1 Pairings and polygons

A *set-partition* of a set X is a collection $\{I_1, \ldots, I_r\}$ of pairwise disjoint, non-empty subsets, the union of which is equal to X. A *pairing* (or, alternatively, *pair-partition*) of X is a set-partition into pairs.

Let us consider now two pairings $\mathcal{B}, \mathcal{W}$ of the same set X consisting of $2n$ elements. We consider the following bicolored, edge-labeled graph $\mathcal{L}(\mathcal{B}, \mathcal{W})$:

- it has n black vertices indexed by the pairs of $\mathcal{B}$ and n white vertices indexed by the pairs of $\mathcal{W}$;

- its edges are labeled by the elements of X. The extremities of the edge labeled i are the unique pair of $\mathcal{B}$ containing i and the unique pair of $\mathcal{W}$ containing i.

Note that each vertex has degree 2 and each edge has one white and one black extremity. Besides, if we erase the indices of the vertices, it is easy to recover them from the labels of the edges (the index of a vertex is the set of the two labels of the edges incident to this vertex). Thus, in the following we forget the indices of the vertices and view $\mathcal{L}(\mathcal{B}, \mathcal{W})$ as an edge-labeled graph.

As every vertex has degree 2, the graph $\mathcal{L}(\mathcal{B}, \mathcal{W})$ can be seen as a collection of polygons.

Example 1.1. For partitions

$$\mathcal{B} = \big\{\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}, \{9,10\}, \{A,B\}, \{C,D\}\big\},$$
$$\mathcal{W} = \big\{\{2,3\}, \{4,5\}, \{6,7\}, \{8,9\}, \{10,1\}, \{B,C\}, \{D,A\}\big\},$$

the corresponding polygons $\mathcal{L}(\mathcal{B}, \mathcal{W})$ are shown in Figure 7.

1.2.2 Non-oriented maps

Definition 1.2. A *non-oriented map* is a triple $M = (\mathcal{B}, \mathcal{W}, \mathcal{E})$ of pairings of the same base set X.

The terminology comes from the fact that it is possible to represent such a triple of pair-partitions as a bicolored graph embedded in a non-oriented (and possibly non-connected) surface. Let us explain how this works.

First, we consider the union of the polygons $\mathcal{L}(\mathcal{B}, \mathcal{W})$ defined above in Section 1.2.1. The edges of these polygons, that is the elements of the set X, are called *edge-sides*.

We consider the union of the interiors of these polygons as a (possibly disconnected) surface with a boundary. If we consider two edge-sides, we can *glue* them: that means that we identify with each other their white extremities, their black extremities, and the edge-sides themselves.

For any pair in the pairing $\mathcal{E}$, we glue the two corresponding edge-sides. In this way we obtain a (possibly disconnected, possibly non-orientable) surface S without boundary. After the gluing, the edges of the polygons form a bicolored graph G embedded in the surface. For instance, with the pairings $\mathcal{B}$ and $\mathcal{W}$ from Example 1.1 and

$$\mathcal{E} = \big\{ \{1, 3\}, \{2, 10\}, \{4, 9\}, \{5, D\}, \{6, C\}, \{7, B\}, \{8, A\} \big\}, \tag{1.1}$$

we get the graph from Figure 2 embedded in the projective plane.

In general, the graph G has as many connected components as the surface S. Besides, the connected components of $S \setminus G$ correspond to the interiors of the collection of polygons we are starting from, and, thus, they are homeomorphic to open discs. These connected components are called *faces*.

This makes the link with the more common definition of maps: usually, a (bicolored) map is defined as a (bicolored) *connected* graph G embedded in a (non-oriented) surface S in such a way that each connected component of $S \setminus G$ is homeomorphic to an open disc. It should be stressed that with our definition — contrary to the traditional convention — we do not require the map to be *connected*.

Note that our maps have labeled edge-sides, and each element of the label set X is used exactly once as a label.

The pairing $\mathcal{B}$ (respectively $\mathcal{W}$) indicates which edge-sides share the same corner around a black (respectively white) vertex. This explains the names of these pairings.

This encoding of non-oriented maps by triples of pairings is of course not new. It can for instance be found in [15]; the presentation in that paper is nevertheless a bit different as the authors consider there *connected monochromatic* maps.

1.3 Face-type. Summation over non-oriented maps

Let $2\pi_1, 2\pi_2, \ldots, 2\pi_\ell$ be the numbers of edges of the polygons, or — equivalently — the numbers of edges of the faces of the map. We say that $\pi = (\pi_1, \ldots, \pi_\ell)$ is the *face-type* of the collection of polygons or the *face-type* of the map.

The *summation over non-oriented maps M with face-type (n)* should be understood as follows: we fix a bicolored polygon $\mathcal{L}$ with $2n$ labeled edges and consider all pair-partitions $\mathcal{E}$ of its edges; we sum over the resulting collection of maps $M = M(\mathcal{E})$. We will refer to this kind of summation as *conservative summation*.

1.4 Edge liberation for non-oriented maps

Sometimes it will be convenient to consider a different way of summing over maps with the face-type (n) in which the arrangement of the labels on the polygon $\mathcal{L}$ is not fixed. To be more specific: we consider all maps $M = (\mathcal{B}', \mathcal{W}', \mathcal{E}')$ where $\mathcal{B}'$, $\mathcal{W}'$ and $\mathcal{E}'$ are pair-partitions of the same base set $X = [2n]$ with the property that $\mathcal{L}(\mathcal{B}', \mathcal{W}')$ consists of a single polygon. In order to differentiate this kind of summation, we will call it *liberal summation* over non-oriented maps with the face-type (n).

Proposition 1.3. *For each integer $n \geqslant 1$ the following two formal linear combinations of unlabeled maps are equal:*

$$\sum_{\substack{M \\ \text{liberal summation}}} M = (2n - 1)! \sum_{\substack{M \\ \text{conservative summation}}} M, \tag{1.2}$$

where on the left-hand side we consider a liberal summation *over non-oriented maps with the face-type (n) and on the right-hand side we consider a* conservative summation *over non-oriented maps with the face-type (n). On both sides of the equality we remove the labels of the edge-sides of the maps.*

Proof. We fix the pair partitions $\mathcal{B}$ and $\mathcal{W}$ of the same set $X = [2n]$ in such a way that $\mathcal{L} = \mathcal{L}(\mathcal{B}, \mathcal{W})$ is a single polygon. Let us consider the collection

$$\big\{ (\pi, \mathcal{E}) : \pi \in \mathfrak{S}(2n) \text{ and } \mathcal{E} \text{ is a pair-partition of } X = [2n] \big\}, \tag{1.3}$$

where $\mathfrak{S}(2n)$ denotes the symmetric group which we view as the set of permutations of X. A formal sum of

$$M := M(\mathcal{E}) \tag{1.4}$$

over this collection clearly corresponds to the conservative summation over non-oriented maps M with face type (n), with each summand taken with the multiplicity $|\mathfrak{S}(2n)| = (2n)!$; thus this formal sum is equal to $2n$ times the right-hand side of (1.2).

For a pair partition I of the base set $X = [2n]$ and a permutation $\pi \in \mathfrak{S}(2n)$ we denote by $\pi(I)$ the pair-partition of the same base set X defined as follows: for each $a, b \in S$ we have

$$\{a, b\} \in I \iff \{\pi(a), \pi(b)\} \in \pi(I).$$

To each pair $(\pi, \mathcal{E})$ we shall associate the map

$$M' := (\mathcal{B}', \mathcal{W}', \mathcal{E}') \tag{1.5}$$

with $\mathcal{B}' := \pi(\mathcal{B})$, $\mathcal{W}' := \pi(\mathcal{W})$, $\mathcal{E}' := \pi(\mathcal{E})$. The map M' can be viewed as the map $M = (\mathcal{B}, \mathcal{W}, \mathcal{E})$ with the permuted labels of the edge-sides; in particular after removal of the labels of the edge-sides the maps M and M' are equal. Thus the following two formal linear combinations of maps *with removed labels* are equal:

$$\sum M = \sum M',$$

where both sums run over (1.3) while M, M' should be understood as in (1.4), (1.5).

Clearly, the map M' has one face and each non-oriented map M' with one face on the base set X can be obtained in this way. Furthermore, two pairs $(\pi, \mathcal{E})$ and $(\sigma, \mathcal{F})$ have the same image $M' = (\mathcal{B}', \mathcal{W}', \mathcal{E}')$ if and only if $\mathcal{F} = \sigma^{-1}\pi(\mathcal{E})$ and the permutation $\sigma^{-1}\pi$ leaves each of the partitions $\mathcal{B}$ and $\mathcal{W}$ invariant; the set of such permutations forms the dihedral group D_n of the isometries of a regular polygon with n edges. It follows that there is a bijective correspondence between the preimage of a given map M' (we assume that M' has a single face) and the dihedral group D_n. Therefore a formal sum of $M' := (\mathcal{B}', \mathcal{W}', \mathcal{E}')$ over the collection (1.3) corresponds to the liberal summation with each summand taken with the multiplicity $|D_n| = 2n$; in other words it is equal to the left-hand side of (1.2) multiplied by $2n$.

The comparison of the above two conclusions about the conservative and the liberal ways of summing finishes the proof. $\qquad\square$

1.5 Removal of edges

If E is an edge of a map M, we denote by $M \setminus E = M \setminus \{E\}$ *the map M with the edge E removed*. This definition is a bit subtle; for example since we do not allow maps having isolated vertices, if some endpoint of E is a leaf, we remove it as well. Removal of an edge might change the topology of the surface on which the map is drawn; for this reason instead of figures of the type presented in Figure 2 it is more convenient to consider for this purpose *ribbon graphs*, see Figure 3.

For a more rigorous treatment see [9, Section 3.6].

1.6 Three kinds of edges

Let a map M with some selected edge E be given (see example in Figure 8). There are three possibilities:

- *Both sides of the edge E are lying on the boundary of the same face F. This means that if we travel along the boundary of the face F then we visit the edge E twice. Assume that the directions in which we travel twice along the edge E are* opposite, *see Figure 9a.*

  In this case the edge E is called *straight* and we associate to it the weight 1.

- *Both sides of the edge E are lying on the boundary of the same face F. This means that if we travel along the boundary of the face F then we visit the edge E twice.*
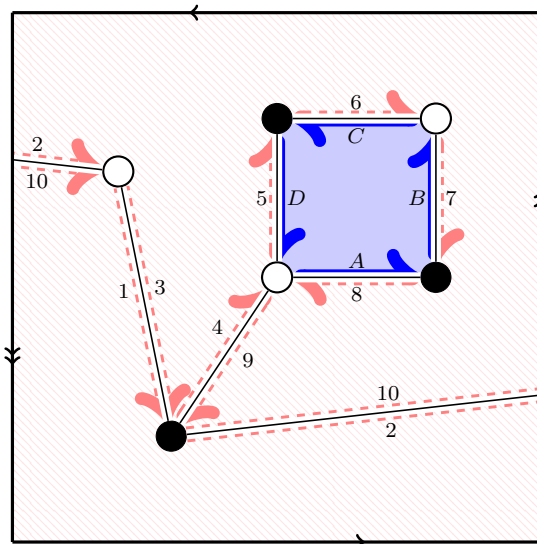
Figure 8: The non-oriented map from Figure 2. On the boundary of each face some arbitrary orientation was chosen, as indicated by arrows. The edge $\{4, 9\}$ is an example of a *straight edge*, the edge $\{1, 3\}$ is an example of a *twisted edge*, the edge $\{6, C\}$ is an example of an *interface edge*.
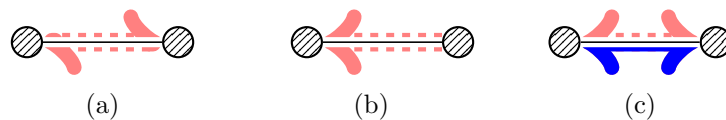

Figure 9: Three possible kinds of edges in a map (see Figure 8): (a) *straight edge*: both edge-sides of the edge belong to the same face and have opposite orientations, (b) *twisted edge*: both edge-sides of the edge belong to the same face and have the same orientation, (c) *interface edge*: the edge-sides of the edge belong to two different faces; their orientations are not important. In all three cases the colors of the vertices are not important.

> Assume that the directions in which we travel twice along the edge E are the same, see Figure 9b.

> In this case the edge E is called *twisted* and we associate to it the weight γ.

- *The edge E is lying on the boundary of two different faces, see Figure 9c.*

  In this case the edge E is called *interface* and we associate to it the weight $\frac{1}{2}$.

The weight given by the above convention will be denoted $\mathrm{mon}_{M,E}$; we will need this notion much later in Section 5.1. The classification of edges into three types (i.e. *straight* versus *twisted* versus *interface*) will be necessary immediately.

1.7 Top-degree histories

The definitions presented in this section are directly related to the quantity $\mathrm{mon}_M^{\mathrm{top}}$ from Definition 0.1.

Definition 1.4. We say that a non-oriented map M is a *top-degree map* if each connected component of M is one of the faces of M.

For a given map M we will say that a *history* is an arbitrary linear order $\prec$ on the set of edges of M. Let $E_1, \ldots, E_n$ be the sequence of edges of M, listed according to the linear order $\prec$. We set $M_i = M \setminus \{E_1, \ldots, E_i\}$; in other words $M_0, M_1, \ldots, M_n$ is the sequence of maps obtained from M by removing the edges, one by one, in the order prescribed by the history $\prec$.

Definition 1.5. We say that $(M, \prec)$ is a *top-degree pair* if M is a non-oriented map and $\prec$ is a history with the property that each of the maps $M_0, \ldots, M_n$ defined above is a top-degree map.

With the above definitions, for a given non-oriented map M of face-type (n), the corresponding quantity $\mathrm{mon}_M^{\mathrm{top}}$ from Definition 0.1 can be reformulated as the probability that for a uniformly random choice of the history $\prec$, the pair $(M, \prec)$ is a top-degree pair.

We will not make use of the following lemma (but we will make use of its extension, Lemma 5.2). Nevertheless, we decided to state it here because it provides a natural and intuitive interpretation for the notion of *top-degree pairs*.

Lemma 1.6. *Let M be a map with n edges and $\prec$ be a history. We use the notations introduced above.*

Then the following conditions are equivalent:

(A) the pair $(M, \prec)$ is top-degree;

(B) for each $0 \leqslant i \leqslant n - 1$ the edge E_{i+1} of the map M_i is either:

- *a twisted edge, or,*
- *a bridge or a leaf.*

Note that an alternative proof of this result (in a wider generality) will be given in Section 5.2.

Proof. Suppose that the condition (B) holds true. It is easy to prove by backward induction that for each $0 \leqslant i \leqslant n - 1$ the map M_i is a top-degree map which implies the condition (A).

Assume now that the condition (B) does not hold true. Let i be the maximal number with the property that the edge E_{i+1} of the map M_i is neither a twisted edge, nor a bridge, nor a leaf. Clearly $i < n$. The reasoning from the previous paragraph shows that the map M_{i+1} is top-degree. The edge E_{i+1} is not a bridge, so its endpoints must be located in the same connected component of M_{i+1} therefore they belong to the same face. Since the edge E_{i+1} is not a twisted edge of M_i, it follows that E_{i+1} is an interface edge of M_i.

It follows that the connected component of the map M_i which contains the edge E_{i+1} consists of two faces. In this way we proved that M_i is not top-degree and the condition (A) does not hold true. □

1.8 Top degree histories and orientable maps

Let $M = (\mathcal{B}, \mathcal{W}, \mathcal{E})$ be a non-oriented map and let E be one of its edges. We denote by $\mathrm{twist}_E(M)$ the non-oriented map which is obtained from M by twisting the edge E, see Figure 6. Formally, $\mathrm{twist}_E(M) = (\mathcal{B}, \mathcal{W}', \mathcal{E})$ is the map obtained by changing the structure of the pairings of the white vertices, as explained in Figure 6.

Lemma 1.7. *Let M be a map, let E be one of its edges and let $M_1 := M \setminus E$. Assume that M_1 is a top-degree map (respectively, an orientable map).*

If E is a bridge or a leaf in the map M, then M is a top-degree map (respectively, an orientable map).

If E is neither a bridge nor a leaf in the map M, then exactly one of the following two maps: M and $\mathrm{twist}_E(M)$ is a top-degree map (respectively, an orientable map).

The proof is immediate.

Theorem 1.8. *Let an integer $n \geqslant 1$ be fixed and X be an arbitrary label set with $|X| = 2n$. There exists a bijection Ψ between:*

(a) *the set of top-degree pairs $(M, \prec)$ where M is a non-oriented map with n edges and labels from X;*

(b) *the set of pairs $(M, \prec)$, where M is an orientable, non-oriented map with n edges and labels from X, and $\prec$ is a history on M.*

This bijection has the property that if $\Psi : (M, \prec) \mapsto (M', \prec')$ then the maps M and M' regarded as bicolored graphs are isomorphic.

Proof. The bijection which we will construct will have the following additional property: for each $(M, \prec)$ there exists some subset $\{e_1, \dots, e_l\}$ of the set of the edges of M such that

$$\Phi(M, \prec) = \left(\mathrm{twist}_{e_1} \cdots \mathrm{twist}_{e_l} M, \prec \right).$$

We will use induction with respect to the variable n. For $n = 1$ both sets (a) and (b) consist of a single element and there is an obvious bijection between them.

Consider the case $n \geqslant 2$. Let $(M, \prec)$ be a top-degree pair. Let E be the first of the edges of M, according to the linear order $\prec$; let $M_1 := M \setminus E$ and let $\prec_1$ be the restriction of the linear order $\prec$ to the edges of M_1; and let X_1 be the set X with the labels of the edge-sides of E removed.

The pair $(M_1, \prec_1)$ is also a top-degree pair and thus $\Phi(M_1, \prec_1)$ has been already constructed by the inductive assertion; there exists some set of the edges $\{e_1, \ldots, e_l\}$ of M_1 with the property that

$$\Phi(M_1, \prec_1) = \left(M_1', \prec_1\right)$$

with

$$M_1' = \mathrm{twist}_{e_1} \cdots \mathrm{twist}_{e_l} M_1.$$

By twisting the same set of edges in the bigger map M we define

$$\widetilde{M} := \mathrm{twist}_{e_1} \cdots \mathrm{twist}_{e_l} M.$$

There are the following two possibilities.

- *The case when the edge E is a bridge or a leaf in the map M.*

  We define

  $$\Phi(M, \prec) := \left(\widetilde{M}, \prec\right).$$

  Note that $M_1' = \widetilde{M} \setminus E$ is, by definition, an orientable map and E is a bridge or a leaf in the map $\widetilde{M}$; thus Lemma 1.7 implies that the map $\widetilde{M}$ is orientable, as required.

- *The case when the edge E is neither a bridge nor a leaf in the map M.*

  Since $M_1' = \widetilde{M} \setminus E$ is an orientable map and E is neither a bridge nor a leaf in the map $\widetilde{M}$, Lemma 1.7 implies that exactly one of the maps $\widetilde{M}$ and $\mathrm{twist}_E \widetilde{M}$ is orientable; we denote this orientable map by M'.

  Finally, we define

  $$\Phi(M, \prec) := \left(M', \prec\right).$$

This concludes the inductive construction. It remains to show that the map Φ constructed above has an inverse. The construction of the inverse Φ^{-1} is quite analogous to that of Φ and we skip it. $\qquad\square$

2 Oriented maps

2.1 Oriented maps

We define an *oriented* map as a bicolored graph G embedded in an *oriented* surface S in such a way that each connected component of $S \setminus G$ is homeomorphic to an open disc. If the number of edges is equal to n, we shall assume that the edges are labeled by the elements of $[n]$ in such a way that each label is used exactly once.

There is a bijective correspondence between such oriented maps with n edges and the set of pairs (σ_1, σ_2), where $\sigma_1, \sigma_2 \in \mathfrak{S}(n)$ are permutations. This correspondence follows from the observation that the structure of such an oriented map is uniquely determined

by the counterclockwise cyclic order of the edges around the white vertices (which we declare to be encoded by the disjoint cycle decomposition of the permutation σ_1) and by the counterclockwise cyclic order of the edges around the black vertices (which we declare to be encoded by the disjoint cycle decomposition of the permutation σ_2). The corresponding oriented map will be denoted by $M(\sigma_1, \sigma_2)$.

Example 2.1. The oriented map shown in Figure 1 corresponds to the pair

$$\sigma_1 = (1, 4, 9, 5, 7)(2, 6)(3, 8), \qquad \sigma_2 = (1, 9)(2, 3, 5)(4, 7)(6, 8).$$

2.2 Edge liberation for oriented maps

For $\sigma_1, \sigma_2 \in \mathfrak{S}(n)$ we say that *"$\langle \sigma_1, \sigma_2 \rangle$ is transitive"* if the group generated by σ_1 and σ_2 acts transitively on the underlying set $[n]$. It is easy to see that the oriented map $M(\sigma_1, \sigma_2)$ is *connected* if and only if $\langle \sigma_1, \sigma_2 \rangle$ is transitive.

The following result shows that a summation over *oriented, connected* maps can be alternatively viewed as a summation over *orientable* (but not oriented), *connected* maps.

Proposition 2.2. *Let $n \geqslant 1$. Then the following two formal linear combinations of non-oriented unlabeled maps are equal:*

$$(2n)! \sum_{\substack{\sigma_1, \sigma_2 \in \mathfrak{S}(n) \\ \langle \sigma_1, \sigma_2 \rangle \text{ is transitive}}} M(\sigma_1, \sigma_2) = 2 \cdot n! \sum_M M, \qquad (2.1)$$

where the sum on the right-hand side runs over non-oriented maps $M = (\mathcal{B}, \mathcal{W}, \mathcal{E})$ where $\mathcal{B}, \mathcal{W}, \mathcal{E}$ are arbitrary pair-partitions of the set $X = [2n]$ such that M is orientable and connected.

Proof. Consider the set

$$\big\{ (\sigma_1, \sigma_2, f) : \sigma_1, \sigma_2 \in \mathfrak{S}(n) \text{ and } \langle \sigma_1, \sigma_2 \rangle \text{ is transitive,}$$
$$f \colon [n] \times \{1, 2\} \to [2n] \text{ is a bijection} \big\}. \quad (2.2)$$

A formal sum of $M(\sigma_1, \sigma_2)$ over this set is clearly equal to the left-hand side of (2.1).

Let us consider some triple which belongs to (2.2). The permutations σ_1 and σ_2 define an *oriented* map $M(\sigma_1, \sigma_2)$ with the edges labeled by the elements of $[n]$; in the following we will show how to view this map as a *non-oriented* map. This will be done by defining the labels associated to all edge-sides of the original map M; these new labels belong to $[2n]$. In other words, to each edge e of M (i.e., to each label $e \in [n]$) we need to associate an ordered pair of two labels from $[2n]$; we declare this pair to be $\big(f(e, 1), f(e, 2)\big)$. More specifically, if we go counterclockwise around the white endpoint of e and read the labels of the edge-sides, then immediately after the label $f(e, 1)$ we should read the label $f(e, 2)$, see Figure 10.
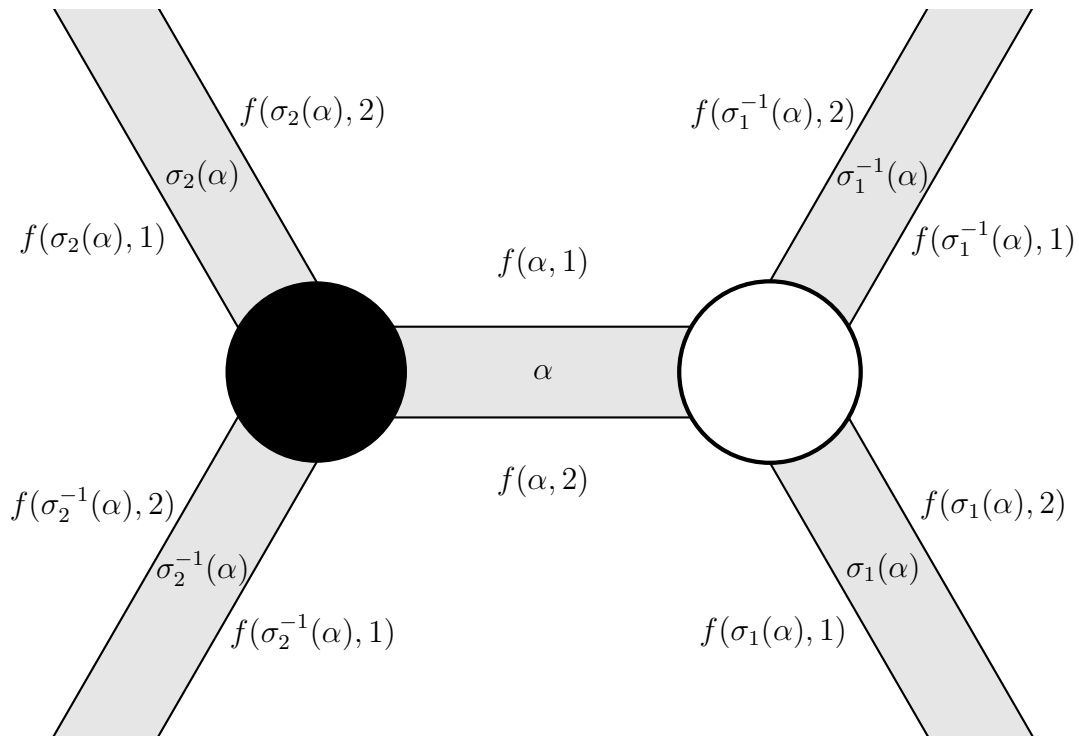
Figure 10: The convention for defining the labels of the edge-sides. The symbol written directly on the edge is its label; as usual the counterclockwise cycle structure of the edges around the white vertices is given by the permutation σ_1 while the counterclockwise cycle structure of the edges around the black vertices is given by the permutation σ_2. The two symbols written next to the edge are the labels of the edge-sides.

More formally speaking, the above construction corresponds to the non-oriented map $M = (\mathcal{B}, \mathcal{W}, \mathcal{E})$, where

$$
\begin{aligned}
\mathcal{B} &= \Big\{ \big\{ f(k,1), f\big(\sigma_2(k),2\big) \big\} : k \in [n] \Big\}, \\
\mathcal{W} &= \Big\{ \big\{ f(k,2), f\big(\sigma_1(k),1\big) \big\} : k \in [n] \Big\}, \\
\mathcal{E} &= \Big\{ \{ f(k,1), f(k,2) \} : k \in [n] \Big\}.
\end{aligned}
$$

The resulting non-oriented map M is orientable and connected. Furthermore, each such an orientable, connected, non-oriented map can be obtained in this way in $2 \cdot n!$ ways (the factor 2 comes from the two possible ways of choosing the orientation of the map, the factor $n!$ counts the possible choices of the labeling of the edges of the oriented map from which we start).

The comparison of the multiplicities arising from the two ways of interpreting the summation over (2.2) concludes the proof. □

3 Proof of the first main result, Theorem 0.3

Proof of Theorem 0.3. Recall that the left-hand side of (0.1) runs over oriented, unlabeled, rooted, connected maps M_1 with n edges. Each such a map can become a *labeled* map in $(n-1)!$ distinct ways as follows: we attach to the root edge the label 1 and we attach the remaining labels from $[n] \setminus \{1\}$ to the other edges. Thus the left-hand side of (0.1) is equal to

$$\frac{1}{(n-1)!} \sum_{\substack{\sigma_1, \sigma_2 \in \mathfrak{S}(n), \\ \langle \sigma_1, \sigma_2 \rangle \text{ is transitive}}} M(\sigma_1, \sigma_2) \tag{3.1}$$

which is viewed as linear combination of *unlabeled* maps.

Edge liberation for oriented maps (Proposition 2.2) implies that (3.1) is equal to

$$\frac{1}{(2n-1)!} \sum_M M, \tag{3.2}$$

where the sum runs over non-oriented maps M over the base set $X = [2n]$ such that M is orientable and connected. Our analysis of the left-hand side of (0.1) is now completed and we shall turn to its right-hand side.

Recall that the sum on the right-hand side of (0.1) runs over non-oriented, unlabeled, rooted maps M_2 with n edges and one face. Note that there is a canonical way of defining the labels on the edge-sides of this map, as follows. We perform a walk along the boundary of the unique face of map, with the first step along the decorated edge-side in the direction from its black to its white extremity. We attach the label 1 to this decorated edge-side, and we continue to label the edge-sides in the order in which we visit them with the remaining elements of the label set $[2n] \setminus \{1\}$. This means that the sum on the right-hand side of (0.1) is in fact a conservative summation over non-oriented maps with face-type (n).

From Definition 0.1 it follows now that the right-hand side of (0.1) is equal to

$$\frac{1}{n!} \sum_{M=(G,S)} \sum_{\substack{\prec: \\ (M, \prec) \text{ is top-degree}}} G, \tag{3.3}$$

where the first sum is a *conservative* sum over non-oriented maps with face-type (n) and the second sum runs over the histories such that $(M, \prec)$ is a top-degree pair. We apply edge liberation for non-oriented maps (Proposition 1.3); in this way (3.3) takes the form

$$\frac{1}{n! \, (2n-1)!} \sum_{M=(G,S)} \sum_{\substack{\prec: \\ (M, \prec) \text{ is top-degree}}} G, \tag{3.4}$$

where the first sum is this time a *liberal* sum over non-oriented maps with face-type (n).

We apply the bijection provided by Theorem 1.8; in this way we see that (3.4) is equal to

$$\frac{1}{(2n-1)!} \sum_{M=(G,S)} G, \tag{3.5}$$

where the sum runs over all connected, orientable non-oriented maps over the base set $[2n]$.

Since (3.2) and (3.5) are equal as formal linear combinations of unlabeled graphs, this concludes the proof. $\qquad\square$

4 Background and history of the result: Jack characters

In this section we present the background and the motivations for Theorem 0.3, in particular the problems related to *Jack characters*.

4.1 Jack characters

4.1.1 Jack polynomials

Henry Jack [18] introduced a family $\left(J_\pi^{(\alpha)}\right)$ (indexed by an integer partition π) of symmetric functions which depend on an additional parameter α. During the last forty years, many connections of these *Jack polynomials* with various fields of mathematics and physics were established: it turned out that they play a crucial role in understanding Ewens random permutations model [4], generalized β-ensembles and some statistical mechanics models [25], Selberg-type integrals [19], certain random partition models [2], and some problems of algebraic geometry [24], among many others. Better understanding of Jack polynomials is also very desirable in the context of generalized β-ensembles and their discrete counterpart model [7]. Jack polynomials are a special case of the celebrated *Macdonald polynomials* which *"have found applications in special function theory, representation theory, algebraic geometry, group theory, statistics and quantum mechanics"* [14].

4.1.2 Dual combinatorics of Jack polynomials

Lassalle [21, 22] initiated investigation of a kind of *dual combinatorics* of Jack polynomials. More specifically, one expands Jack polynomial in the basis of power-sum symmetric functions:

$$J_\lambda^{(\alpha)} = \sum_\pi \theta_\pi^{(\alpha)}(\lambda) \ p_\pi. \tag{4.1}$$

The above sum runs over partitions π such that $|\pi| = |\lambda|$. The coefficient $\theta_\pi^{(\alpha)}(\lambda)$ is called *unnormalized Jack character*; with the right choice of the normalization it becomes *the normalized Jack character* $\mathrm{Ch}_\pi(\lambda)$ (the details of this relationship will be given in Definition 4.1). An interesting feature of Jack characters is that for the special choice of

the deformation parameter $\alpha = 1$ they coincide with the usual characters of the symmetric groups.

The above approach is referred to as *dual* because one fixes π and views the character $\lambda \mapsto \mathrm{Ch}_\pi(\lambda)$ as a function of the Young diagram λ, opposite to the usual approach in the representation theory where one usually fixes the irreducible representation (the diagram λ) and views the character as a function of the conjugacy class (the partition π). In the context of the representation theory of the symmetric groups (which corresponds to the special case $\alpha = 1$) this dual approach was initiated by Kerov and Olshanski [20] and it soon turned out to be highly successful; its applications include, for example, random Young diagrams [17, 1, 26]. Lassalle [21, 22] adapted this idea to the framework of Jack characters.

Jack characters $\mathrm{Ch}_\pi(\lambda)$ are in the focus of the current paper. Our motivations for investigating them are threefold. Firstly, since Jack characters are related to Jack polynomials, a better understanding of the former might shed some light on the latter. Secondly, they can be used in order to investigate some natural deformations of classical random Young diagrams [28, 10]. Thirdly, numerical data [23] as well as some partial theoretical results [21, 22, 9, 27, 28] indicate that they might have a rich algebraic-combinatorial or representation-theoretic structure.

Our ultimate goal would be to find some convenient closed formula for Jack characters. This goal is beyond our reach; a more modest goal would be to find a closed formula for the dominant part of Jack characters in the suitable asymptotic scaling. We shall address this issue later on.

4.1.3 Definition of Jack characters

In order for this dual approach to be successful (both with respect to the usual characters of the symmetric groups and for the Jack characters) one has to choose the most convenient normalization constants. In the current paper we will use the normalization introduced by Dołęga and Féray [7] which offers some advantages over the original normalization of Lassalle. Thus, with the right choice of the multiplicative constant, the unnormalized Jack character $\theta_\lambda^{(\alpha)}(\pi)$ from (4.1) becomes the *normalized Jack character* $\mathrm{Ch}_\pi^{(\alpha)}(\lambda)$. Regretfully, their definition is quite technical and not very enlightening. On the bright side, this definition is not relevant for the purposes of the current paper and the Readers faint at heart are cordially invited to fast forward to Section 4.1.4.

Definition 4.1. Let $\alpha > 0$ be given and let π be a partition. The *normalized Jack character* $\mathrm{Ch}_\pi(\lambda)$ is given by:

$$\mathrm{Ch}_\pi(\lambda) := \begin{cases} \alpha^{-\frac{|\pi|-\ell(\pi)}{2}} \binom{|\lambda|-|\pi|+m_1(\pi)}{m_1(\pi)} \, z_\pi \, \theta_{\pi,1^{|\lambda|-|\pi|}}^{(\alpha)}(\lambda) & \text{if } |\lambda| \geqslant |\pi|; \\ 0 & \text{if } |\lambda| < |\pi|, \end{cases} \quad (4.2)$$

where

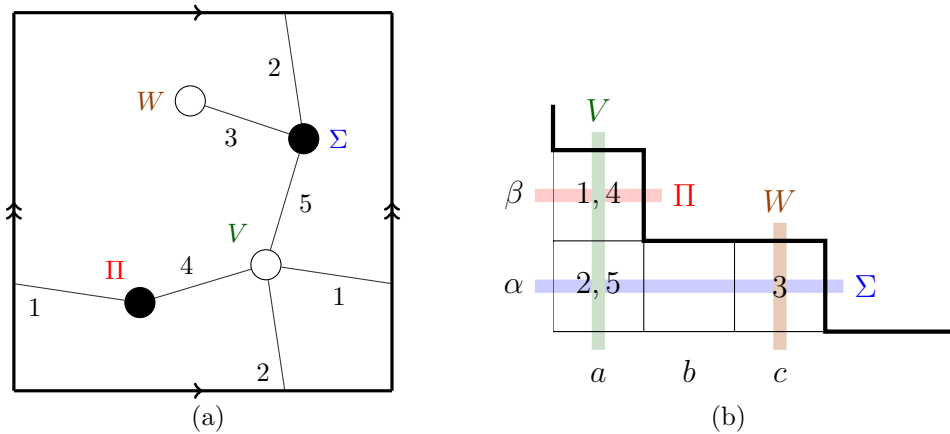$$z_\pi = \prod_i i^{m_i(\pi)} \, m_i(\pi)!$$

Figure 11: (a) Example of a bicolored graph (drawn on the torus: the left side of the square should be glued to the right side, as well as bottom to top, as indicated by arrows) and (b) an example of its embedding $F(\Sigma) = \alpha$, $F(\Pi) = \beta$, $F(V) = a$, $F(W) = c$. $F(1) = F(4) = (a\beta)$, $F(2) = F(5) = (a\alpha)$, $F(3) = (c\alpha)$. The columns of the Young diagram were indexed by small Latin letters, the rows by small Greek letters.

is the standard numerical factor.

Jack character $\mathrm{Ch}_\pi(\lambda)$ depends on the deformation parameter α, but to keep the notation light we shall usually make this dependence implicit.

4.1.4   The deformation parameter

In order to avoid dealing with the square root of the variable α, we introduce an indeterminate A such that
$$A^2 = \alpha.$$

In this way Jack character $\mathrm{Ch}_\pi(\lambda) \in \mathbb{Q}[A, A^{-1}]$ becomes a Laurent polynomial in the indeterminate A. This is the viewpoint which we will *usually* take in this paper, with the exception of Sections 4.3.1 and 4.3.2 where A will be some fixed real number.

4.2   Bicolored graphs and their embeddings

An *embedding F of a bicolored graph G to a Young diagram* λ is a function which maps the set $V_\circ(G)$ of white vertices of G to the set of columns of λ, which maps the set $V_\bullet(G)$ of black vertices of G to the set of rows of λ, and maps the edges of G to boxes of λ, see Figure 11. We also require that an embedding preserves the relation of *incidence*, i.e., a vertex V and an incident edge E should be mapped to a row or column $F(V)$ which contains the box $F(E)$. We denote by $N_G(\lambda)$ the number of such embeddings of G to λ. Quantities $N_G(\lambda)$ were introduced by Féray and the second-named author [12] and they proved to be very useful for studying various asymptotic and enumerative problems of the representation theory of symmetric groups [12, 13, 8].

Definition 4.2. For a bicolored graph G and a Young diagram λ we define the *normalized number of embeddings* [27] which is a Laurent polynomial in A:

$$\mathfrak{N}_G(\lambda) := \frac{A^{|V_\circ(G)|}}{(-A)^{|V_\bullet(G)|}} \; N_G(\lambda) \in \mathbb{Q}\left[A, A^{-1}\right]. \tag{4.3}$$

A very similar quantity denoted by $N_G^{(\alpha)}(\lambda)$ — which differs from $\mathfrak{N}_G(\lambda)$ only by the choice of the sign — was considered already in [9].

In the case when $M = (G, S)$ is a map, we denote $\mathfrak{N}_M(\lambda) := \mathfrak{N}_G(\lambda)$.

4.3 Stanley formulas

For some special values of the deformation parameter $A \in \left\{\pm\frac{1}{\sqrt{2}}, \pm 1, \pm\sqrt{2}\right\}$ which correspond to $\alpha \in \left\{\frac{1}{2}, 1, 2\right\}$, Jack characters admit closed formulas in terms of *embeddings* of certain *bicolored maps*. Formulas of this type are called *Stanley formulas* after Richard Stanley, who found such a formula for $\alpha = 1$ as a conjecture [29]. We shall review them in the following.

4.3.1 Stanley formula for $\alpha = 1$ and oriented maps

In the special case of $A = \pm 1$ which corresponds to $\alpha = 1$, Jack polynomials coincide (up to simple multiplicative constants) with Schur polynomials. Using this fact one can show that in this special case the Jack character $\mathrm{Ch}_\pi^{A:=1}$ coincides with the (suitably normalized) character of the symmetric group; for the details see the work of Lassalle [22] (who used a different normalization) as well as the work of Dołęga and Féray [7]. For this reason, for $A = 1$ Jack characters have a much richer algebraic and representation-theoretic structure than for a generic value of A.

In particular, it has been observed in [12] that a certain formula conjectured by Stanley [29] and proved by Féray [11] for the normalized characters of the symmetric groups can be expressed as the sum

$$\mathrm{Ch}_\pi^{A:=1}(\lambda) = (-1)^{\ell(\pi)} \sum_M \mathfrak{N}_M^{A:=1}(\lambda) \tag{4.4}$$

over all *oriented bicolored maps M with face-type π*, where $\ell(\pi)$ denotes the number of parts of the partition π.

4.3.2 Stanley formula for $\alpha \in \{2, \frac{1}{2}\}$ and non-oriented maps

In the special case when $A = \pm\sqrt{2}$ and $\alpha = 2$ (respectively, $A = \pm\frac{1}{\sqrt{2}}$ and $\alpha = \frac{1}{2}$) Jack polynomials coincide with *zonal polynomials* (respectively, *symplectic zonal polynomials*). Thanks to this additional structure it has been proved in a joint work of the second-named

author with Féray [13] that

$$\mathrm{Ch}_\pi^{A:=\sqrt{2}}(\lambda) = (-1)^{\ell(\pi)} \sum_M \left(-\frac{1}{\sqrt{2}}\right)^{|\pi|+\ell(\pi)-|V(M)|} \mathfrak{N}_M^{A:=\sqrt{2}}(\lambda), \qquad (4.5)$$

$$\mathrm{Ch}_\pi^{A:=\frac{1}{\sqrt{2}}}(\lambda) = (-1)^{\ell(\pi)} \sum_M \left(\frac{1}{\sqrt{2}}\right)^{|\pi|+\ell(\pi)-|V(M)|} \mathfrak{N}_M^{A:=1/\sqrt{2}}(\lambda), \qquad (4.6)$$

where the sums run over all *non-oriented maps M with face-type π*, as in Section 1.3. The reader should be advised that the notations and the normalizations in [13] are a bit different; the link between the statements above and the results of [13] is given in [9, Section 5].

4.4 How to prove a closed formula for Jack characters? Orientability generating series

Jack characters admit an abstract characterization [27, Theorem 1.7] which was found in a recent paper of the second-named author (see also Theorem A.2 of Féray in an appendix to the same paper [27]). This abstract characterization opens the following path toward *proving* a closed formula for Jack characters: in the first step one should *guess* the right formula, and in the second step one should *verify* that it indeed fulfills the aforementioned defining properties of Jack characters.

How to make the first step and to guess a closed formula for Jack characters? The three Stanley formulas (4.4), (4.5), (4.6) might suggest that the hypothetical formula for Jack characters in the generic case should be of the form

$$\mathrm{Ch}_\pi(\lambda) = (-1)^{\ell(\pi)} \sum_M \mathrm{weight}_M \, \mathfrak{N}_M(\lambda), \qquad (4.7)$$

where the sum should run over *non-oriented* maps with face-type π. In the above formula $\mathrm{weight}_M \in \mathbb{Q}[A, A^{-1}]$ is some hypothetical quantity which measures the *non-orientability* of the map M [9, Conjecture 1.1].

A joint work of the second-named author with Dołęga and Féray [9] presents an attempt to guess the exact form of this hypothetical quantity weight_M, an attempt which was based on a reverse-engineering of the results of Lassalle [21]. Our candidate quantity, denoted mon_M (which is an acronym for ***m**easure **o**f **n**on-orientability*) was defined as follows: we remove the edges from the ribbon graph of M in a uniformly random order; to each edge which is about to be removed we associate a factor which is related to the topological way in which this edge is attached to the remaining edges (i.e., the edges which have not been removed yet) as we discussed in Section 1.6. The quantity mon_M is defined as the expected value of the product of the aforementioned factors. The details of this construction will be recalled in Section 5.3. This weight mon_M gives rise to the *orientability generating series* [9, Section 1.10] which is defined in analogy to (4.7) as

$$\widehat{\mathrm{Ch}}_\pi(\lambda) := (-1)^{\ell(\pi)} \sum_M \mathrm{mon}_M \, \mathfrak{N}_M(\lambda), \qquad (4.8)$$
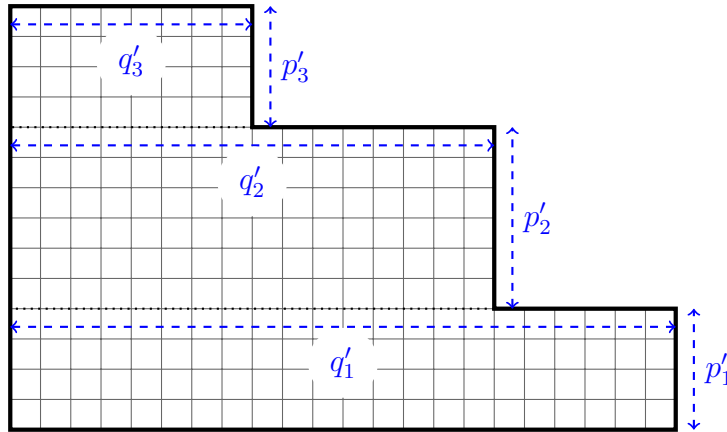
Figure 12: Multirectangular Young diagram $P \times Q$.

where the sum runs over *non-oriented* maps with face-type π.

4.5 Stanley polynomials

In order to be able to speak about the *degree* of some functions on the set of Young diagrams we will need the notion of *Stanley polynomials*. The content of this section is an abridged and less formal version of [27, Section 1.10].

4.5.1 Multirectangular coordinates

We start with *anisotropic multirectangular coordinates*

$$P = (p_1, \ldots, p_\ell),$$
$$Q = (q_1, \ldots, q_\ell).$$

They give rise to *isotropic multirectangular coordinates* given by

$$P' = (p'_1, \ldots, p'_\ell) := (Ap_1, \ldots, Ap_\ell),$$
$$Q' = (q'_1, \ldots, q'_\ell) := \left(\frac{1}{A}q_1, \ldots, \frac{1}{A}q_\ell\right).$$

Note that P' and Q' depend implicitly on P and Q.

Suppose that $P' = (p'_1, \ldots, p'_\ell)$ and $Q = (q'_1, \ldots, q'_\ell)$ are sequences of non-negative integers such that $q'_1 \geqslant \cdots \geqslant q'_\ell$; we consider the *multirectangular* Young diagram

$$P' \times Q' = (\underbrace{q'_1, \ldots, q'_1}_{p'_1 \text{ times}}, \ldots, \underbrace{q'_\ell, \ldots, q'_\ell}_{p'_\ell \text{ times}}).$$

This concept is illustrated in Figure 12.

4.5.2 The deformation parameter γ

We usually view Jack polynomials as functions of the parameter α and Jack characters as functions of another parameter $A = \sqrt{\alpha}$. However, it is convenient to consider yet another deformation parameter

$$\gamma := \frac{1}{A} - A \in \mathbb{Q}\left[A, A^{-1}\right]. \tag{4.9}$$

As we shall see, several quantities can be expressed as polynomials in γ.

4.5.3 Stanley polynomials and degree of functions on $\mathbb{Y}$

Let $F \colon \mathbb{Y} \to \mathbb{Q}\left[A, A^{-1}\right]$ be a function on the set of Young diagrams and let

$$\mathrm{St}_\ell = \mathrm{St}_\ell(\gamma; p_1, \ldots, p_\ell; q_1, \ldots, q_\ell) = \mathrm{St}_\ell(\gamma; P; Q)$$

be a polynomial in $2\ell + 1$ variables.

Suppose that the equality

$$F(P' \times Q') = \mathrm{St}_\ell\left(\gamma; P; Q\right)$$

— with the substitution (4.9) for the variable γ — holds true for all choices of P, Q and $A \neq 0$ for which the multirectangular diagram $P' \times Q'$ is well-defined. Then we say that St_ℓ is the *Stanley polynomial* for F. The above definition is not very precise; on the formal level one should consider Stanley polynomial as an element of some inverse limit for $\ell \to \infty$; for the details we refer to [27, Section 1.10].

Stanley polynomials are a perfect tool for studying asymptotic questions in the setup when the Young diagram $P' \times Q'$ tends to infinity. In particular, we say that $F \colon \mathbb{Y} \to \mathbb{Q}\left[A, A^{-1}\right]$ is a function of degree (at most) d if the corresponding Stanley polynomial is of degree (at most) d.

4.5.4 Stanley polynomials for Ch_π and $\widehat{\mathrm{Ch}}_\pi$

It is a highly non-trivial result of Dołęga and Féray [7, Corollary 3.5] (see also [27, Theorem 2.15, Corollary 2.11] for details how to adapt their result to our setup) that a Stanley polynomial exists for each Jack character Ch_π and that is of degree at most $|\pi| + \ell(\pi)$. For example,

$$\mathrm{Ch}_1(P' \times Q') = \sum_i p_i q_i,$$

$$\mathrm{Ch}_2(P' \times Q') = \sum_i p_i q_i \left[q_i - p_i + \gamma\right] - 2 \sum_{i<j} p_i p_j q_j,$$

$$\mathrm{Ch}_3(P' \times Q') = \sum_i p_i q_i \left[q_i^2 - 3p_i q_i + p_i^2 + 3\gamma(q_i - p_i) + 2\gamma^2 + 1\right] +$$

$$- 3 \sum_{i<j} p_i p_j q_j \left[(q_i - p_i + \gamma) + (q_j - p_j + \gamma)\right] +$$

$$+ \sum_{i<j<k} 6 p_i p_j p_k q_k.$$

One can show that the Stanley polynomial for the orientability generating series $\widehat{\mathrm{Ch}}_\pi$ exists and it is also of degree $|\pi| + \ell(\pi)$; the proof is postponed to Lemma 5.5.

4.6 Orientability generating series versus Jack character

The initial prediction of the authors of [9] was that the quantity mon_M is the right guess for the hypothetical weight$_M$ and thus $\widehat{\mathrm{Ch}}_\pi = \mathrm{Ch}_\pi$. Regretfully, this turns out to be not the case [9, Section 7] and $\widehat{\mathrm{Ch}}_\pi \neq \mathrm{Ch}_\pi$ in general.

This might seem as the end of the story and an example of a failed research, nevertheless the orientability series $\widehat{\mathrm{Ch}}_\pi$ appeared to predict the properties of the Jack character Ch_π *suspiciously well*. The latter statement was supported heuristically in the following two ways.

- Firstly, computer-assisted comparison of the coefficients of Stanley polynomials for the character Ch_π with their counterpart for the orientability generating series $\widehat{\mathrm{Ch}}_\pi$ indicates that *a lot of them* coincide for several concrete examples.

- Secondly, in the case when $\pi = (n)$ is a partition with only one part, a comparison of

  (i) the contribution to the orientability generating series $\widehat{\mathrm{Ch}}_n$ which comes from the maps M with small genus $g \leqslant \frac{3}{2}$ with

  (ii) its counterpart for Jack character Ch_n

  indicate a full match [3, Section 5]. For (i) a closed formula found by the first named author [3] is available. The comparison can be either performed numerically for small values of n using the data provided by Lassalle [23], or by comparing the aforementioned closed formula for (i) with Lassalle's conjectural closed formula [22, Section 11]; they turn out to be identical.

4.7 The second main result: the top-degree of Jack character

One of the main results of the current paper is the following positive result.

Theorem 4.3 (The second main result). *For each integer $n \geqslant 1$*

$$\mathrm{Ch}_n^{\mathrm{top}} = \widehat{\mathrm{Ch}}_n^{\mathrm{top}}. \tag{4.10}$$

Above, $\mathrm{Ch}_n^{\mathrm{top}}$ denotes the *top-degree of Jack character* Ch_n. More explicitly, Stanley polynomial for Ch_n is known to be of degree $n + 1$; the homogeneous part of this Stanley polynomial of degree $n + 1$ defines a function on the set of Young diagrams which will be denoted $\mathrm{Ch}_n^{\mathrm{top}}$. Analogously, the top-degree $\widehat{\mathrm{Ch}}_n^{\mathrm{top}}$ of the orientability generating series is defined as the homogeneous part of $\widehat{\mathrm{Ch}}_n$ of degree $n + 1$.

In the remaining part of this section we will explain the relationship between the two main results of the current paper: Theorem 0.3 and Theorem 4.3.

4.8 Two formulas for the top-degree part of Jack characters

The following result was proved by the second-named author using a modified version of the strategy which we outlined in Section 4.4: guess the right closed formula and then verify that it satisfies some abstract characterization (this abstract characterization of $\mathrm{Ch}_n^{\mathrm{top}}$ turns out to be much more complex than the analogous characterization of Ch_n).

Theorem 4.4 ([27, Theorem 1.21]). *For each integer $n \geqslant 1$*

$$\mathrm{Ch}_n^{\mathrm{top}}(\lambda) = (-1) \sum_M \gamma^{n+1-|V(M)|} \, \mathfrak{N}_M(\lambda), \tag{4.11}$$

where the sum runs over oriented, unlabeled, rooted, connected maps with n edges and with arbitrary face-type.

Our proof of Theorem 4.3 will be based on the following simple observation: our main bijective result (Theorem 0.3) shows equality between the right-hand sides of (4.8) and (4.11). The missing details of the proof will be provided in Section 5.6.

4.9 Top-degree of Jack characters: the real story. Motivations for Theorem 0.3

The (yet unpublished at the time) bijective result of the current paper (Theorem 0.3) preceded the proof of the above-mentioned closed formula for the top-degree of Jack characters from Theorem 4.4. Indeed, the key difficulty in Theorem 4.4 was *to guess the right formula* and this difficulty was overcome by converting a conjectural formula based on the ideas from [9] to a more convenient form thanks to Theorem 0.3. For more discussion on this topic see below.

4.10 Outlook into the future

The Reader interested in Jack characters may wonder: why bother now proving the second main result of the current paper, Theorem 4.3. After all, a simpler closed formula for $\mathrm{Ch}_n^{\mathrm{top}}$ (Theorem 4.4) was already available; the formula (4.10) has served its duty as a source of heuristics and can be retired now.

However, our ultimate goal, finding a closed formula for Jack characters Ch_n (and, more generally, Ch_π), has not been completed. The latter is currently far beyond our reach; a more modest partial accomplishment would be to find a closed formula for the next term in the asymptotic expansion of Jack character Ch_n after its top-degree part $\mathrm{Ch}_n^{\mathrm{top}}$. Once the right candidate formula is found by some heuristic means, the machinery from [27] could be probably relatively easily adapted in order to prove that such a candidate formula indeed holds true.

Regretfully, there is a missing key element to this approach: we have no good candidate formula for such a sub-dominant part of Ch_n of degree $n-1$ (see [27, Section 1.17]). Probably the simplest approach would be to deduce or extrapolate such a formula based on existing formulas for the asymptotically dominant part $\mathrm{Ch}_n^{\mathrm{top}}$. Unfortunately, the final

formula (4.11) for Ch_n^{top} does not seem to offer any hints how to extrapolate it into the sub-dominant regime.

On the other hand, our original starting point, the orientability generating series $\widehat{\text{Ch}}_n$ given by (4.8), offers immediately a candidate formula for such a sub-dominant part. We have to admit: it is a candidate formula which gives *slightly* wrong predictions [9, Section 7], but it is nevertheless a good starting point for some better formula.

Suppose that this happens to be indeed the case and someone, someday finds such a better formula based on the concept of the orientability generating series. In such a scenario some modified version of the bijection behind our main bijective result (Theorem 0.3) might become handy in order to convert this hypothetical formula into a more convenient form, just like we used this bijection in the current paper in order to transform the right-hand side of (4.8) into more convenient the right-hand side of (4.11).

For more discussion on this topic see [27, Section 1.17].

4.11 *b*-conjecture

In this section all maps are *connected*, and *rooted*, that is they posses a marked, oriented corner incident to some black vertex (that is, an angular region around black vertex v delimited by two consecutive edges attached to v).

Goulden and Jackson [16] introduced, using Jack symmetric functions, some multivariate generating series $\psi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}; 1, 1 + \beta)$ with an additional parameter β that might be interpreted as a continuous deformation of the rooted bicolored maps generating series. Indeed, it has the property that for $\beta \in \{0, 1\}$ it specializes to rooted, orientable (for $\beta = 0$) or general, i.e. orientable or not (for $\beta = 1$) bicolored maps generating series. Goulden and Jackson made the following conjecture: coefficients of ψ (indexed by three partitions of the same size) are polynomials in β with positive integer coefficients that can be written as a multivariate generating series of rooted, general bicolored maps, where the exponent of β is an integer-valued statistics that in some sense "measures the non-orientability" of the corresponding bicolored map.

This *b-conjecture* has not yet been proved, but some progress towards determining both algebraic and combinatorial properties of the coefficients in question has been made, and the work is ongoing. Dołęga and Féray [6] proved recently that all coefficients of ψ are polynomials in β with rational coefficients. Based on this result, Dołęga [5] found a combinatorial interpretation of the top-degree of coefficients of ψ indexed by two arbitrary partitions $\mu, \nu \vdash n$, and one partition consisting of only one part (n), which conjecturally should be given by certain maps with only one face. An interesting phenomenon is that, similarly as in our Theorem 0.3, the top-degree part found by Dołęga is given by *orientable* maps with the black, and white, respectively, vertex degrees given by partitions μ, and ν, respectively, and *arbitrary face structure*, and at the same time it is given by certain maps (called *unhandled*) with a unique face, and black and white vertex degrees given by partitions μ, and ν, respectively.

We cannot resist to state that there must be a deep connection between these problems, and understanding it would be of great interest.

5 Top-degree of the orientability generating series

In this section we will present the details of the definition of the polynomial $\text{mon}_M(\gamma)$, its relationship to the quantity $\text{mon}_M^{\text{top}}$ introduced in Definition 0.1, and the missing details of the proof of Theorem 4.3.

5.1 Weight associated to a map with a history

We continue the discussion from Section 1.7.

Let $E_1, \ldots, E_n$ be the sequence of edges of a non-oriented M, listed according to the linear order $\prec$. We set $M_i = M \setminus \{E_1, \ldots, E_i\}$ and define

$$\text{mon}_{M,\prec} := \prod_{0 \leqslant i \leqslant n-1} \text{mon}_{M_i, E_{i+1}} . \tag{5.1}$$

This quantity (5.1) can be interpreted as follows: from the map M we remove (one by one) all the edges, in the order specified by the history. For each edge which is about to be removed we consider its weight $\text{mon}_{M_i, E_{i+1}}$ relative to the current map (recall that the factor $\text{mon}_{M_i, E_{i+1}}$ was defined in Section 1.6 and it depends on the *type* of the edge E_{i+1} in the map M, i.e. *straight* versus *twisted* versus *interface*).

5.2 The top-degree of $\text{mon}_{M,\prec}$

The following result provides some crude information about the polynomial $\text{mon}_{M,\prec}(\gamma)$.

Lemma 5.1 ([9, Lemma 3.7]). *For any map M the weight $\text{mon}_{M,\prec}$ is a polynomial in the variable γ of degree (at most) $2\,\text{genus}(M)$.*

Here we use the term *genus* with a small abuse of notation; usually it is used only for orientable surfaces while we use it also for a *non-orientable* connected map by setting

$$\text{genus}(M) := \frac{2 \cdot (\text{the number of connected components of } M) - \chi(M)}{2},$$

where

$$\chi(M) = |\mathcal{F}(M)| - |\mathcal{E}(M)| + |V(M)|$$

is the Euler characteristic of M.

It follows that the degree of the polynomial $\text{mon}_{M,\prec}$ is bounded from above by

$$\text{degree}\,\text{mon}_{M,\prec} \leqslant 2\,|\mathcal{F}(M)| - \chi(M) = |\mathcal{F}(M)| + |\mathcal{E}(M)| - |V(M)|. \tag{5.2}$$

The remaining part of this section is devoted to the investigation of the corresponding leading coefficient

$$\left[\gamma^{|\mathcal{F}(M)| + |\mathcal{E}(M)| - |V(M)|}\right] \text{mon}_{M,\prec}(\gamma).$$

The following result is an extension of Lemma 1.6; its only new component is condition (C).

Lemma 5.2. *Let M be a non-oriented map and $\prec$ be a history. We use the notations from Section 1.7, i.e. $E_1, \ldots, E_n$ are the sequence of edges of M, listed according to the linear order $\prec$ and $M_i = M \setminus \{E_1, \ldots, E_i\}$.*

Then the following conditions are equivalent:

(A) the pair $(M, \prec)$ is top-degree (Definition 1.5);

(B) for each $0 \leqslant i \leqslant n - 1$ the edge E_{i+1} of the map M_i is either:

- *a twisted edge, or,*
- *a bridge or a leaf;*

(C) the degree of the polynomial $\mathrm{mon}_{M,\prec}$ is equal to $|\mathcal{F}(M)| + |\mathcal{E}(M)| - |V(M)|$.

If the above conditions hold true then the leading coefficient of the polynomial $\mathrm{mon}_{M,\prec}$ is given by

$$\left[\gamma^{|\mathcal{F}(M)| + |\mathcal{E}(M)| - |V(M)|} \right] \mathrm{mon}_{M,\prec} = 1. \tag{5.3}$$

Proof. Condition (C) holds true if and only if the inequalities involved in the proof of the bound (5.2) become equalities and this happens when both of the following two conditions hold true:

(C1) the map M is top-degree (Definition 1.4), and,

(C2) the weight $\mathrm{mon}_{M,\prec}$ is a polynomial in the variable γ of degree *exactly* $2 \, \mathrm{genus}(M)$.

In the following we shall find some equivalent reformulations of the condition (C2).

By revisiting the proof of Lemma 5.1 presented in [9, Lemma 3.7] one can show that the condition (C2) holds true if and only if for each $0 \leqslant i \leqslant n - 1$ the weight $\mathrm{mon}_{M_i, E_{i+1}}$ is a polynomial in the variable γ of degree *exactly*

$$2 \, \mathrm{genus}(M_i) - 2 \, \mathrm{genus}(M_i \setminus E_{i+1}) = 2 \, \mathrm{genus}(M_i) - 2 \, \mathrm{genus}(M_{i+1}).$$

The case-by-case analysis from the proof of [9, Lemma 3.7] shows that this is equivalent to the condition (B), as well as to the following condition:

$$|\mathcal{F}(M_i)| - (\text{number of connected components of } M_i) =$$
$$|\mathcal{F}(M_{i+1})| - (\text{number of connected components of } M_{i+1});$$

in other words, the numbers

$$|\mathcal{F}(M_i)| - (\text{number of connected components of } M_i) \tag{5.4}$$

over $\in \{0, 1, \ldots, n\}$ are all equal. Since for $i = n$ the quantity (5.4) is equal to zero ($M_n = \emptyset$ is the empty map), this is equivalent to

$$|\mathcal{F}(M_i)| = (\text{number of connected components of } M_i) \qquad \text{for each } 1 \leqslant i \leqslant n$$

which is clearly the condition (A). Remember that together with the removal of some leaf we always remove its endpoint, since we do not allow maps having isolated vertices, i.e. connected components consisting of one vertex.

By now we have proved that the conditions (C2), (B), and (A) are all equivalent. On the other hand, condition (A) implies that the map $M = M_0$ is top-degree which is the condition (C1). This concludes the proof of the equivalence.

Condition (B) implies that each of the weights $\mathrm{mon}_{M_i, E_{i+1}}$ is a monic monomial in γ. Condition (C) implies that the product of these monic monomials has the degree equal to $|\mathcal{F}(M)| + |\mathcal{E}(M)| - |V(M)|$. This concludes the proof of (5.3). $\qquad\square$

5.3 Measure of non-orientability of a map

Let M be a map with n edges. We define

$$\mathrm{mon}_M = \mathrm{mon}_M(\gamma) := \frac{1}{n!} \sum_{\prec} \mathrm{mon}_{M,\prec}. \tag{5.5}$$

This quantity can be interpreted as the mean value of the weight associated to the map M equipped with a randomly selected history (with all histories having equal probability). We call mon_M *the measure of non-orientability* of the map M.

Example 5.3. We revisit Example 0.2. For the histories $\{3,6\} \prec \{2,4\} \prec \{1,5\}$ and $\{3,6\} \prec \{1,5\} \prec \{2,4\}$ when the edge $\{3,6\}$ is removed first the corresponding weight is equal to $\mathrm{mon}_{M,\prec} = 1 \cdot \frac{1}{2} \cdot 1$. For the remaining 4 histories the corresponding weight is equal to $\mathrm{mon}_{M,\prec} = \gamma \cdot \gamma \cdot 1$. Finally,

$$\mathrm{mon}_M = \frac{2 \times 1 \cdot \frac{1}{2} \cdot 1 + 4 \times \gamma \cdot \gamma \cdot 1}{6}.$$

Proposition 5.4. *Let M be a non-oriented map with n edges. The corresponding polynomial mon_M is of degree at most $n + |\mathcal{F}(M)| - |V(M)|$. The corresponding leading coefficient*

$$\left[\gamma^{n + |\mathcal{F}(M)| - |V(M)|} \right] \mathrm{mon}_M = \mathrm{mon}_M^{\mathrm{top}}$$

is given by the quantity defined in Definition 0.1.

Proof. This is an immediate application of Lemma 5.2. $\qquad\square$

5.4 Orientability generating series

We recall that the orientability generating series was defined in (4.8) as a weighted sum of the normalized numbers of embeddings, with the weight given by the polynomial mon_M described above:

$$\widehat{\mathrm{Ch}}_\pi(\lambda) := (-1)^{\ell(\pi)} \sum_M \mathrm{mon}_M \, \mathfrak{N}_M(\lambda), \tag{5.6}$$

where the sum is a conservative summation over non-oriented maps with the face-type π.

Lemma 5.5. *For each partition π there exists a Stanley polynomial for $\widehat{\mathrm{Ch}}_\pi$; this Stanley polynomial is of degree $|\pi| + \ell(\pi)$.*

Proof. It is relatively easy to show that for any bicolored graph G the corresponding normalized number of embeddings $\lambda \mapsto \mathfrak{N}_G(\lambda)$ has a Stanley polynomial which is homogeneous of degree $|V(G)|$.

Proposition 5.4 provides an upper bound on the degree of the polynomial mon_M.

By combining the above two observations it follows that the Stanley polynomial for the product $\mathrm{mon}_M \, \mathfrak{N}_M$ exists and its degree is bounded from above by $|\mathcal{E}(M)| + |\mathcal{F}(M)| = |\pi| + \ell(\pi)$, as required. $\qquad\square$

5.5 The top-degree of the orientability generating series

Corollary 5.6. *For any integer $n \geqslant 1$ the top-degree homogeneous part of the orientability generating series is given by*

$$\widehat{\mathrm{Ch}}_n^{\mathrm{top}} = \sum_M \mathrm{mon}_M^{\mathrm{top}} \, \gamma^{n+1-|V(M)|} \, \mathfrak{N}_M,$$

where the sum on the right-hand side runs over non-oriented maps with face-type (n).

Proof. By Lemma 5.5 Stanley polynomial for $\widehat{\mathrm{Ch}}_n$ is of degree (at most) $n+1$; out goal now is to extract its homogeneous part of degree $n+1$. This can be done by revisiting the proof of Lemma 5.5 and using Proposition 5.4. $\qquad\square$

5.6 Proof of Theorem 4.3

Proof of Theorem 4.3. The left-hand side of (4.10) is given by Theorem 4.4. The right-hand side of (4.10) is given by Corollary 5.6. Now it is enough to apply Theorem 0.3 to show that they are equal. $\qquad\square$

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