Coloring graphs with no even hole $\geqslant 6$: the triangle-free case

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Abstract

In this paper, we prove that the class of graphs with no triangle and no induced cycle of even length at least 6 has bounded chromatic number. It is well-known that even-hole-free graphs are χ-bounded but we allow here the existence of C_4. The proof relies on the concept of Parity Changing Path, an adaptation of Trinity Changing Path which was recently introduced by Bonamy, Charbit and Thomassé to prove that graphs with no induced cycle of length divisible by three have bounded chromatic number.

Keywords: graph coloring; forbidding cycles; Trinity Changing Path

1 Introduction

A *hole* in a graph is an induced cycle of length at least four. A *proper coloring* of a graph is a function that assigns to each vertex a color with the constraint that two adjacent vertices are not colored the same. The *chromatic number* of G, denoted by $\chi(G)$, is the smallest number of colors needed to color the graph properly. All the colorings considered in the sequel are proper, so we just call them colorings. The size of the largest clique of G is denoted $\omega(G)$. We obviously have $\omega(G) \leqslant \chi(G)$, and one may wonder whether the equality holds. In fact, it does not hold in the general case, and the simplest counter-examples are *odd holes*, i.e. holes of odd length, for which $\omega(G) = 2$ but $\chi(G) = 3$. Graphs for which the equality $\chi(G') = \omega(G')$ holds for every induced subgraph G' of G are called *perfect*, and the Strong Perfect Graph Theorem [4] proved that a graph if perfect if and

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only if it is *Berge*, that is to say there is no odd hole in G nor in its complement. In order to get some upper bound on $\chi(G)$, Gyárfás [11] introduced the concept of χ-bounded class: a family $\mathcal{G}$ of graphs is called *χ-bounded* if there exists a function f such that $\chi(G') \leqslant f(\omega(G'))$ whenever G' is an induced subgraph of $G \in \mathcal{G}$.

This notion has been widely studied since then, in particular in hereditary classes (*hereditary* means closed under taking induced subgraph). A classical result of Erdős [9] asserts that there exist graphs with arbitrarily large *girth* (that is, the length of the shortest induced cycle) and arbitrarily large chromatic number. Thus forbidding only one induced subgraph H may lead to a χ-bounded class only if H is acyclic. It is conjectured that this condition is also sufficient [10, 24], but it is proved only if H is a path, a star [11] or a tree of radius two [12] (or three, with additional conditions [13]). Scott [19] also proved it for any tree H, provided that we forbid every induced subdivision of H, instead of just H itself.

Consequently, forbidding holes in order to get a χ-bounded class is conceivable only if we forbid infinitely many hole lengths. Two parameters should be taken into account: first, the length of the holes, and secondly, the parity of their lengths. In this respect, Gyárfás [11] made a famous series of three conjectures. The first one asserts that the class of graphs with no odd hole is χ-bounded. The second one asserts that, for every k, the class of graphs with no hole of length at least k is χ-bounded. The last one generalizes the first two conjectures and asserts that for every k, the class of graphs with no odd hole of length at least k is χ-bounded. After several partial results [17, 20, 5], the first and the second conjectures were recently solved by Chudnovsky, Scott and Seymour [23, 6]. Moreover, we learned while writing this article that Scott and Seymour have proved a very general result implying the triangle-free case of the third conjecture (which also implies the result of this paper): for every $k \geqslant 0$, every triangle-free graph with large enough chromatic number admits a sequence of holes of k consecutive lengths [22][1].

The class of even-hole-free graphs has been extensively studied from a structural point of view. A decomposition theorem together with a recognition algorithm have been found by Conforti, Cornuéjols, Kapoor and Vušković [7, 8, 3]. Reed conjectured [15] that every even-hole-free graphs has a vertex whose neighborhood is the union of two cliques (called a *bisimplicial* vertex), which he and his co-authors proved [1] a few years later. As a consequence, they obtained that every even-hole-free graph G satisfies $\chi(G) \leqslant 2\omega(G) - 1$.

Forbidding C_4 is in fact a strong restriction since C_4 can also be seen as the complete bipartite graph $K_{2,2}$: Kühn and Osthus [14] proved that for every graph H and for every integer s, every graph of large average degree (with respect to H and s) with no $K_{s,s}$ as a (non-necessarily induced) subgraph contains an induced subdivision of H, where each edge is subdivided at least once. This strong result implies that the chromatic number is bounded in any class $\mathcal{C}$ defined as graphs with no triangles, no induced C_4 and no cycles of length divisible by k, for any fixed integer k. Indeed, let $G \in \mathcal{C}$ be a minimal counter-example to $\chi(G) \leqslant t$ (with t chosen large enough with respect to k), then it has

[1]Bibliography update: while this article was under review process, Scott and Seymour finally managed to prove that, for every $c, k \geqslant 0$, every graph with clique number at most c and sufficiently large chromatic number has a hole of every possible length modulo k [21]. This implies Gyárfás' third conjecture.

large minimum degree. Moreover it has neither induced C_4 nor triangles, consequently it has no C_4 subgraphs. By Kühn and Osthus' theorem, there exists an induced subdivision H of K_ℓ for some well-chosen integer ℓ depending on k. Consider K_ℓ as an auxiliary graph where we color each edge with $c \in \{1, \dots, k\}$ if this edge is subdivided c times modulo k in H. By Ramsey's theorem [16], if ℓ is large enough, then we can find a monochromatic clique K of size k. Let C_0 be a Hamiltonian cycle through K and call C the corresponding cycle in the subdivided edges in H. Since K was monochromatic in K_ℓ, the edges used in C_0 are subdivided the same number of times modulo k, consequently C has length divisible by k. Moreover, it is an induced cycle since each edge is subdivided at least once in H.

This is why we are interested in finding a χ-boundedness result when every even hole except C_4 is forbidden, which was conjectured by Reed [18]. In this paper, we achieve a partial result by forbidding also triangles[2]. This is a classical step towards χ-boundedness, and Thomassé *et al.* [25] even asked whether this could always be sufficient, namely: does there exist a function f such that for every class $\mathcal{C}$ of graphs and any $G \in \mathcal{C}$, $\chi(G) \leqslant f(\chi_T(G), \omega(G))$, where $\chi_T(G)$ denotes the maximum chromatic number of a triangle-free induced subgraph of G?

The result of this paper is closely related to the following recent one, by Bonamy, Charbit and Thomassé, answering to a question by Kalai and Meshulam on the sum of Betti numbers of the stable set complex (see [2] for more details):

Theorem 1 ([2])**.** *There exists a constant c such that every graph G with no induced cycle of length divisible by 3 satisfies $\chi(G) < c$.*

Indeed, the so-called Parity Changing Path (to be defined below) is directly inspired by their Trinity Changing Path. The structure of the proofs also have several similarities.

Contribution We prove the following theorem:

Theorem 2. *There exists a constant c such that every graph G with no triangle and no induced cycle of even length at least 6 satisfies $\chi(G) < c$.*

The outline is to prove the result when the 5-hole is also forbidden (see Lemma 3 below), which should intuitively be easier, and then deduce the theorem for the general case.

To begin with, let us introduce and recall some notations: the class under study, namely graphs with no triangle and no induced C_{2k} with $k \geqslant 3$ (meaning that every even hole is forbidden except C_4) will be called $\mathcal{C}_{3,2k\geqslant 6}$ for short. Moreover, we will consider in Section 2 the subclass $\mathcal{C}_{3,5,2k\geqslant 6}$ of $\mathcal{C}_{3,2k\geqslant 6}$ in which the 5-hole is also forbidden. For two subsets of vertices $A, B \subseteq V$, A *dominates* B if $B \subseteq N(A)$. A *major connected component* of G is a connected component C of G for which $\chi(C) = \chi(G)$. Note that such a component always exists. For any induced path $P = x_1 x_2 \cdots x_\ell$ we say that P is

[2]The aforementioned recent result of Scott and Seymour [21], in addition to proving Gyárfás' third conjecture, also proves the general case of Reed's conjecture.

a path from its *origin* x_1 to its *end* x_ℓ or an *x_1x_ℓ-path*. Its *interior* is $\{x_2, \dots, x_{\ell-1}\}$ and its *length* is $\ell - 1$.

Moreover, we use a rather common technique called a *levelling* [23, 5] : given a vertex v, the *v-levelling* is the partition $(N_0, N_1, \dots, N_k, \dots)$ of the vertices according to their distance to v: N_k is the set of vertices at distance exactly k from v and is called the *k-th level*. In particular, $N_0 = \{v\}$ and $N_1 = N(v)$. We need two more facts about levellings: if x and y are in the same part N_k of a v-levelling, we call an *upper xy-path* any shortest path from x to y among those with interior in $N_0 \cup \dots \cup N_{k-1}$. Observe that it always exists since there is an xv-path and a vy-path (but it may take shortcuts; in particular, it may be just one edge). Moreover, in any v-levelling, there exists k such that $\chi(N_k) \geqslant \chi(G)/2$: indeed, if t is the maximum of $\chi(N_i)$ over all levels N_i, one can color G using $2t$ colors by coloring $G[N_i]$ with the set of colors $\{1, \dots, t\}$ if i is odd, and with the set of colors $\{t+1, \dots, 2t\}$ if i is even. Such a level with chromatic number at least $\chi(G)/2$ is called a *colorful* level. Observe that, if N_k is a colorful level in a triangle-free graph G with $\chi(G) \geqslant 3$, then $k \geqslant 2$.

Let us now introduce the main tool of the proof, called *Parity Changing Path* (*PCP* for short) which, as already mentioned, is inspired by the *Trinity Changing Path (TCP)* appearing in [2]: intuitively (see Figure 1 for an unformal diagram), a PCP is a sequence of induced subgraphs and paths $(G_1, P_1, \dots, G_\ell, P_\ell, H)$ with no "bad" chord between them, such that each block G_i can be crossed by two possible paths of different parities, and the last block H typically is a "stock" of big chromatic number, in which we can find whichever structure always appears in a graph with high chromatic number. Formally, a *PCP of order ℓ* in G is a sequence of induced subgraphs $G_1, \dots, G_\ell, H$ (called *blocks*; the G_i are the *regular* blocks) and induced paths $P_1, \dots, P_\ell$ such that the origin of P_i is some vertex y_i in G_i, and the end of P_i is some vertex x_{i+1} of G_{i+1} (or of H if $i = \ell$). Apart from these special vertices which belong to exactly two subgraphs of the PCP, the blocks and paths $G_1, \dots, G_\ell, H, P_1, \dots, P_\ell$ composing the PCP are pairwise disjoint. The only possible edges have both endpoints belonging to the same block or path. We also have one extra vertex $x_1 \in G_1$ called the *origin* of the PCP. Moreover in each block G_i, there exists one induced x_iy_i-path of odd length, and one induced x_iy_i-path of even length (these paths are not required to be disjoint one from each other). In particular $x_i \neq y_i$ and x_iy_i is not an edge. For technical reasons that will appear later, we also require that H is connected, every G_i has chromatic number at most 4 and every P_i has length at least 2. Finally the chromatic number of H is called the *leftovers*.

In fact in Section 3, we need a slightly stronger definition of PCP: a *strong PCP* is a PCP for which every G_i contains an induced C_5.

We first bound the chromatic number in $\mathcal{C}_{3,5,2k\geqslant 6}$ (see Lemma 3 below), which is easier because we forbid one more cycle length, and then deduce the theorem for $\mathcal{C}_{3,2k\geqslant 6}$. The proofs for $\mathcal{C}_{3,2k\geqslant 6}$ and $\mathcal{C}_{3,5,2k\geqslant 6}$ follow the same outline, which we informally describe here:

(i) If $\chi(G)$ is large enough, then for every vertex v we can grow a PCP whose origin is v and whose leftovers are large (Lemmas 4, 5 and then Lemma 12).

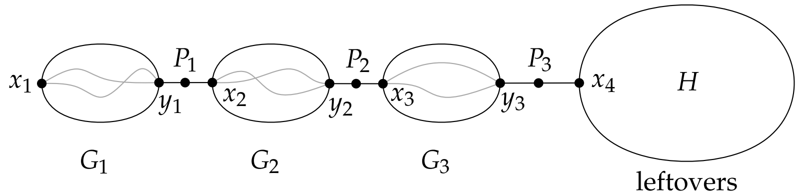


Figure 1: An informal diagram for a PCP of order 3. Grey curved lines stand for the even and odd length x_iy_i-paths.

(ii) Using (i), if $\chi(G)$ is large enough and $(N_0, N_1, \dots)$ is the v-levelling, we can grow a *rooted* PCP: it is a PCP in a level N_k, which has a *root*, *i.e.* a vertex in the previous level N_{k-1} whose unique neighbor in the PCP is the origin (Lemma 6 and then Lemma 13).

(iii) Given a rooted PCP in a level N_k, if a vertex $x \in N_{k-1}$ has a neighbor in some block, then it has a neighbor in every preceding regular block (Lemma 7).

(iv) Given a rooted PCP of order ℓ in a level N_k and a stable set S in N_{k-1}, the chromatic number of $N(S) \cap N_k$ is bounded. Consequently, the *active lift* of the PCP, defined as $N(G_\ell) \cap N_{k-1}$, has high chromatic number (Lemmas 8, 9 and 10 and then Lemmas 9, 14, 15).

(v) The final proofs put everything together: consider a graph of $G \in \mathcal{C}_{3,5,2k\geqslant 6}$ (resp. $\mathcal{C}_{3,2k\geqslant 6}$) with chromatic number large enough. Then pick a vertex v, let $(N_0, N_1, \dots)$ be the v-levelling and N_k be a colorful level. By (ii), grow inside N_k a rooted PCP P. Then by (iv), get an active lift A of P inside N_{k-1} with big chromatic number. Grow a rooted PCP P' inside A, and get an active lift A' of P' inside N_{k-2} with chromatic number big enough to find an edge xy (resp. a 5-hole C) in A'. Then "clean" P' in order to get a stable set S inside the last regular block of P', dominating this edge (resp. hole). Now find an even hole of length $\geqslant 6$ in $\{x, y\} \cup S \cup P$ (resp. $C \cup S \cup P$), a contradiction.

2 Forbidding 5-holes

This section is devoted to the proof of the following lemma :

Lemma 3. *There exists a constant c' such that every graph $G \in \mathcal{C}_{3,5,2k\geqslant 6}$ satisfies $\chi(G) < c'$.*

We follow the outline described above. Let us start with step (i):

Lemma 4. *Let $G \in \mathcal{C}_{3,5,2k\geqslant 6}$ be a connected graph and v be any vertex of G. For every δ such that $\chi(G) \geqslant \delta \geqslant 18$, there exists a PCP of order 1 with origin v and leftovers at least $h(\delta) = \delta/2 - 8$.*

Proof. The proof is illustrated on Figure 2(a). Let $(N_0, N_1, \ldots)$ be the v-levelling and N_k be a colorful level (hence $k \geqslant 2$ since G is triangle-free). Let N'_k be a major connected component of $G[N_k]$, so $\chi(N'_k) \geqslant \delta/2$. Let xy be an edge of N'_k, and x' (resp. y') be a neighbor of x (resp. y) in N_{k-1}. Let $Z' = N(\{x', y', x, y\}) \cap N'_k$ and $Z = Z' \setminus \{x, y\}$. Let $z \in Z$ be a vertex having a neighbor z_1 in a major connected component M_1 of $N'_k \setminus Z'$. Observe that $N'_k \setminus Z'$ is not empty since $\chi(Z') \leqslant 6$ (the neighborhood of any vertex is a stable set since G is triangle-free). The goal is now to find two vz-paths P and P' of different parities with interior in $G[N_0 \cup \ldots \cup \{x', y'\} \cup \{x, y\}]$. Then we can set $G_1 = G[P \cup P']$, $P_1 = G[\{z, z_1\}]$ and $H = G[M_1]$ as parts of the wanted PCP. In practice, we need to be a little more careful to ensure the condition on the length of P_1 and the non-adjacency between z and H, which is described after finding such a P and a P'.

Let P_0 (resp. P'_0) be a vx'-path (resp. vy'-path) of length $k-1$ (with exactly one vertex in each level). By definition of Z, z is connected to $\{x', y', x, y\}$.

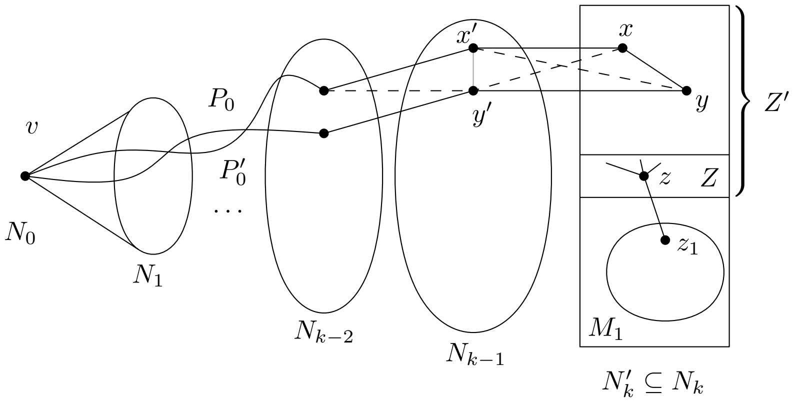
1. (see Figure 2(b)) If z is connected to x or y, say x, then z is connected neither to x' nor to y, otherwise it creates a triangle. We add the path $x'xz$ to P_0 to form P. Similarly, we add either the edge $y'z$ if it exists, or else the path $y'yxz$ to P'_0 to form P'. Observe that P' is indeed an induced path since there is no triangle. Moreover, the lenghts of P and P' differ by exactly one, so P and P' have indeed distinct parities.

2. (see Figure 2(c)) Otherwise, z is connected neither to x nor to y, thus z is connected to exactly one of x' and y', since otherwise it would either create a triangle x', y', z or a 5-hole $zx'xyy'$, so say $zx' \in E$ and $zy' \notin E$. We add the edge $x'z$ to P_0 to form P. We add the path $y'x'z$ if $y'x' \in E$, otherwise add the path $y'yxx'z$ to P'_0 to form P'. Observe that this is an induced path since G has no triangle and no 5-hole. Moreover, the lenghts of P and P' differ by either one or three, so P and P' have indeed distinct parities.

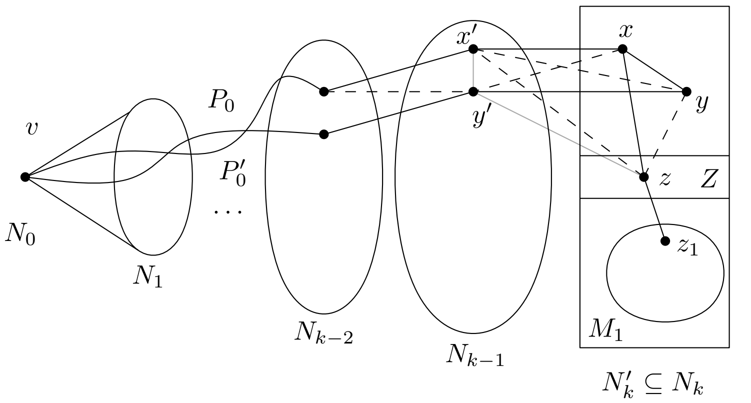
Now comes the fine tuning. Choose in fact $z_1 \in M_1 \cap N(z)$ so that z_1 is connected to a major connected component M_2 of $M_1 \setminus N(z)$. Choose z_2 a neighbor of z_1 in M_2 such that z_2 is connected to a major connected component M_3 of $M_2 \setminus N(z_1)$. We redefine $H = G[\{z_2 \cup M_3\}]$ and $P_1 = G[\{z, z_1, z_2\}]$. Then P_1 is a path of length 2, G_1 is colorable with 4 colors as the union of two induced paths, and H is connected. Moreover H has chromatic number at least $\chi(N'_k) - \chi(Z') - \chi(N(z)) - \chi(N(z_1))$. Since the neighborhood of any vertex is a stable set, $\chi(Z') \leqslant 6$ and $\chi(N(z)), \chi(N(z_1)) \leqslant 1$. Thus $\chi(H) \geqslant \delta/2 - 8$. ☐

We can iterate the previous process to grow some longer PCP. In the following, for a function f and an integer k, $f^{(k)}$ denotes the k-th iterate of f, that is to say that $f^{(k)}(x) = \underbrace{(f \circ \ldots \circ f)}_{k \text{ times}}(x)$.

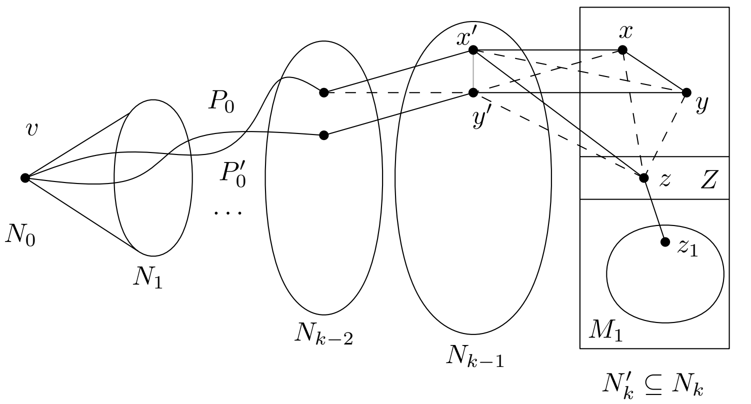
Lemma 5. *Let $h(x) = x/2 - 8$ be the function defined in Lemma 4. For every positive integers $\ell, \delta \in \mathbb{Z}_+$, if $G \in \mathcal{C}_{3,5,2k \geqslant 6}$ is connected and satisfies $\chi(G) \geqslant \delta$ and $h^{(\ell-1)}(\delta) \geqslant 18$, then from any vertex x_1 of G, one can grow a PCP of order ℓ with leftovers at least $h^{(\ell)}(\delta)$.*



(a) Overview of the situation



(b) Case 1



(c) Case 2

Figure 2: Illustrations for the proof of Lemma 4. Dashed edges stand for non-edges, and grey edges stand for edges that may or may not exist.

Proof. We prove the result by induction on ℓ. For $\ell = 1$, the result follows directly from Lemma 4. Now suppose it is true for $\ell - 1$, and let $G \in \mathcal{C}_{3,5,2k\geqslant 6}$ be such that $\chi(G) \geqslant \delta$ and $h^{(\ell-1)}(\delta) \geqslant 18$. Then $\delta \geqslant h^{(\ell-1)}(\delta) \geqslant 18$, so we can apply Lemma 4 to get a PCP of order 1 and leftovers at least $h(\delta)$ from any vertex x_1. Let x_2 be the unique vertex in the intersection of P_1 and the last block H of the PCP (as in the definition). Now apply the induction hypothesis to H, knowing that H is connected, $\chi(H) \geqslant h(\delta) = \delta'$ and $h^{(\ell-2)}(\delta') \geqslant 18$. Then we obtain a PCP of order $\ell - 1$ with origin x_2 and leftovers at least $h^{(\ell-2)}(\delta')$, which finishes the proof by gluing the two PCP together. □

Now we grow the PCP in a level N_k of high chromatic number, and we want the PCP to be rooted (*i.e.* there exists a root $u' \in N_{k-1}$ that is adjacent to the origin u of the PCP, but to no other vertex of the PCP). This is step (ii).

Lemma 6. *Let $G \in \mathcal{C}_{3,5,2k\geqslant 6}$ be a connected graph, $v \in V(G)$ and $(N_0, N_1, \ldots)$ be the v-levelling. Let h be the function defined in Lemma 4. For every k, δ such that $\chi(N_k) \geqslant \delta+1$ and $h^{(\ell-1)}(\delta) \geqslant 18$, there exists a rooted PCP of order ℓ in N_k with leftovers at least $h^{(\ell)}(\delta)$.*

Proof. Let N'_k be a major connected component of N_k and let u' be a vertex in $N_{k-1} \cap N(N'_k)$ (which is non-empty). Since there is no triangle, $N'_k \setminus N(u')$ still has big chromatic number (at least δ), and let N''_k be a major connected component of $N'_k \setminus N(u')$. Let z be a vertex of $N(u') \cap N'_k$ having a neighbor in N''_k. Then we apply Lemma 5 in $\{z\} \cup N''_k$ to grow a PCP of order ℓ from z with leftovers at least $h^{(\ell)}(\delta)$. Now u' has an only neighbor z on the PCP, which is the origin. □

Let us observe the properties of such a rooted PCP. We start with step (iii):

Lemma 7. *Let v be a vertex of a graph $G \in \mathcal{C}_{3,2k\geqslant 6}$, $(N_0, N_1, \ldots)$ be the v-levelling. Let $(G_1, P_1, \ldots, G_\ell, P_\ell, H)$ be a rooted PCP of order ℓ in a level N_k for some k. If $x' \in N_{k-1}$ has a neighbor x in some regular block G_{i_0} (resp. in H), then x has a neighbor in every G_i for $1 \leqslant i \leqslant i_0$ (resp. for $1 \leqslant i \leqslant \ell$).*

Proof. If x' has a neighbor in H, we set $i_0 = \ell + 1$. We proceed by contradiction. Let u be the origin of the PCP and u' its root. Since $x' \neq u'$ by definition of the root, there exists an upper $x'u'$-path P_{up} of length at least one. Consider a ux-path inside the PCP. Let $v_1, \ldots, v_r$ be the neighbors of x' on this path, different from x (if any), in this order (from u to x). Now we can show that any regular block G_i with $1 \leqslant i \leqslant i_0 - 1$ contains at least one v_j: suppose not for some index i, let j be the greatest index such that v_j is *before* x_i, *i.e.* $v_j \in G_1 \cup P_1 \cup \cdots \cup G_{i-1} \cup P_{i-1}$.

If such an index does not exist (*i.e.* all the v_j are after G_i), then there is an odd and an even path from u to v_1 of length at least 3 by definition of a regular block, and this path does not contain any neighbor of x'. Close them to build two induced cycles by going through x', P_{up} and u': one of them is an even cycle, and its length is at least 6.

If $j = r$ (*i.e.* all the v_j are before G_i), then we can use the same argument with a path of well-chosen parity from v_r to x, crossing G_i.

Otherwise, there is an odd and an even path in the PCP between v_j and v_{j+1}, crossing G_i, and its length is at least 4 because x_i and y_i are at distance at least 2 one from each other. We can close the even path by going back and forth to x: this gives an even hole of length at least 6. □

Note that, in the lemma above, G is taken in $\mathcal{C}_{3,2k\geqslant 6}$ and not in $\mathcal{C}_{3,5,2k\geqslant 6}$. In particular, we will use Lemma 7 in the next section as well. Let us now continue with step (iv):

Lemma 8. *Let v be a vertex of a graph $G \in \mathcal{C}_{3,5,2k\geqslant 6}$ and $(N_0, N_1, \ldots)$ be the v-levelling. Let $S \subseteq N_{k-1}$ be a stable set. Then $\chi(N(S) \cap N_k) \leqslant 52$.*

Proof. Let $\delta = \chi(N(S) \cap N_{k-1}) - 1$. Suppose by contradiction that $\delta \geqslant 52$, then $h(\delta) \geqslant 18$ hence by Lemma 6, we can grow a rooted PCP of order 2 inside $N(S) \cap N_k$. Let u be the origin of the PCP and u' its root. Observe in particular that S dominates G_2. Let xy be an edge of G_2, and let x' (resp. y') be a neighbor of x (resp. y) in S. By Lemma 7, both x' and y' have a neighbor in G_1. This gives an $x'y'$-path P_{down} with interior in G_1. In order not to create an even hole nor a 5-hole by closing it with $x'xyy'$, we can ensure that P_{down} is an even path of length at least 4. Moreover, there exists an upper $x'y'$-path P_{up}. Then either the hole formed by the concatenation of P_{up} and $x'xyy'$, or the one formed by the concatenation of P_{up} and P_{down} is an even hole of length $\geqslant 6$, a contradiction. □

The previous lemma allows us to prove that one can *lift* the PCP up into N_{k-1} to get a subset of vertices with high chromatic number. We state a lemma that will be reused in the next section:

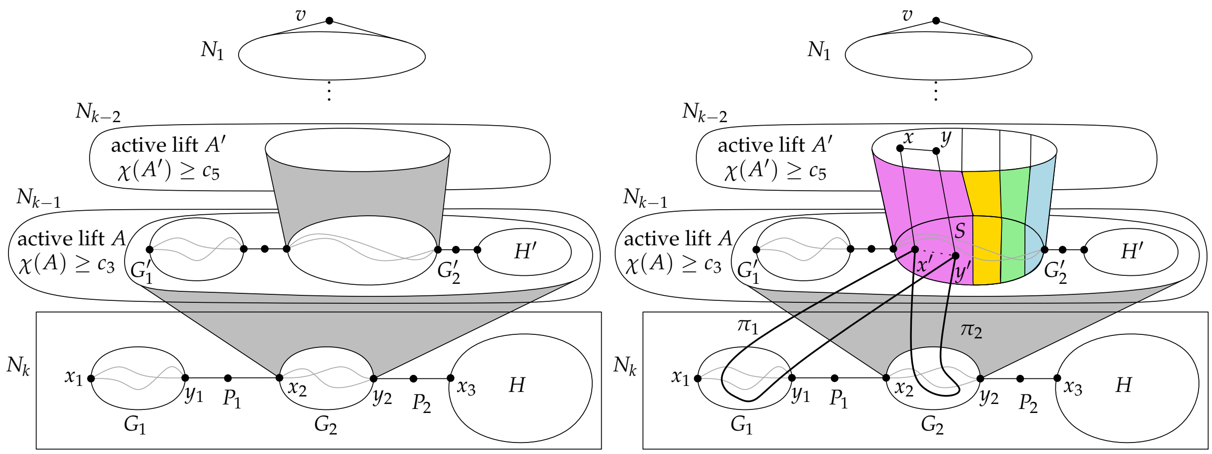
Lemma 9. *Let v be a vertex of a graph $G \in \mathcal{C}_{3,2k\geqslant 6}$ and $(N_0, N_1, \ldots)$ be the v-levelling. Let P be a rooted PCP of order $\ell \geqslant 1$ in a level N_k with leftovers at least δ (hence $k \geqslant 2$). Let $A = N(G_\ell) \cap N_{k-1}$ (called the active lift of the PCP). Suppose that for every stable set $S \subseteq A$, we have $\chi(N(S) \cap N_k) \leqslant \gamma$, then $\chi(A) \geqslant \delta/\gamma$.*

Proof. Let $r = \chi(A)$, suppose by contradiction that $r < \delta/\gamma$ and decompose A into r stable sets $S_1, \ldots, S_r$. Then $N(A) \cap N_k$ is the (non-necessarily disjoint) union of r sets $N(S_1) \cap N_k, \ldots, N(S_r) \cap N_k$, and each of them has chromatic number at most γ by assumption. Consequently $\chi(N(A) \cap N_k) \leqslant r\gamma < \delta$ and hence $\chi(H \setminus N(A)) \geqslant \chi(H) - \chi(N(A) \cap N_k) \geqslant 1$. Let x be any vertex of $H \setminus N(A)$ and x' be a neighbor of x in N_{k-1}. By construction, $x' \notin A$ so x' has no neighbor in G_ℓ. This is a contradiction with Lemma 7. □

By Lemmas 8 and 9 with $\gamma = 52$, we can directly deduce the following:

Lemma 10. *Let v be a vertex of a graph $G \in \mathcal{C}_{3,5,2k\geqslant 6}$ and $(N_0, N_1, \ldots)$ be the v-levelling. Let P be a rooted PCP of order $\ell \geqslant 1$ in a level N_k with leftovers at least δ (hence $k \geqslant 2$). Let $A = N(G_\ell) \cap N_{k-1}$ be the active lift of the PCP, then we have $\chi(A) \geqslant g(\delta) = \delta/52$.*

We can now finish the proof, this is step (v). Recall that a sketch was provided, and it may help to understand the following proof. Moreover, Figure 3 illustrates the proof.



(a) First paragraph of the proof

(b) End of the proof

Figure 3: Illustrations for the proof of Lemma 3.

Proof of Lemma 3. Let c' be a constant big enough so that

$$g\left(h^{(2)}\left(g\left(h^{(2)}\left(\frac{c'}{2}-1\right)\right)-1\right)\right) \geqslant 5 \ .$$

Suppose that $\chi(G) \geqslant c'$. Pick a vertex v, let $(N_0, N_1, \ldots)$ be the v-levelling and N_k be a colorful level, so $\chi(N_k) \geqslant \chi(G)/2 \geqslant c_1 + 1$ where $c_1 = c'/2 - 1$. By Lemma 6, grow a rooted PCP $P = (G_1, P_1, G_2, P_2, H)$ inside N_k of order 2 with leftovers at least $c_2 = h^{(2)}(c_1)$. Then apply Lemma 10 and get an active lift A of P inside N_{k-1} with chromatic number at least $c_3 = g(c_2)$. By definition of c' and c_3, we can show that $c_3 \geqslant 2$, which in particular implies that $k - 1 \geqslant 2$, since N_1 is a stable set. Since $h(c_3 - 1) \geqslant 18$, apply again Lemma 6 to get a rooted PCP $P' = (G'_1, P'_1, G'_2, P'_2, H')$ of order 2 inside N_{k-1} with leftovers at least $c_4 = h^{(2)}(c_3 - 1)$. Now apply Lemma 10 to get an active lift A' of P' inside N_{k-2} with chromatic number at least $c_5 = g(c_4)$. The situation at this point is described in Figure 3(a).

Because of the chromatic restriction in the definition of the PCP, one can color G'_2 with 4 colors. Moreover, G'_2 dominates A' by definition. Thus there exists a stable set $S \subseteq G'_2$ such that $\chi(N(S) \cap A') \geqslant c_6 = c_5/4$ (since A' is the union of the $N(S') \cap A'$ for the four stable sets S' that partition G'_2).

Now $c_6 > 1$ so there is an edge xy inside $N(S) \cap A'$. Call x' (resp. y') a vertex of S dominating x (resp. y). Both x' and y' have a neighbor in G_2 by definition of A and, by Lemma 7, both x' and y' also have a neighbor in G_1. This gives an $x'y'$-path π_1 (resp. π_2) with interior in G_1 (resp. G_2). Due to the path $x'xyy'$ of length 3, π_1 and π_2 must be even paths of length at least 4 (see Figure 3(b)). Thus the concatenation of π_1 and π_2 is an even hole of length at least 6, a contradiction. □

3 General case

This section aims at proving Theorem 2, using the result of the previous section. As already mentioned, we follow the same outline, except that we now need the existence of a C_5 several times. Let us start by a technical lemma to find both an even and an odd path out of a 5-hole and its dominating set:

Lemma 11. *Let G be a triangle-free graph inducing a 5-hole C. Let $S \subseteq V(G)$ be a minimal dominating set of C, assumed to be disjoint from C. If we delete the edges with both endpoints in S, then for every vertex $t \in S$, there exists a vertex $t' \in S$ such that one can find an induced tt'-path of length 4 and an induced tt'-path of length 3 or 5, both with interior in C.*

Proof. Let $t \in S$. Since S is a minimal dominating set of C, there exists $v_1 \in C$ such that $N(v_1) \cap S = \{t\}$ (otherwise, t is useless and can be removed from S). Number the other vertices of C with $v_2, \ldots, v_5$ (following the adjacency on the cycle). Since G is triangle-free, t can not be adjacent to both v_3 and v_4, so up to relabeling the cycle in the other direction we assume that t is not adjacent to v_3. Let $t' \in S$ be a vertex dominating v_3. Then $tv_1v_2v_3t'$ is an induced path of length 4 between t and t'. Moreover, $tv_1v_5v_4v_3t'$ is a (non-necessarily induced) path of length 5 between t and t'. If this path is not induced, the only possible chords are tv_4 and $t'v_5$ since G is triangle-free, which in any case gives an induced tt'-path of length 3. □

Recall that in this section, we are interesting in strong PCP, *i.e.* PCP, all regular blocks G_i of which contain an induced C_5. We start with step (i):

Lemma 12. *Let c' be the constant of Lemma 3, let $G \in \mathcal{C}_{3,2k\geqslant 6}$ and v be any vertex of G. For every $\delta \in \mathbb{N}$ such that $\chi(G) \geqslant \delta \geqslant 2c'$, there exists a strong PCP of order 1 with origin v and leftovers at least $f(\delta) = \delta/2 - 15$.*

Proof. Let $(N_0, N_1, \ldots)$ be the v-levelling, N_k be a colorful level (hence $k \geqslant 2$ since G is triangle-free) and let N'_k be a major connected component of $G[N_k]$, so $\chi(N'_k) \geqslant c'$. Using Lemma 3, there exists a 5-hole C in $G[N'_k]$. Consider a minimum dominating set D of C inside N_{k-1}.

From now on, the proof is very similar to the one of Lemma 4. Similarly, we define $Z' = N(D \cup C) \cap N'_k$ and $Z = Z' \setminus C$. Let $z \in Z$ be a vertex having a neighbor z_1 in a major connected component M_1 in $N'_k \setminus Z'$. The goal is now to find two vz-paths P and P' of different parity with interior in $N_0 \cup \ldots \cup D \cup C$, then we can set $G_1 = G[P \cup P' \cup C]$, $P_1 = G[\{z, z_1\}]$ and $H = G[M_1]$ as parts of the wanted PCP. In practice, we need to be a little more careful to ensure the condition on the length of P_1 and the non-adjacency between z and H.

Let us now find those two paths P and P'. By definition of Z, z also has a neighbor in D or in C.

1. If z has a neighbor $x \in C$, let $y \in C$ be a vertex adjacent to x on the hole. Let x' and y' be respectively a neighbor of x and a neighbor of y in D, in particular $x' \neq y'$

since G is triangle-free. Observe that z is connected neither to x' nor to y, otherwise it creates a triangle. We grow P by starting from an induced path of length $k-1$ from v to x' and then add the path $x'xz$. Similarly, we grow P' by starting from an induced path of length $k-1$ from v to y', and then add the edge $y'z$ if it exists, or else the path $y'yxz$. Observe that P' is indeed an induced path since there is no triangle. Moreover P and P' have distinct parities.

2. If z has no neighbor in C, then it has at least one neighbor x' in D. Apply Lemma 11 to get a vertex $y' \in D$ such that there exists an $x'y'$-path of length 3 or 5, and another one of length 4, both with interior in C. Observe that x' and y' cannot have a common neighbor u in $N_{k-2} \cup \{z\}$, otherwise there would be either a triangle x', u, y' (if $x'y' \in E$), or a C_6 using the $x'y'$-path of length 4 with interior in C. Now we grow P by starting from an induced path of length $k-1$ from v to x', and add the edge $x'z$. We grow P' by starting from an induced path of length $k-1$ from v to y', and then add the edge $x'y'$ if it exists, otherwise add the $x'y'$-path of length 3 or 5 with interior in C, and then finish with the edge $x'z$.

Now comes the fine tuning. Choose in fact $z_1 \in M_1 \cap N(z)$ so that z_1 is connected to a major connected component M_2 of $M_1 \setminus N(z)$. Choose z_2 a neighbor of z_1 in M_2 such that z_2 is connected to a major connected component M_3 of $M_2 \setminus N(z_1)$. We redefine $H = G[\{z_2 \cup M_3\}]$ and $P_1 = G[\{z, z_1, z_2\}]$. Then P_1 is a path of length 2, H is connected and G_1 is colorable with 4 colors (it is easily 7-colorable as the union of a 5-hole and two paths; a careful case analysis shows that it is 4-colorable). Moreover H has chromatic number at least $\chi(N'_k) - \chi(Z') - \chi(N(z)) - \chi(N(z_1))$. Since the neighborhood of any vertex is a stable set, $\chi(Z') \leqslant |D| + |C| + \chi(C) \leqslant 13$ and $\chi(N(z)), \chi(N(z_1)) \leqslant 1$. Thus $\chi(H) \geqslant \delta/2 - 15$. □

We go on with step (ii): find a strong rooted PCP. The following lemma is proved in the same way as Lemma 6 by replacing the use of Lemma 5 by Lemma 12, so we omit the proof here.

Lemma 13. *Let $G \in \mathcal{C}_{3,2k\geqslant 6}$ be a connected graph, f be the function defined in Lemma 12, v be a vertex of G and $(N_0, N_1, \ldots)$ be the v-levelling. For every k, δ such that $\chi(N_k) \geqslant \delta + 1 \geqslant 2c' + 1$, there exists a strong rooted PCP of order 1 in N_k with leftovers at least $f(\delta)$.*

Step (iii) is proved by Lemma 7 from the previous section, and was valid not only for $G \in \mathcal{C}_{3,5,2k\geqslant 6}$ but also for $G \in \mathcal{C}_{3,2k\geqslant 6}$. So we continue with step (iv):

Lemma 14. *Let v be a vertex of a graph $G \in \mathcal{C}_{3,2k\geqslant 6}$, $(N_0, N_1, \ldots)$ be the v-levelling. Let S be a stable set inside N_{k-1}. Then $\chi(N(S) \cap N_k) \leqslant 2c'$.*

Proof. Suppose by contradiction that $\chi(N(S) \cap N_k) \geqslant 2c' + 1$. By Lemma 13, we can grow in $N(S) \cap N_k$ a rooted PCP of order 1, and in particular S dominates G_1. By definition of a strong PCP, there is a 5-hole C in G_1. Since S is a dominating set of C, we can apply Lemma 11 to get two vertices $t, t' \in S$ such that one can find both an even and an odd

tt'-path with interior in C and length at least 3. Then any upper tt'-path close a hole of even length $\geqslant 6$. $\square$

In fact, as in previous section, we can directly deduce from Lemmas 9 and 14 that one can lift the PCP up into N_{k-1} to get a subset of vertices with high chromatic number:

Lemma 15. *Let $G \in \mathcal{C}_{3,2k\geqslant 6}$, $v \in V(G)$ and $(N_0, N_1, \dots)$ be the v-levelling. Let P be a strong rooted PCP of order 1 in a level N_k (hence $k \geqslant 2$) with leftovers δ. Let $A = N(G_1) \cap N_{k-1}$ be the active lift of the PCP. If $\delta \geqslant 2c'$, then $\chi(A) \geqslant \varphi(\delta) = \frac{\delta}{2c'}$.*

We are now ready to finish the proof, this is step (v). The proof follow the same outline as the proof of Lemma 3, which was sketched at the end of the Introduction.

Recall that a sketch was given and may be useful to have a less technical overview of the proof.

Proof of Theorem 2. Let c be a constant such that

$$\varphi\left(f\left(\varphi\left(f\left(\frac{c}{2}-1\right)\right)-1\right)\right) \geqslant 4c' \ .$$

Suppose that $G \in \mathcal{C}_{3,2k\geqslant 6}$ has chromatic number $\chi(G) \geqslant c$. Then pick a vertex v, let $(N_0, N_1, \dots)$ be the v-levelling and N_k be a colorful level, consequently $\chi(N_k) \geqslant c_1 + 1 = c/2$. Apply Lemma 13 and grow inside N_k a strong rooted PCP $P = (G_1, P_1, H)$ of order 1 with leftovers at least $c_2 = f(c_1)$. Then apply Lemma 15 and get an active lift $A = N(G_1)$ of P inside N_{k-1} with chromatic number at least $c_3 = \varphi(c_2)$. By Lemma 13, we can obtain a strong rooted PCP $P' = (G'_1, P'_1, H')$ inside A with leftovers at least $c_4 = f(c_3 - 1)$, and by Lemma 15 we obtain an active lift A' of P' inside N_{k-2} with chromatic number at least $c_5 = \varphi(c_4)$. Because of the chromatic restriction in the definition of the PCP, one can color G'_1 with 4 colors. Moreover, G'_1 dominates A' by definition. Thus there exists a stable set $S \subseteq P'$ such that $\chi(N(S) \cap A') \geqslant c_6 = c_5/4$. Now $c_6 \geqslant c'$ thus Lemma 3 proves the existence of a 5-hole C inside $N(S) \cap A'$. Let us give an overview of the situation: we have a 5-hole C inside N_{k-2}, dominated by a stable set S inside N_{k-1}, and every pair of vertices t, t' of S can be linked by a tt'-path P_{down} with interior in $G_1 \subseteq N_k$. Lemma 11 gives the existence of two vertices $t, t' \in S$ linked by both an odd path and an even path of length $\geqslant 3$ with interior in C. Closing one of these paths with P_{down} gives an induced even hole of length $\geqslant 6$, a contradiction. $\square$

Concluding remark

Observe that no optimization was made on the constants c' and c from Lemma 3 and Theorem 2. The proof gives the following upper bounds:

- $\chi(G) \leqslant 435122$ for every $G \in \mathcal{C}_{3,5,2k\geqslant 6}$, and
- $\chi(G) \leqslant 12 \cdot 10^{18}$ for every $G \in \mathcal{C}_{3,2k\geqslant 6}$.

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