The gonality sequence of complete graphs

Filip Cools
KU Leuven
Department of Mathematics
Belgium
f.cools@kuleuven.be

Marta Panizzut
TU Berlin
Institut für Mathematik
Germany
panizzut@math.tu-berlin@de

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Abstract

The gonality sequence $(\gamma_r)_{r\geqslant 1}$ of a finite graph/metric graph/algebraic curve comprises the minimal degrees γ_r of linear systems of rank r. For the complete graph K_d, we show that $\gamma_r = kd - h$ if $r < g = \frac{(d-1)(d-2)}{2}$, where k and h are the uniquely determined integers such that $r = \frac{k(k+3)}{2} - h$ with $1 \leqslant k \leqslant d-3$ and $0 \leqslant h \leqslant k$. This shows that the graph K_d has the gonality sequence of a smooth plane curve of degree d. The same result holds for the corresponding metric graphs.

Keywords: Gonality sequence; complete graphs; plane curves

1 Introduction

Baker and Norine in [BN07] introduced a theory of linear systems on graphs, later generalized by several authors to metric graphs and other combinatorial objects [MZ08, GK08]. It presents strong analogies with the one on algebraic curves. Many remarkable theorems have been proven to have a combinatorial counterpart, for example the Riemann-Roch Theorem and Clifford's Theorem, see [BN07, MZ08, Cop16].

With the notation g^r_s we indicate a linear system of degree s and rank r. We refer to Section 2 for the definitions. The *gonality sequence* $(\gamma_r)_{r\geqslant 1}$ of a finite graph is defined as

$$\gamma_r := \min\{s \text{ such that there exists a } g^r_s\}.$$

The gonality sequence was first introduced in [LM12] in the context of algebraic curves and the terminology comes from the integer γ_1, which is called the *gonality*. By Riemann-Roch, it follows that $\gamma_r = g + r$ if $r \geqslant g$.

The gonality sequence is known for general chains of loops, as it follows from the results in [CDPR11]. Except for graphs with low gonality, the gonality sequence is still undetermined for other graphs. The main result proven in this paper is the following:

Theorem 1. The gonality sequence of K_d is given by

$$\gamma_r = \begin{cases} kd - h & \text{if } r < g \\ g + r & \text{if } r \geqslant g \end{cases},$$

where $g = \frac{(d-1)(d-2)}{2}$ is the genus of K_d, and k and h are the uniquely determined integers with $1 \leqslant k \leqslant d-3$ and $0 \leqslant h \leqslant k$ such that

$$r = \frac{k(k+3)}{2} - h.$$

The proof consists of two parts, respectively presented in Section 3 and in Section 4. In Section 3 we verify the existence of divisors of degree $s = kd - h$ and rank (at least) $r = \frac{k(k+3)}{2} - h$. In other words, we prove that $kd - h$ is an upper bound for γ_r. Herefore, we use an algorithm, presented by Cori and Le Borgne in [CB16], for computing the rank of divisors on complete graphs. The more involved part is showing that the upper bound $kd-h$ for γ_r is in fact sharp, which is done in Section 4. We translate this problem into a property of sequences of integers $(\alpha_i)_{i=1,\ldots,d-1}$ satisfying certain hypotheses. In both parts, reduced divisors play an essential role. We provide an easy characterization of reduced divisors on complete graphs.

In Section 5, we extend the result to the complete metric graph K_d with edge lengths equal to one. To be precise, we show that the gonality sequence of this metric graph is the same as the one of the corresponding ordinary graph; hereby answering [Bak08, Conjecture 3.14] in the case of complete graphs. The existence part of the theorem follows immediately from [HKN13, Theorem 1.3], which relates the rank of divisors on graphs and their corresponding metric graphs. The second part is proven in an analogous way as in Section 4, again utilizing the property of integer sequences.

We conclude by remarking that our arguments do not directly extend to complete metric graphs with arbitrary edge lengths. This is done in Section 6, where we give a quick account on the problems we encountered.

While the proof is purely combinatorial, the motivation for considering the question comes from plane curves and the specialization of divisors from curves to graphs.

Baker in [Bak08] showed that there is a close connection between linear systems on curves and on graphs. Indeed, let R be a discrete valuation ring with field of fractions K. A model for a curve X over K is a surface over R whose generic fiber is X. Given a smooth curve X over K and a strongly semistable regular model $\mathfrak{X}$ over R, it is possible to specialize a divisor on the curve to a divisor on the dual graph of the special fiber of $\mathfrak{X}$. The complete graph K_d pops up if one takes a model of a smooth plane curve of degree d degenerating to a union of d lines.

The gonality sequence of smooth plane curves has been computed by Ciliberto in [Cil83], and Hartshorne in [Har86]. Therefore a natural question in this setting is whether the complete graph and the smooth plane curve have the same gonality sequence. Theorem 1 provides a positive answer. As we point out in Remark 9, the linear systems with degree $s = dk-h$ and rank $r = \frac{k(k+3)}{2} - h$ on plane curves specialize to linear systems of the same degree and rank on the graphs. So the first part of Theorem 1 can in fact be deduced from the result on plane curves.

2 Linear systems and reduced divisors

Let G be a finite graph without loop edges. We denote its vertex set by $V(G)$ and its edge set by $E(G)$.

Definition 2. A *divisor* on G is an element of the free abelian group $\mathrm{Div}(G)$ on the set $V(G)$. Any divisor can be represented in a unique way as a finite formal combination of vertices of G with integer coefficients:

$$D = \sum_{v \in G} a_v\,(v), \quad \text{with } a_v \in \mathbb{Z}.$$

The *degree* $\deg(D)$ of a divisor D is the sum $\sum_{v \in V(G)} a_v$ of its coefficients. If $a_v \geqslant 0$ for every $v \in G$, the divisor is said to be *effective*, and this is indicated with $D \geqslant 0$.

Let $f : V(G) \to \mathbb{Z}$ be an integer-valued function on the vertices of G. We define the *principal divisor* corresponding to f as

$$\mathrm{div}(f) = \sum_{v \in V(G)} \Big(\sum_{e = vw \in E(G)} \big(f(v) - f(w)\big) \Big)(v).$$

Two divisor $D_1, D_2 \in \mathrm{Div}(\Gamma)$ are *linearly equivalent*, denoted by $D_1 \sim D_2$, if there exists a function f such that

$$D_1 - D_2 = \mathrm{div}(f).$$

The *linear system* corresponding to D, indicated with $|D|$, is the set of the effective divisors linearly equivalent to D. In symbols,

$$|D| = \Big\{ E \in \mathrm{Div}(G) \,\Big|\, E \geqslant 0,\, E \sim D \Big\}.$$

The *rank* $\mathrm{rk}_G(D)$ of a divisor D is defined as -1 if D is not equivalent to any effective divisor, otherwise

$$\mathrm{rk}_G(D) = \max \Big\{ r \in \mathbb{Z}_{\geqslant 0} \Big|\, |D - E| \neq \varnothing \quad \forall\, E \in \mathrm{Div}(G),\ E \geqslant 0,\ \deg(E) = r \Big\}.$$

We will often omit the subscript G in $\mathrm{rk}_G(D)$ when it is clear from the context on which graph G we are working.

Reduced divisors will play an important role in proving the results of this paper. We briefly repeat the definition presented in [BN07].

Definition 3. Let A be a subset of $V(G)$. Given $v \in A$, the *outgoing degree* $\mathrm{outdeg}_A(v)$ *of A at v* is defined as the number of edges having v as one endpoint and whose other endpoint lies in $V(G) \setminus A$. Let D be a divisor on G and v be a vertex of G. The divisor D is *v-reduced* if it is effective in $V(G) \setminus \{v\}$ and each non-empty subset $A \subseteq V(G) \setminus \{v\}$ contains a point w such that $D(w) < \mathrm{outdeg}_A(w)$.

The following result is proven in [BN07, Proposition 3.1]:

Proposition 4. Let v be a vertex on a graph G. Then for every divisor D on G, there exists a unique v-reduced divisor D' such that $D' \sim D$.

Let K_d be the complete graph on d vertices.

Lemma 5. A divisor D on K_d is reduced with respect to a vertex v if and only if there exists an ordering $v_1, v_2, \ldots, v_{d-1}$ of the vertices in $V(K_d)\backslash\{v\}$ such that $0 \leqslant D(v_i) \leqslant i-1$.

Proof. For every $A \subseteq V(K_d)\backslash\{v\}$ and for every $w \in A$, we have that $\text{outdeg}_A(w) = d-|A|$. Therefore D is v-reduced if and only if D is effective when restricted to $K_d \setminus \{v\}$, and each non-empty subset A contains an element w such that $D(w) < d - |A|$.

Suppose that there exists an ordering such that $0 \leqslant D(v_i) \leqslant i - 1$. Let A be a subset of $V(K_d) \setminus \{v\}$, and let $j = \min\{i \mid v_i \in A\}$. We have that $j \leqslant d - |A|$. It follows that

$$D(v_j) \leqslant j - 1 < d - |A|,$$

therefore D is v-reduced.

For the other implication, if we take $A = V(K_d) \setminus \{v\}$, then there exists a vertex v_1 in A such that $D(v_1) = 0$. For $A = V(K_d) \setminus \{v, v_1\}$, there exists a vertex v_2 such that $D(v_2) \leqslant 1$. Iterating this argument, we obtain that if $A = V(K_d) \setminus \{v_0, \ldots, v_i\}$, there exists a vertex v_{i+1} in A such that $D(v_{i+1}) \leqslant i$. □

Remark 6. In particular, if D is a v-reduced divisor on K_d, then there exists a vertex $v_1 \neq v$ such that $D(v_1) = 0$.

3 An upper bound for the gonality sequence

We provide a completely combinatorial proof of the inequality $\gamma_r \leqslant kd - h$. Since

$$\text{rk}(D - (v)) \geqslant r(D) - 1$$

for each divisor D and each vertex v (see [Bak08, Lemma 2.7]), it suffices to handle the case $h = 0$, i.e. there exists a divisor D of degree kd and rank $r = \frac{k(k+3)}{2}$. We will explicitly construct such a divisor: if $V(K_d) = \{v_1, \ldots, v_d\}$, then the divisor

$$D = kd\,(v_d) \sim \sum_{i=1}^{d} k(v_i)$$

will do the job.

In [CB16] the authors provide an algorithm to compute the rank of a divisor on the complete graph K_d. Although it is stated in a different terminology, it is possible to present it in terms of divisors.

Lemma 7 (Lemma 3 in [CB16])**.** Let D be a v_d-reduced effective divisor on K_d. Let $v_i \neq v_d$ be a vertex for which $D(v_i) = 0$ (see Remark 6). The divisor $D' = D - (v_i)$ has rank $\mathrm{rk}(D) - 1$.

The algorithm takes a divisor as input. The first step consists of computing the v_d-reduced divisor D equivalent to it. If $D(v_d) < 0$, then the divisor has rank -1. Otherwise we take a vertex v_i such that $D(v_i) = 0$ and consider the divisor $D_1 = D - (v_i)$. By the previous lemma $\mathrm{rk}(D) = \mathrm{rk}(D_1) + 1$. We iterate until we find a divisor of rank -1. The algorithm terminates after at most $\deg(D)$ steps.

Theorem 8. Let $1 \leqslant k \leqslant d-1$. The rank of the divisor $D = kd\,(v_d)$ is $\frac{k(k+3)}{2}$.

Proof. We use Lemma 7 and the algorithm described above. For convenience, we write the subsequent divisors appearing in the algorithm as $D_{s,t}$ where $s \geqslant 1$ and $0 \leqslant t \leqslant s$. In this way, we obtain a sequence of divisors where the indexes are in lexicographic order:

$$D_{1,0},\ D_{1,1},\ D_{2,0},\ D_{2,1},\ D_{2,2},\ D_{3,0},\ \ldots$$

The first two divisors are:

$$\begin{aligned} D_{1,0} = D - (v_1) \quad &\sim \quad \Big[kd - (d-1)\Big](v_d) + \textstyle\sum_{i=2}^{d-1}(v_i) \\ D_{1,1} = D_{1,0} - (v_1) \quad &\sim \quad \Big[kd - (d-1) - 1\Big](v_d) + (d-2)(v_1) \end{aligned}$$

At every step we subtract the divisor (v_i) corresponding to the vertex with zero coefficient and smallest index i, and we compute the linearly equivalent v_d-reduced divisor. The latter is the divisor taken as input in the following step. So,

$$D_{2,0} = D_{1,1} - (v_2) \sim \Big[kd - 2(d-1)\Big](v_d) + \sum_{i=3}^{d-1} 2(v_i) + (v_2).$$

At the step (s,t) (with $s \leqslant k$), the divisor $D_{s,t}$ is linearly equivalent to the following v_d-reduced divisor:

$$\Big[kd - s(d-1) - t\Big](v_d) + \sum_{i=s+1}^{d-1}(s-t)(v_i) + \sum_{i=t+1}^{s}(i-t-1)(v_i) + \sum_{i=1}^{t}(d-2-t+i)(v_i).$$

For $(s,t) = (k,k)$, we have

$$D_{k,k} \sim (d-2)(v_k) + \cdots + (d-k-1)(v_1),$$

which is effective because $k \leqslant d-1$. The divisor obtained at the next step is

$$D_{k+1,0} = D_{k,k} - (v_{k+1}) = (-1)(v_{k+1}) + (d-2)(v_k) + \cdots + (d-k-1)(v_1).$$

Since it is v_{k+1}-reduced, it is not equivalent to any effective divisor, so it has rank -1. We can conclude that the rank of D is $\frac{(k+1)(k+2)}{2} - 1 = \frac{k(k+3)}{2}$. $\square$

Remark 9. For the reader familiar with the theory of specialization of divisors from curves to graphs, we remark that it is possible to prove the first part of Theorem 1 using the gonality sequence of smooth plane curves [Cil83] and Baker's Specialization Lemma [Bak08]. In fact, the complete graph K_d is the dual graph of the special fiber of a regular strongly semistable model of a smooth plane curve of degree d. In [Cil83], Ciliberto showed that, given an effective divisor E' of degree $h \leqslant k$ on X and H the generic divisor cut out by a line, the complete linear system

$$|kH - E'|$$

has rank $\frac{k(k+3)}{2} - h$.

The divisors $kH - E'$ described above specialize to divisors of the form

$$D = (k - a_1)(v_1) + (k - a_2)(v_2) + \cdots + (k - a_d)(v_d),$$

on K_d, with $a_1 + a_2 + \cdots + a_d = h$. By the Specialization Lemma, the rank of these divisors is at least $\frac{k(k+3)}{2} - h$.

It is not difficult to prove that it cannot be strictly bigger. By a relabeling of the vertices, we can assume that $a_i = 0$ for $i > h$ and $a_i \geqslant a_j$ if $i < j$. We consider the effective divisor

$$E = \sum_{i=1}^{k+1} \Big(k - a_i - (i-2)\Big)(v_i)$$

of degree $\frac{k(k+3)}{2} - h + 1$. We have that

$$D - E = -1(v_1) + 0(v_2) + (v_3) + \cdots + (k-1)(v_{k+1}) + k(v_{k+2}) + \cdots + k(v_d).$$

This divisor is v_1-reduced, therefore $|D - E| = \varnothing$ and the rank of D is $\frac{k(k+3)}{2} - h$.

4 Sharpness of the upper bound

This section is concerned with proving the inequality $\gamma_r \geqslant kd - h$ for $r = \frac{k(k+3)}{2} - h$, with $k \leqslant d - 3$ and $0 \leqslant h \leqslant k$. In fact, we show that each divisor D of degree $dk - h - 1$ has rank strictly smaller than $\frac{k(k+3)}{2} - h$. It suffices to do this for $h = k$. This follows from $\mathrm{rk}(D + (v)) \leqslant \mathrm{rk}(D) + 1$ for each divisor D and vertex v.

Let D be any divisor of degree $k(d-1) - 1$ on K_d. By the above argumentation, we need to show that its rank is strictly smaller than

$$\frac{k(k+3)}{2} - k = \frac{k(k+1)}{2}.$$

We may assume that D is reduced with respect to the vertex v_d. By Lemma 5, we can label the vertices so that the coefficients satisfy the following inequalities:

$$D(v_1) = 0 \leqslant D(v_2) \leqslant \cdots \leqslant D(v_{d-1}) \leqslant d - 2 \quad \text{and} \quad D(v_i) \leqslant i - 1 \text{ for } i \leqslant d - 1.$$

If $D(v_d) < \frac{k(k+1)}{2}$, we can already conclude that the rank of the divisor cannot be $\frac{k(k+1)}{2}$, so assume that $D(v_d) \geqslant \frac{k(k+1)}{2}$.

We write

$$D(v_d) = a(d-1) + b \quad \text{with} \quad a, b \in \mathbb{Z}_{\geqslant 0} \text{ and } 0 \leqslant b \leqslant d-2.$$

Since $D(v_d) \leqslant \deg(D) = k(d-1) - 1$, we have that $a < k$. Consider the divisor

$$D' = \Big(b - (d-1)\Big)(v_d) + \sum_{i=1}^{d-1} \Big(D(v_i) + a + 1\Big)(v_i) \sim D.$$

Note that D' has a negative coefficient at v_d. If we are able to construct an effective divisor E of degree at most $\frac{k(k+1)}{2}$ which is supported on the vertices $v_1, \dots, v_{d-1}$ and such that $D' - E$ is v_d-reduced, then we can conclude that the rank of D cannot be $\frac{k(k+1)}{2}$. By Lemma 5, this happens if

$$\sum_{i=1}^{d-1} \max\Big\{0, D(v_i) + a - (i-2)\Big\} \leqslant \frac{k(k+1)}{2}.$$

To simplify the notation, we define

$$\alpha_i := D(v_i) + a - (i-2) \text{ and } \alpha_i^+ := \max\Big\{0, D(v_i) + a - (i-2)\Big\}.$$

We have that

$$\begin{aligned} \sum_{i=1}^{d-1} \alpha_i &= \Big[\deg(D) - D(v_d)\Big] + a(d-1) - \sum_{i=1}^{d-1}(i-2) \\ &= k(d-1) - \frac{(d-2)(d-3)}{2} - b. \end{aligned}$$

From the above arguments, it follows that the following inequalities are satisfied for every index $i \leqslant d-1$:

$$\alpha_i \leqslant a+1, \quad \alpha_i \geqslant a - i + 2, \quad \alpha_{i+1} \geqslant \alpha_i - 1. \tag{$*$}$$

In particular, we can deduce that $\alpha_1 = a+1$ and $\alpha_{a+2} \geqslant 0$.

The inequality $\sum_{i=1}^{d-1} \alpha_i^+ \leqslant \frac{k(k+1)}{2}$ is not always satisfied, as the following example illustrates:

Example 10. Consider the v_d-reduced divisor

$$D = \Big((k-1)(d-1)\Big)(v_d) + (0(v_1)) + \sum_{i=2}^{d-1}(v_i).$$

In this case, $a = k-1$, $b = 0$ and

$$D' = -(d-1)(v_d) + k(v_1) + \sum_{i=2}^{d-1}(k+1)(v_i) \sim D.$$

Furthermore, we have $\alpha_1 = k$ and $\alpha_i = k - (i-2)$ for $i \geqslant 2$, so

$$\sum_{i=1}^{d-1} \alpha_i^+ = k + \frac{k(k+1)}{2} > \frac{k(k+1)}{2}.$$

It is still possible to show that the divisor does not have rank $\frac{k(k+1)}{2}$, by considering instead the divisor

$$D'' = (k-1)(v_1) + \sum_{i=2}^{d-1} k(v_i) \sim D.$$

We remark that in this case the coefficient at v_d is not negative. Let E be the following divisor of degree $\frac{k(k+1)}{2}$:

$$E = (v_d) + (k-1)(v_1) + \sum_{i=2}^{k+1} (k-i+1)(v_i).$$

The divisor $D' - E$ has a negative coefficient at the vertex v_d and is v_d-reduced. Therefore, also for this example we can conclude that D has not rank $\frac{k(k+1)}{2}$.

We can generalize Example 10 as follows: if $t_1 = \sum_{i=1}^{d-1} \alpha_i^+ > \frac{k(k+1)}{2}$, instead of the divisor D', we consider

$$D'' = b(v_d) + \sum_{i=1}^{d-1} \Big(D(v_i) + a\Big)(v_i) \sim D.$$

If we are able to construct an effective divisor E of degree at most $\frac{k(k+1)}{2}$ with coefficient $b+1$ at v_d and such that $D'' - E$ is v_d-reduced, or equivalently, if

$$t_2 = b + 1 + \sum_{i=1}^{d-1} \max\Big\{0, \alpha_i - 1\Big\} \leqslant \frac{k(k+1)}{2},$$

then we can conclude that the rank of D cannot be $\frac{k(k+1)}{2}$.

We claim that (at least) one of the two terms t_1 and t_2 is at most $\frac{k(k+1)}{2}$.

Claim 11. Let $\alpha = (\alpha_i)_{i=1,\dots,d-1}$ be a sequence of integers satisfying the rules in $(*)$ and such that

$$\sum_{i=1}^{d-1} \alpha_i = k(d-1) - \frac{(d-2)(d-1)}{2} - b.$$

Let $\alpha_i^+ = \max\{0,\, \alpha_i\}$ and $(\alpha_i - 1)^+ = \max\{0,\, \alpha_i - 1\}$. Define

$$t_1 := \sum_{i=1}^{d-1} \alpha_i^+ \quad \text{and} \quad t_2 := b + 1 + \sum_{i=1}^{d-1} \Big(\alpha_i - 1\Big)^+.$$

Then $\min\{t_1, t_2\} \leqslant \frac{k(k+1)}{2}$.

Remark 12. To summarize our strategy, we add the principal divisor

$$-(d-1)(v_d) + \sum_{i=1}^{d-1}(v_i)$$

to D until it either has a negative value at v_d, or one time before that, and (at least) one of the two resulting divisors is v_d-reduced after subtracting an effective divisor of degree $\frac{k(k+1)}{2}$.

The idea of proof of Claim 11 is as follows: first we introduce a specific integer sequence, which we show to be the "worst-case scenario". Afterwards, we prove the claim for this particular sequence.

For each $p \in \{1, \ldots, d-2\}$ and $q \in \{p+2, \ldots, d\}$, we define the sequence $\beta^{(p,q)} = \big(\beta_i\big)_{i=1,\ldots,d-1}$ as follows:

$$\begin{aligned} &\beta_i = a+1 && \text{for } 1 \leqslant i \leqslant p, \\ &\beta_{p+j} = a+1-j && \text{for } 1 \leqslant j \leqslant q-1-p, \\ &\beta_{p+j} = a+1-(j-1) && \text{for } q-p \leqslant j \leqslant d-1-p. \end{aligned} \tag{$\triangle$}$$

In case $p = d-1$ we are slightly abusing the notation, since there is no admissible value for q. The sequence $\beta^{(d-1,q)}$ becomes

$$\beta^{(d-1)} = (a+1, a+1, \ldots, a+1)$$

.

If $q = d$, then $\beta_{d-1} = p + a - d + 2$. For example, the sequence $\beta^{(1,d)}$ is given by $(a+1, a, a-1, \ldots, a-d+3)$.

Note that $\beta^{(p,q)}$ satisfies the rules $(*)$.

Remark 13. Let $\alpha = (\alpha_i)_{i=1,\ldots,d-1}$ be a sequence of integers satisfying $(*)$. For every i we have the following inequalities:

$$a - i + 2 = \beta_i^{(1,d)} \leqslant \alpha_i \leqslant \beta_i^{(d-1)} = a+1$$

Moreover for $q > p+2$ it holds that

$$\sum_{i=1}^{d-1} \beta_i^{(p,q-1)} = \sum_{i=1}^{d-1} \left(\beta_i^{(p,q)}\right) + 1.$$

Similarly, for $p < d-2$ we have

$$\sum_{i=1}^{d-1} \beta_i^{(p+1,d)} = \sum_{i=1}^{d-1} \left(\beta_i^{(p,p+2)}\right) + 1,$$

and for $p = d-1$
$$\sum_{i=1}^{d-1} \beta_i^{(d-1)} = \sum_{i=1}^{d-1} \left(\beta_i^{(d-2,d)}\right) + 1.$$

It follows that for each sequence α satisfying $(*)$, there exists a unique sequence $\beta^{(p,q)}$ with $\sum_{i=1}^{d-1} \beta_i = \sum_{i=1}^{d-1} \alpha_i$.

Remark 14. From the formula
$$\sum_{i=1}^{d-1} \beta_i = k(d-1) - \frac{(d-2)(d-3)}{2} - b,$$

it is possible to compute p and q. Indeed, since
$$\sum_{i=1}^{d-1} \beta_i = (a+1)(d-1) - \frac{(d-1-p)(d-p)}{2} + (d-q),$$

it follows that
$$q = (d-1)\Big[p - (k-a)\Big] - \frac{p(p-1)}{2} + (b+2).$$

Since the difference between two subsequent terms (corresponding to $p+1$ and p) is
$$(d-1) - \frac{p(p+1)}{2} + \frac{(p-1)p}{2} = (d-1) - p,$$

there is a unique p for which $q \in \{p+2, \dots, d\}$. Note that $p \geqslant k-a$, since $q \geqslant p+2$.

Lemma 15. Let k, d, a, b be integers such that
$$d-2 \geqslant k+1 \geqslant 2, \quad 0 \leqslant a \leqslant k-1 \quad \text{and} \quad 0 \leqslant b \leqslant d-2.$$

Let $\alpha = (\alpha_i)_{i=1,\dots,d-1}$ be a sequence of integer numbers that satisfies the conditions $\alpha_i \leqslant a+1$, $\alpha_{i+1} \geqslant \alpha_i - 1$ and
$$\sum_{i=1}^{d-1} \alpha_i = k(d-1) - \frac{(d-2)(d-3)}{2} - b.$$

If p, q are integers such that the sequence $\beta^{p,q}$ satisfies $\sum_{i=1}^{d-1} \beta_i = \sum_{i=1}^{d-1} \alpha_i$, then
$$\sum_{i=1}^{d-1} \alpha_i^+ \leqslant \sum_{i=1}^{d-1} \beta_i^+ \quad \text{and} \quad \sum_{i=1}^{d-1} (\alpha_i - 1)^+ \leqslant \sum_{i=1}^{d-1} (\beta_i - 1)^+.$$

Proof. Consider sequences α and $\beta^{(p,q)}$ that satisfy the conditions in the statement. By construction, the sequence $\beta^{(p,q)} = (\beta_i)_{i=1,\dots,d-1}$ is taken in such a way that, within the sequences satisfying $(*)$ and having fixed sum equal to $\sum_{i=1}^{d-1} \alpha_i$, it minimizes the number of places i with positive values for β_i, fixing the value for $\sum_{i=1}^{d-1} \beta_i^+$.

Suppose that $q < a + p + 2$, hence $\beta_q > 0$. Let $i_1, \dots, i_c$ be the indexes i, ordered from small to big, for which α_i has negative values, and $j_1, \dots, j_{d-a-p-3}$ the same for β_i (so $j_h = h + a + p + 2$). Then $d - a - p - 3 \geqslant c$ since the β_i are negative in as much places as possible. Moreover, we have that $\alpha_{i_h} \geqslant \beta_{j_h}$ for each $h \in \{1, \dots, c\}$. Indeed, if $\alpha_{i_h} < \beta_{j_h} = -h$, then $\alpha_{i_1} < -1$ by repeatedly using the rule $\alpha_{i+1} \geqslant \alpha_i - 1$, hence $\alpha_{i_1 - 1} < 0$, a contradiction. Therefore, the sum of the negative α_i is at least the sum of the negative β_i. Since $\sum_{i=1}^{d-1} \beta_i = \sum_{i=1}^{d-1} \alpha_i$, we can conclude that

$$\sum_{i=1}^{d-1} \beta_i^+ \geqslant \sum_{i=1}^{d-1} \alpha_i^+.$$

Moreover,

$$\sum_{i=1}^{d-1} (\beta_i - 1)^+ = \sum_{i=1}^{d-1} \beta_i^+ - (a + p + 1) \geqslant \sum_{i=1}^{d-1} \alpha_i^+ - (a + p + 1) \geqslant \sum_{i=1}^{d-1} (\alpha_i - 1)^+.$$

Now suppose that $q \geqslant a + p + 2$, so $\beta_q \leqslant 0$ (if $q \neq d$). The number of $\beta_i \geqslant 0$ is $a + p + 1$. If $\sum_{i=1}^{d-1} \alpha_i^+ > \sum_{i=1}^{d-1} \beta_i^+$, then the number of places i where $\alpha_i \geqslant 0$ is more than $a + p$. From this, it follows that the sum of the negative α_i is at least the sum of the negative β_i. We obtain a contradiction with $\sum_{i=1}^{d-1} \alpha_i = \sum_{i=1}^{d-1} \beta_i$. Analogously, we can see that $\sum_{i=1}^{d-1} (\alpha_i - 1)^+ \leqslant \sum_{i=1}^{d-1} (\beta_i - 1)^+$. □

We conclude this section by proving Claim 11.

Proof. Because of Lemma 15, the sequence $\beta^{(p,q)} = (\beta_i)_{i=1,\dots,d-1}$ defined by $(\triangle)$ maximizes $\sum_{i=1}^{d-1} \beta_i^+$ and $b + 1 + \sum_{i=1}^{d-1} (\beta_i - 1)^+$. Hence it suffices to check the claim for these kind of sequences. We distinguish three cases:

1. $\beta_{d-1} > 0$;
2. $\beta_{d-1} \leqslant 0$ and $q < a + p + 2$;
3. $\beta_{d-1} \leqslant 0$ and $q \geqslant a + p + 2$.

Case (1). Remark that $\beta_i^+ = \beta_i$ for every i. We compute t_2:

$$\begin{aligned} t_2 &= \sum_{i=1}^{d-1} \Big(\beta_i - 1\Big)^+ + b + 1 = \sum_{i=1}^{d-1} \beta_i^+ - (d-1) + b + 1 \\ &= \sum_{i=1}^{d-1} \beta_i - (d-1) + b + 1 = (k-1)(d-1) - \frac{(d-2)(d-3)}{2} + 1 \end{aligned}$$

It follows that $t_2 \leqslant \frac{k(k+1)}{2}$ if and only if

$$\frac{(d-k-1)(d-k-2)}{2} \geqslant 0.$$

This condition is satisfied since $k \leqslant d-3$.

Case (2). Remark that $\beta_q \geqslant 1$. We compute t_2 also in this case:

$$\begin{aligned}
t_2 &= \sum_{i=1}^{d-1} \Big(\beta_i - 1\Big)^+ + b + 1 = \sum_{i=1}^{d-1} \beta_i^+ - (a+p+1) + b + 1 \\
&= \sum_{i=1}^{d-1} \beta_i + \frac{(d-p-a-3)(d-p-a-2)}{2} - (a+p) + b \\
&= \Big[(k-1)(d-1) - \frac{(d-2)(d-3)}{2} + 1\Big] + \frac{(d-p-a-2)(d-p-a-1)}{2}
\end{aligned}$$

It follows that $t_2 \leqslant \frac{k(k+1)}{2}$ if and only if

$$\frac{(d-p-a-1)(d-p-a-2)}{2} \leqslant \frac{(d-k-1)(d-k-2)}{2}.$$

This is satisfied since $p \geqslant k-a$.

Case (3). First we consider the special situation $p = k-a$. We compute the term t_1:

$$\begin{aligned}
t_1 &= \sum_{i=1}^{d-1} \beta_i^+ = p(a+1) + \frac{a(a+1)}{2} = \frac{(p+a)(p+a+1)}{2} - \frac{p(p-1)}{2} \\
&= \frac{k(k+1)}{2} - \frac{p(p-1)}{2} \leqslant \frac{k(k+1)}{2}.
\end{aligned}$$

If $p > k-a$, we compute t_2:

$$\begin{aligned}
t_2 &= \sum_{i=1}^{d-1} \Big(\beta_i - 1\Big)^+ + b + 1 = \sum_{i=1}^{d-1} \beta_i^+ - (a+p) + b + 1 \\
&= \sum_{i=1}^{d-1} \beta_i + \frac{(d-p-a-1)(d-p-a-2)}{2} - (d-q) - (a+p) + b + 1 \\
&= \Big[(k-1)(d-1) - \frac{(d-2)(d-3)}{2} + 1\Big] + \frac{(d-p-a-1)(d-p-a-2)}{2} \\
&\quad + q - p - a - 1.
\end{aligned}$$

It follows that $t_2 \leqslant \frac{k(k+1)}{2}$ if and only if

$$\frac{(d-p-a-1)(d-p-a-2)}{2} + q - p - a - 1 \leqslant \frac{(d-k-1)(d-k-2)}{2}.$$

This condition is satisfied since $p > k-a$. $\square$

5 Metric graphs

If G is a graph, we define the *metric graph* Γ *corresponding* to G as the metric graph with $V(\Gamma) = V(G)$, $E(\Gamma) = E(G)$ and edge lengths $l(e) = 1$, see [HKN13]. In this section we explain how to extend the previous results to the complete metric graphs corresponding to the graphs K_d, which we will also denote by K_d. We start by briefly recalling the main definitions regarding linear systems on metric graphs. We refer to [MZ08, GK08, HKN13] for further reading.

Definition 16. Let Γ be a metric graph. A *divisor* on Γ is an element of the free abelian group $\mathrm{Div}(\Gamma)$ on the points of the graph, so

$$D = \sum_{p \in \Gamma} a_p\,(p) \quad \text{with } a_p \in \mathbb{Z}.$$

The *degree* of D, denoted by $\deg(D)$, is the sum of its coefficients. As before, if $a_p \geqslant 0$ for every $p \in \Gamma$, the divisor is said to be *effective*. The *support of a divisor* D is the set of points p of Γ such that $a_p \neq 0$ and it is indicated with $\mathrm{supp}(D)$.

A *rational function* $f : \Gamma \to \mathbb{R}$ on Γ is continuous, piecewise linear with integer slopes and only finitely many pieces. The *principal divisor* $\mathrm{div}(f)$ associated to f is the divisor whose coefficient at p is given by the sum of the incoming slopes of f at p. Only at a finite number of points, the coefficients are not zero.

Two divisors D_1, $D_2 \in \mathrm{Div}(\Gamma)$ are *linearly equivalent*, $D_1 \sim D_2$, if there exists a rational function f such that

$$D_1 - D_2 = \mathrm{div}(f).$$

The *linear system* of D, indicated with $|D|$, is the set of the effective divisors linearly equivalent to D,

$$|D| = \Big\{ E \in \mathrm{Div}(\Gamma) \Big|\ E \geqslant 0, E \sim D \Big\}.$$

The *rank* $\mathrm{rk}_\Gamma(D)$ *of a divisor* is defined as -1 if D is not equivalent to any effective divisor, otherwise

$$\mathrm{rk}_\Gamma(D) = \max \Big\{ r \in \mathbb{Z}_{\geqslant 0} \Big|\ |D - E| \neq \varnothing \quad \forall\ E \in \mathrm{Div}(\Gamma),\ E \geqslant 0,\ \deg(E) = r \Big\}.$$

Again, reduced divisor on metric graphs will play an important role. We recall the definition from [HKN13] and [Luo11].

Definition 17. Let Γ be a metric graph and X be a closed connected subset of Γ. Given $p \in \partial X$, the *outgoing degree* $\mathrm{outdeg}_X(p)$ *of* X *at* p is defined as the maximum number of internally disjoint segments in $\Gamma \setminus X$ with an open end in p. Let D be a divisor on Γ. A boundary point $p \in \partial X$ is *saturated with respect to* X *and* D if $D(p) \geqslant \mathrm{outdeg}_X(p)$, and *non-saturated* otherwise. A divisor D is *p-reduced* if it is effective in $\Gamma \setminus \{p\}$ and each closed connected subset $X \subseteq \Gamma \setminus \{p\}$ contains a non-saturated boundary point.

Theorem 18 (Proposition 7 in [MZ08])**.** Let D be a divisor on a metric graph Γ. For every point $p \in \Gamma$ there exists a unique p-reduced divisor linearly equivalent to D.

Let K_d be the complete metric graphs on d vertices. We present a similar characterization of reduced divisors on K_d as in Lemma 5. Given a vertex v of K_d and an ordering $v_1, \ldots, v_{d-1}$ of the vertices in $V(K_d) \setminus \{v\}$, for every $1 \leqslant i \leqslant d-1$ we define

$$A_i = (v, v_i] \sqcup \bigsqcup_{j<i} (v_j, v_i].$$

Note that $K_d \setminus \{v\} = \sqcup_{i=1}^{d-1} A_i$. See Figure 1 for an example.

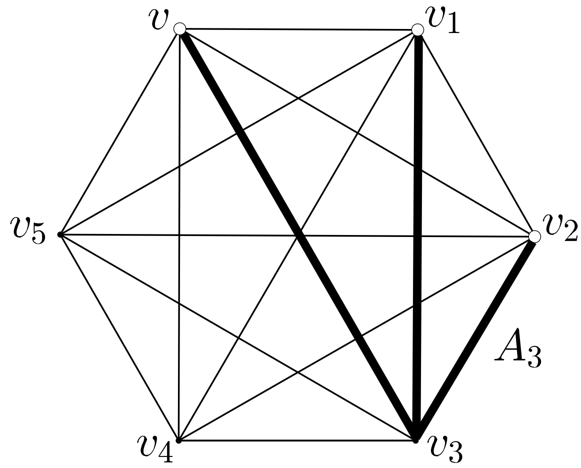

Figure 1: The thickened edges represent the set A_3. The vertices v, v_1 and v_2 are not contained in A_3.

Lemma 19. Let D be a divisor on the metric graph K_d. It is reduced with respect to a vertex v if and only if the following conditions are satisfied:

1. D is effective in $K_d \setminus \{v\}$;
2. for every edge $e \in E(K_d)$,
$$\sum_{p \in e^\circ} D(p) \leqslant 1;$$
3. there exists an ordering $v_1, \ldots, v_{d-1}$ of the vertices in $V(K_d) \setminus \{v\}$ such that
$$\mathfrak{D}(v_i) := \deg(D|_{A_i}) \leqslant i-1.$$

Proof. We start by proving the 'if' part, so assume that K_d satisfies the three conditions. We need to show that every closed connected subset X of $K_d \setminus \{v\}$ contains a non-saturated boundary point. If X contains a boundary point p that is not in the support of D, than

p is non-saturated. So we focus on the case that the boundary points of X are all in the support of D. In case $X = \{p\}$, with p a point in the interior of an edge, then p is non-saturated because of condition (2).

Suppose that X satisfies the hypothesis that if two vertices are contained in X, then also the edge connecting them is contained in X. We define

$$A := V(K_d) \cap X \quad \text{and} \quad A^{\complement} = (K_d \setminus X) \cap V(K_d) = V(K_d) \setminus A.$$

A vertex w of X is not-saturated if and only if

$$D(w) + \left| \left\{ q \in \partial X \,:\, q \in (w, w') \text{ with } w' \in A^{\complement} \right\} \right| < d - |A|.$$

Let $j = \min\{i \mid v_i \in A\}$, so $j \leqslant d - |A|$. By condition (3), we know that $\mathfrak{D}(v_j) \leqslant j - 1$. We show that v_j is non-saturated:

$$\begin{aligned}
&D(v_j) + \left| \left\{ q \in \partial X \,:\, q \in (v_j, w') \text{ with } w' \in A^{\complement} \right\} \right| \\
&= \mathfrak{D}(v_j) + \left| \left\{ q \in \partial X \,:\, q \in (v_j, v_i) \text{ with } i > j \text{ and } v_i \in A^{\complement} \right\} \right| \\
&\leqslant \mathfrak{D}(v_j) + d - |A| - j \\
&\leqslant j - 1 + d - |A| - j = d - |A| - 1.
\end{aligned}$$

If the hypothesis on the closed subset X is not satisfied, consider a closed connected subset $X' \subset K_d \setminus \{v\}$ containing X and that satisfies the hypothesis. For each vertex $v \in X$ it holds that $\operatorname{outdeg}_{X'}(v) \leqslant \operatorname{outdeg}_X(v)$. Therefore if v is non-saturated for X', it is also non-saturated for X.

For the 'only if' part, suppose that D is v-reduced. It is clear that the conditions (1) and (2) are satisfied. We now show condition (3). As a first step, we claim that there exists a vertex $v_1 \neq v$ such that

$$(v, v_1] \cap \operatorname{Supp}(D) = \varnothing.$$

Indeed, if not, there is a point $p \in (v, w] \cap \operatorname{Supp}(D)$ for each vertex $w \neq v$, and we can consider a connected subset $X \subset K_d \setminus \{v\}$ for which the boundary consists of all these points p. Note that X contains the complete subgraph on the vertices in $V(K_d) \setminus \{v\}$. Since each boundary point $p \in \partial X$ is saturated, this is in contradiction with the fact that D is v-reduced.

In a similar way, we can find a vertex $v_2 \neq v, v_1$ such that the sum of the coefficients $D(p)$ with $p \in (v, v_2] \cup (v_1, v_2]$ is at most 1. In order to show this, we work with a connected component X for which the boundary consists of points $p \in \operatorname{Supp}(D)$ lying on either $(v, w]$ or $(v_1, w]$ for some vertex $w \neq v, v_1$. This component contains the complete subgraph on the vertices in $V(K_d) \setminus \{v, v_1\}$.

Iterating this argument, we find back condition (3). □

Example 20. Consider the graph and the divisor D drawn in Figure 2. The divisor is v-reduced, since we have

$$\mathfrak{D}(v_1) = 0, \quad \mathfrak{D}(v_2) = 0, \quad \mathfrak{D}(v_3) = 2, \quad \mathfrak{D}(v_4) = 2, \quad \text{and} \quad \mathfrak{D}(v_5) = 4.$$

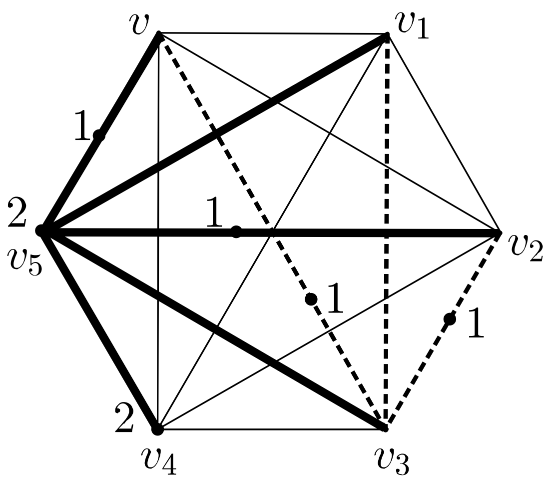

Figure 2: The thickened edges represent the set A_5, and the dashed ones the set A_3.

The definitions of linear systems g^r_s and gonality sequences translate to the setting of metric graphs. We prove that the gonality sequence of complete metric graphs K_d is the same as the one of the ordinary complete graphs.

Theorem 21. The gonality sequence of the metric graph K_d is

$$\gamma_r = \begin{cases} kd - h & \text{if } r < g \\ g + r & \text{if } r \geqslant g. \end{cases}$$

where k and h are the uniquely determined integers with $1 \leqslant k \leqslant d - 3$ and $0 \leqslant h \leqslant k$ such that

$$r = \frac{k(k+3)}{2} - h.$$

In the proof of the theorem we will use the following result from [HKN13], which relates the rank on graphs with the rank on metric graphs:

Theorem 22. Let D be a divisor on a loopless graph G and let Γ be the metric graph corresponding to G. Then,

$$\mathrm{rk}_G(D) = \mathrm{rk}_\Gamma(D).$$

Proof of Theorem 21. Again for every $k \geqslant 1$ and for every $0 \leqslant h \leqslant k$ there are two statements that need to be shown:

- there exists a divisor of degree $kd - h$ and rank $\frac{k(k+3)}{2} - h$;
- there does not exist a divisor of degree strictly smaller than $kd-h$ and rank $\frac{k(k+3)}{2} - h$.

The first statement follows directly from Theorem 1 and Theorem 22. We consider the second statement. As before, it is enough to set $h = k$ and to show that the rank of every divisor D of degree $dk - k - 1$ is strictly smaller than $\frac{k(k+3)}{2} - k = \frac{k(k+1)}{2}$.

Let D be v_d-reduced. By Lemma 19, we can label the vertices so that the following inequalities are satisfied:

$$\mathfrak{D}(v_1) = 0 \leqslant \mathfrak{D}(v_2) \leqslant \cdots \leqslant \mathfrak{D}(v_{d-1}) \leqslant d-2, \quad \text{and} \quad \mathfrak{D}(v_i) \leqslant i-1, \ \text{for } i \leqslant d-1.$$

We write

$$D(v_d) = a(d-1) + b, \ \text{with } a, b \in \mathbb{Z}_{\geqslant 0}, \ 0 \leqslant b \leqslant d-2 \text{ and } a \leqslant k-1.$$

Consider the divisors

$$D' = \Big(b - (d-1)\Big)(v_d) + \sum_{i=1}^{d-1} \Big[D|_{A_i} + (a+1)(v_i)\Big] \sim D, \tag{1}$$

$$D'' = b\,(v_d) + \sum_{i=1}^{d-1} \Big[D|_{A_i} + a\,(v_i)\Big] \sim D. \tag{2}$$

We define $\mathfrak{a}_i := \mathfrak{D}(v_i) + a - (i-2)$ and $\mathfrak{a}_i^+ := \max\{0, \mathfrak{a}_i\}$. The sequence $\mathfrak{a} = (\mathfrak{a}_i)_{i=1,\dots,d-1}$ will play the role of the sequence α in Section 4. In particular, we have that

$$\sum_{i=1}^{d-1} \mathfrak{a}_i = k(d-1) - \frac{(d-2)(d-3)}{2} - b.$$

and the inequalities in $(*)$ are satisfied.

Because of Claim 11, either

$$\mathfrak{t}_1 := \sum_{i=1}^{d-1} \mathfrak{a}_i^+ \quad \text{or} \quad \mathfrak{t}_2 := b + 1 + \sum_{i=1}^{d-1} \Big(\mathfrak{a}_i - 1\Big)^+$$

is at most $\frac{k(k+1)}{2}$. This allows us to construct an effective divisor E of degree $\frac{k(k+1)}{2}$ such that either $D' - E$ or $D'' - E$ is v_d-reduced and has a negative coefficient at v_d, hence $\mathrm{rk}(D) \leqslant \frac{k(k+1)}{2}$. □

6 What about arbitrary edge lengths?

In this section, we will focus on complete metric graphs with arbitrary edge lengths, which we will denote by $K_d(\ell)$ for a vector $\ell \in (\mathbb{R}_{>0})^{\binom{d}{2}}$. The arguments presented in the previous sections do not directly extend to the graphs $K_d(\ell)$. Therefore, computing the gonality sequence of $K_d(\ell)$ is still an open problem in general. Below, we briefly explain the obstructions we encountered.

In Section 3, we proved that the divisor $D = \sum_{i=1}^{d} k(v_i)$ on the graph $G = K_d$ has rank $\frac{k(k+3)}{2}$ for $1 \leqslant k \leqslant d-3$. Theorem 22 allowed us to conclude that the divisor D viewed on the corresponding metric graph $\Gamma = K_d$ has the same rank. This result does not hold if the lengths of the edges are arbitrary. Moreover, our computation of the rank on $G = K_d$ relies on [CB16, Lemma 3], which uses the symmetries of complete graphs.

Even though we can not use the arguments from Section 3, we still expect that the divisor $D = \sum_{i=1}^{d} k(v_i)$ on $\Gamma = K_d(\ell)$ has rank $\frac{k(k+3)}{2}$, if $1 \leqslant k \leqslant d-3$. This statement holds for $k = d-3$: in this case, the divisor D coincides with the canonical divisor K_Γ, which has rank $g - 1 = \frac{d(d-3)}{2}$. It is an easy exercise to check this statement for $k = 1$ and (by using Riemann-Roch) $k = d-4$. The following results provide extra motivation.

Lemma 23. The divisor $D = \sum_{i=1}^{d} k(v_i)$ on $\Gamma = K_d(\ell)$ has rank at most $\frac{k(k+3)}{2}$.

Proof. Consider the effective divisor

$$E = \sum_{j=1}^{k+2} (k+2-j)(v_j),$$

which has degree $\frac{k(k+3)}{2} + 1$. The divisor $D - E$ is v_1-reduced (Lemma 19 also holds for arbitrary edge lengths) and has a negative coefficient at v_1, hence $|D - E| = \varnothing$ and $\mathrm{rk}_\Gamma(D) \leqslant \frac{k(k+3)}{2}$. □

Proposition 24. If $d \geqslant 5$, the divisor $D = \sum_{i=1}^{d} 2(v_i)$ on $\Gamma = K_d(\ell)$ has rank 5.

Proof. We need to show that $|D-E| \neq \varnothing$ for every effective divisor E of degree 5. We may assume that E is supported on the vertices, since the vertex set is rank-determining (see [Luo11]). If $E(v_i) \leqslant 2$ for all i, then $D - E \geqslant 0$ and the statement follows directly. Hence, we may assume that $E(v_i) \geqslant 3$ for some index i, so E is of the form $3(v_i) + (v_j) + (v_k)$, $3(v_i) + 2(v_j)$, $4(v_i) + (v_j)$ or $5(v_i)$ for some $i \neq j \neq k \neq i$. We will only handle the latter case; the others are proven in an analogous manner.

Suppose that $E = 5(v_i)$. Denote by δ the minimal length of the edges (v_i, v_j) with $j \neq i$ and by $p_j \in (v_i, v_j)$ the point on distance δ from v_j. Then

$$D - E = -3(v_i) + \sum_{j \neq i} 2(v_j) \quad \sim \quad F := -3(v_i) + \sum_{j \neq i} 2(p_j).$$

If $p_j = v_i$ for more than one index j, then we are done since $F \geqslant 0$. Otherwise, this means that the divisor F has coefficient -1 at v_i. Say that k is the unique index for which $p_k = v_i$. Now we consider the linearly equivalent divisor

$$F' := -(v_i) + \sum_{j \neq i,\ j \neq k} \Big((s_j) + (t_j) \Big),$$

with $s_j, t_j \in (v_j, v_i)$ the points on equal and maximal distance from p_j, where s_j is in between p_j and v_j, and t_j is in between p_j and v_i. Because of the maximality assumption, either $t_j = v_i$ or $s_j = v_j$ for all $j \neq i$. If $t_j = v_i$ for at least one j, we can conclude since $F' \geqslant 0$. Otherwise, we have that $s_j = v_j$ for all j. Then we can consider a linearly equivalent divisor

$$F'' = -(v_i) + \sum_{j \neq i,\ j \neq k} \Big((s'_j) + (t'_j) \Big)$$

with $t'_j \in (v_j, v_i)$, s'_j on the path (v_j, v_k, v_i) and $t'_j = v_i$ or $s'_j = v_i$ for a certain j, hence $F'' \geqslant 0$. □

Also the proof of the sharpness of the upper bound cannot directly be extended to the graphs $\Gamma = K_d(\ell)$. Indeed, the divisor D' and D'' introduced in (1) and (2) are no longer linearly equivalent to D. It is natural to consider instead the following divisors

$$D' = \Big(b - (d-1)\Big)(v_d) + \sum_{i=1}^{d-1} \Big[D|_{A_i} + (a+1)(p_i)\Big] \sim D, \tag{3}$$

$$D'' = b\,(v_d) + \sum_{i=1}^{d-1} \Big[D|_{A_i} + a\,(p_i)\Big] \sim D, \tag{4}$$

with $p_i \in (v_d, v_i)$ and $p_j = v_j$ for at least one index j. But then we run into problems while constructing the effective divisor E of degree $\frac{k(k+1)}{2}$, since in general $D' - E$ and $D'' - E$ are not v_d-reduced.

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